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ON A CONNECTION BETWEEN ALGEBRA, LOGIC AND LINGUISTICS

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Une *grammaire indépendante du contexte* (avec flèches renversées) consiste en dérivations $A_1 \ldots A_n \to A_{n+1}$; on appelle les A_i *types*. On peut considérer une telle grammaire comme un calcul de séquences à la Gentzen, mais sans ses trois règles structurelles: permutation, contraction et atténuation. (Les trois règles sont permises dans la logique intuitioniste, les deux premières dans la "logique pertinente", et la première dans la "logique linéaire" de Girard.)

L'idée d'une *multicatégorie* trouve son origine dans la méthode des applications bi-linéaires de Bourbaki. Formellement, une multicatégorie est une grammaire indépendante du contexte avec une relation d'équivalence appropriée entre dérivations (séquences). Elle peut être caractérisée par son langage interne. Si on postule des équations appropriées correspondantes aux trois règles structurelles, on obtient une *théorie algébrique* (multisortale).

Une *grammaire catégorielle* est une grammaire indépendante du contexte avec une opération binaire / (= sur) entre types; la notation logique est $\Leftarrow$ (= si). Dans une multicatégorie, A/B est le $Hom(B, A)$ interne; dans le cas spécial, $A \Leftarrow B$ est l'exponentiation A^B. Le langage interne d'une multicatégorie avec / est une version unilatérale du λ-calcul; avec les règles structurelles de Gentzen, elle devient le λK-calcul usuel de Church. Pour une grammaire catégorielle (de l'anglais, par exemple) le passage du premier au second est essentiellement ce que fait la sémantique de Montague.

It is claimed that logic is to linguistics as universal algebra is to multilinear algebra, the common thread being Gentzen's notion of "sequent", which goes under different names in the four fields: deduction, derivation, operation and multilinear form.

1. Contextfree grammars as multicategories.

A *contextfree derivation* has the form $f : A_1 \ldots A_n \to A$, where A_i and A are *syntactic types* (also called "grammatical categories"). The arrow here

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follows the convention in categorial grammar; in generative grammar the arrow is reversed.

Among the derivations there is the reflexive law

$$1_A : A \to A$$

and a rule for constructing new derivations from old:

$$\frac{f : \Lambda \to A \quad g : \Gamma A \Delta \to B}{g < f >: \Gamma \Lambda \Delta \to B},$$

called the *cut rule*, which represents substitution of f into g. Here capital Greek letters denote strings of types, e.g. $\Lambda = A_1 \ldots A_n, n \geq 0$. The notation $g < f >$ for the new derivation is somewhat ambiguous, inasmuch as it does not show where f has been substituted; but we shall use it just the same, as the place of substitution will usually be clear from the context and because a more precise notation would appear to be rather cumbersome.

Among the syntactic types will be s for "statement" and others to be discussed in Section 6 below. It is sometimes convenient to include English words among the types, think of them as one-element types. An analysis would yield two distinct derivations

$$experience\ teaches\ spiders\ that\ time\ flies\ \to \mathrm{s},$$

interpreting the sentence to the left of the arrow in two different ways. To see this compare the passive sentences *spiders that time flies are taught by experience* and *spiders are taught by experience that time flies.* On the other hand, there seems to be no need for distinguishing the derivations

$$John\ (\ likes\ Jane)\ \to \mathrm{s}$$

and

$$(John\ likes)\ Jane\ \to \mathrm{s},$$

assuming that both derivations are possible.

Anyway, equality of derivations is something that should be considered. If this is done, a contextfree grammar becomes what has been called a *multicategory* [L1969]. However, the rules of equality are a bit awkward to state directly, so I now [L1989] prefer to do this with the help of the *internal language*: we adjoin countably many variables of each type, say x_i is one such variable of type A_i, then $fx_1 \ldots x_n$ is to be regarded as a term of type A

whenever $f : A_1 \ldots A_n \to A$ is a derivation. If also $g : A_1 \ldots A_n \to A$, we shall say that $f = g$ provided we can prove $fx_1 \ldots x_n = gx_1 \ldots x_n$ from the usual rules of equality, including the rule of substitution of terms of a certain type for variables of that type.

While we are here mainly interested in applications to linguistics, it should be pointed out that derivations $f : \Lambda \to A$ have appeared in multilinear algebra under a different name: *multilinear forms.*

2. The rôle of Gentzen's structural rules.

The reader will have noticed the similarity between derivations in linguistics and *deductions* in logic, called "sequents" by Gentzen, who imposed three *structural rules* on deductions in intuitionistic logic, which are absent in linguistics:

$$\frac{f : \Gamma A \Lambda B \Delta \to C}{f^i : \Gamma B \Lambda A \Delta \to C} \quad \textit{interchange},$$

$$\frac{f : \Gamma A A \Delta \to C}{f^c : \Gamma A \Delta \to C} \quad \textit{contraction},$$

$$\frac{f : \Gamma \Delta \to C}{f^t : \Gamma A \Delta \to C} \quad \textit{thinning}.$$

Again our notation f^i, f^c, f^t ignores the place where the interchange, contraction and thinning occurs. The capital letters now denote "formulas"; recall the popular slogan: "formulas as types".

While logicians are not usually interested in the question when two deductions are equal, this is sometimes important. For example, one ought to distinguish between the two deductions $A \wedge A \to A$ obtained by putting $B = A$ in $A \wedge B \to A$ and $A \wedge B \to B$ respectively.

Going over to the internal language of a "Gentzen multicategory" [see Szabo 1978], one may think of the variable x_i of type A_i as an assumption that A_i holds. If $f : A_1 \ldots A_n \to A$ is a deduction à la Gentzen, one may think of $fx_1 \ldots x_n$ as a proof of A from the assumptions $x_i (i = 1, \ldots, n)$ à la Prawitz. Note that the same formula may be assumed several times in one proof.

Gentzen's structural rules may now be viewed as definitions of new deductions, when we write $x, y, z, \ldots$ for variables of types $A, B, C, \ldots$ respectively, and $u = u_1 \ldots u_m, v, w, \ldots$ for strings of variables of types $\Gamma, \Delta, \Lambda, \ldots$

respectively:

$$\begin{aligned} f^{i}uywxv &= fuxwyv, \\ f^{c}uxv &= fuxxv, \\ f^{t}uxv &= fuv. \end{aligned}$$

Actually, a Gentzen multicategory may be considered to be a many-sorted algebraic theory, in which the formulas are usually called *sorts* and the deductions are called *operations*.

To adopt a neutral terminology, let us use Gentzen's word *sequent* for $f : A_1 \ldots A_n \to A$. We may then summarize our observations by saying that sequents appear in linguistics as *derivations*, in linear algebra as *forms*, in logic as *deductions* and in universal algebra as *operations*. Curiously, in linguistics, logic and algebra, people have considered variants in which some of Gentzen's structural rules are permitted while others are forbidden.

In linguistics, the three structural rules should be absent from syntax, at least the way I see it, but present in semantics, as conceived by Curry [1958] and Montague [1974]. However some linguists, notably van Benthem [1983, 1988], have advocated that the interchange rule should be admitted when one discusses languages with free word order. Steedman [1988] and Szabolcsi [1989] seem to forbid thinning and to allow contraction.

In intuitionistic logic, all three rules are permitted, in "relevance logic" only interchange and contraction and, in Girard's linear logic as originally formulated [1987], only interchange. However, one can conceive of a non-symmetric linear logic in which all structural rules are forbidden [see Girard 1989].

In algebra, I am only aware of multilinear algebra, in which all three rules are forbidden, and universal algebra, in which all three are allowed. I haven't come across any other variants, unless one wishes to consider as an algebraic theory Church's λ-calculus [1941], which forbids thinning, as opposed to his λK-calculus, which allows it.

3. Type forming operations.

Let us now look at operation on types, formulas or sorts as studied in linguistics, logic and algebra respectively.

We are primarily concerned with binary and nullary operations, presented in their most common notation in algebra and linguistics, as opposed to logic, where they are called *connectives* and occur in a rival notation. In this connection, one may also consider "quantifiers" with respect to variable types.

algebra:	$\otimes$	$/$	$\backslash$	$\times$	$+$	I	1	0	$\prod_X$	$\coprod_X$
logic :		$\Leftarrow$	$\Rightarrow$	$\wedge$	$\vee$		$\top$	$\perp$	$\forall_X$	$\exists_X$

We shall henceforth refer to these type forming operations as "connectives". They are usually introduced in terms of the set $[\Gamma, A]$ of all sequents $f : \Gamma \to A$. To do full justice to the quantifiers, one must declare the free variables and should write $f : \Gamma \xrightarrow{\mathcal{X}} A$ or $f \in [\Gamma, A]_{\mathcal{X}}$, where $\mathcal{X}$ is a set of variables which includes all the variables occurring freely in Γ and A. Such free variables are of course subject to the usual law of substitution, to which I plan to return on another occasion. In principle, all connectives are introduced by means of certain biunique correspondences, namely as follows:

$$\begin{aligned}
[\Gamma(A \otimes B)\Delta, C] &\cong [\Gamma A B \Delta, C], \\
[\Gamma, A/B] &\cong [\Gamma B, A], \\
[\Gamma, B\backslash A] &\cong [B\Gamma, A], \\
[\Gamma, A \times B] &\cong [\Gamma, A] \times [\Gamma, B], \\
[\Gamma(A + B)\Delta, C] &\cong [\Gamma A \Delta, C] \times [\Gamma B \Delta, C], \\
[\Gamma I \Delta, C] &\cong [\Gamma\Delta, C], \\
[\Gamma, 1] &\cong \{*\}, \\
[\Gamma 0 \Delta, C] &\cong \{*\}, \\
[\Gamma, \prod_X A(X)] &\cong [\Gamma, A(X)]_X, \\
[\Gamma \coprod_X A(X)\Delta, C] &\cong [\Gamma A(X)\Delta, C]_X.
\end{aligned}$$

On the right side, $\times$ denotes the cartesian product of sets and $\{*\}$ is a typical one-element set.

If the reader is puzzled by the absence of $I, \otimes$ and $\Leftarrow$ in the usual logical systems, linear logic excepted, this comes from the fact that, in the presence of Gentzen's three structural rules, $I = 1$, $A \otimes B = A \times B$ and $B\backslash A = A/B$. [See L1989].

More precisely, the connectives are introduced by specifying a particular sequent and providing additional information to ensure that it induces a biunique correspondence as required:

(1) $\mathbf{m}_{AB} : AB \to A \otimes B$;
if $f : \Gamma AB\Delta \to C$ then $\exists! f^{\S} : \Gamma(A \otimes B)\Delta \to C$ such that
$f^{\S} u \mathrm{m} xyv = fuxyv$.

(2) $\mathbf{e}_{AB} : (A/B)B \to A$;
if $f : \Gamma B \to A$ then $\exists! f^* : \Gamma \to A/B$ such that
$f^* \mathrm{e} uy = fuy$.

(2') $\mathbf{e'}_{AB} : B(B\backslash A) \to A$;
if $f : B\Gamma \to A$ then $\exists! f^+ : \Gamma \to B\backslash A$ such that
$\mathrm{e'} y f^+ u = fyu$.

(3) $\mathbf{p}_{AB} : A \times B \to A$, $\mathbf{q}_{AB} : A \times B \to B$;
if $f : \Gamma \to A$ and $g : \Gamma \to B$ then $\exists! < f, g >: \Gamma \to A \times B$ such that
$\mathrm{p}(< f, g > u) = fu$, $\mathrm{q}(< f, g > u) = gu$.

(4) $\mathbf{k}_{AB} : A \to A + B$, $\boldsymbol{\ell}_{AB} : B \to A + B$;
if $f : \Gamma A\Delta \to C$ and $g : \Gamma B\Delta \to C$ then $\exists! [f, g] : \Gamma(A + B)\Delta \to C$ such that
$[f, g]u\mathrm{k}xv = fuxv$, $[f, g]u\, \boldsymbol{\ell} yv = guyv$.

(5) $\mathbf{i} : \quad \to I$;
if $f : \Gamma\Delta \to C$ then $\exists! f^{\#} : \Gamma I\Delta \to C$ such that
$f^{\#} u \mathrm{i} v = fuv$.

(6) $\mathrm{o}_\Gamma : \Gamma \to 1$,
$\exists! g : \Gamma \to 1$.

(7) $\Box^{C}_{\Gamma\Delta} : \Gamma 0\Delta \to C$;
$\exists! g : \Gamma 0\Delta \to C$.

(8) $\boldsymbol{\pi}_X : \prod_X A(X) \xrightarrow{X} A(X)$;
if $\varphi(X) : \Gamma \xrightarrow{X} A(X)$ then $\exists! \varphi^\wedge : \Gamma \to \prod_X A(X)$ such that
$\boldsymbol{\pi}_X \varphi^\wedge u =_X \varphi(X)u$.

(9) $\boldsymbol{\kappa}_X : A(X) \xrightarrow{X} \coprod_X A(X)$;
if $\varphi(X) : \Gamma A(X)\Delta \xrightarrow{X} C$ then $\exists! \varphi^\vee : \Gamma \coprod_X A(X)\Delta \to B$ such that
$\varphi^\vee u(\boldsymbol{\kappa}_X z)v =_X \varphi(X)uzv$.

Note that, in principle, $\mathbf{m}_{AB}, \mathbf{e}_{AB}$ etc should carry subscripts; but, in practice, we have usually omitted them. We have also been negligent in failing to declare free variables other than variable types in the above equation. This becomes more important when Gentzen's structural rules are admitted.

It should be pointed out that, in all cases considered, the existence and uniqueness of the sequent with a certain property can be expressed by means of equations which do not contain free variables, except variable types:

(1) $f^{\S} < \mathbf{m} >= f,\ (g < \mathbf{m} >)^{\S} = g.$
(2) $\mathbf{e} < f^* >= f,\ (\mathbf{e} < g >)^* = g.$
(2') $\mathbf{e}' < f^{\dagger} >= f,\ (\mathbf{e}' < g >)^{\dagger} = g.$
(3) $\mathbf{p} \ll f, g \gg= f,\ \mathbf{q} \ll f, g \gg= g,\ < \mathbf{p} < h >, \mathbf{q} < h \gg= h.$
(4) $[f, g] < \mathbf{k} >= f,\ [f, g] < \boldsymbol{\ell} >= g,\ [h < \mathbf{k} >, h < \boldsymbol{\ell} >] = h.$
(5) $f^{\#} < \mathbf{i} >= f,\ (g < \mathbf{i} >)^{\#} = g.$
(6) $\circ_{\Gamma} = g.$
(7) $\square^{C}_{\Gamma\Delta} = g.$
(8) $\boldsymbol{\pi}_X < \varphi^{\wedge} >=_X \varphi(X),\ (\boldsymbol{\pi}_X < g >)^{\wedge} = g.$
(9) $\varphi^{\vee} < \boldsymbol{\kappa}_X >=_X \varphi(X),\ (g < \boldsymbol{\kappa}_X >)^{\vee} = g.$

As regards (2), the notation for $\mathbf{e}_{AB}ty$ in [LS 1986] was $t`y$ (read t of y) and the usual notation for f^*u is $\lambda_{y \in B} fuy$. The two equations in (2) then take the more familiar form:

$$\lambda_{y\in B} fuy \text{ `}y =_{u,y} fuy,\ \lambda_{y\in B}(gu \text{ `}y) =_u gu,$$

where, for once, we have declared the variables. Moreover, in the first of these equations, we may substitute any term b of type B for y. In particular, if b contains no free variables other than such as appear in the string u, we obtain:

$$\lambda_{y\in B} fuy \text{ `}b =_u fub.$$

A similar, but distinct, notation could be adopted for (2'). But is it worth the trouble to introduce an analogous notation for (1)?

As regards (6) and (7), I am not entirely happy with the way they have been stated.

4. Freely generated categorial grammars.

Let us concentrate on the applications to linguistics and multilinear algebra, so we can forget about Gentzen's structural rules. The question now arises: given a contextfree grammar $\mathcal{G}$ (that is, a multicategory), what happens when we freely adjoin some or all of the connectives $\otimes, /, \backslash, \times, \ldots$? More precisely, we wish to construct the left adjoint F to the forgetful functor U from contextfree grammars with connectives to contextfree grammars without. Thus $F(\mathcal{G})$ is the contextfree grammar with $\otimes, /$, etc. freely generated by $\mathcal{G}$. We hope that the multifunctor $\mathcal{G} \to UF(\mathcal{G})$ will turn out to be full and faithful.

The main problem we face is to calculate the set $[\Gamma, A]$ of all sequents $\Gamma \to A$ in $UF(\mathcal{G})$. The problem splits into two parts:

(I) Find all sequents $f : \Gamma \to A$.
(II) Given sequents $f, g : \Gamma \to A$, decide when $f = g$.

The solution to Problem I will show that $\mathcal{G} \to UF(\mathcal{G})$ is full, while the solution to Problem II, sometimes called the "coherence problem", should imply that it is faithful.

I have looked at Problem I repeatedly, using Gentzen's method of cut elimination, first ignoring equations between sequents [L1958] and later bringing them in [L1969]. I intend to look at it once more, exploiting the internal language to get a better insight into what is going on. What this means, essentially, is that one replaces Gentzen style deductions by Prawitz style proofs.

My attacks on Problem II [L1968, 1969] contained what I still believe were some good ideas, but also contained some mistakes. More successful attacks, sometimes for related systems, were carried out by many people; in particular Minc [1977] and Szabo [1978] made use of normalization and the Church-Rosser property, which I believe to be the natural method. To this also I hope to return on another occasion.

Here we shall merely illustrate how Problem I is to be solved, by confining attention to one connective only, and also give a hint for attacking Problem II. Suppose we have a sequent $\Theta \to C$ in $UF(\mathcal{G})$ containing at most the connective $/$, it must be of one of the following forms:

(i) it is already a sequent in $\mathcal{G}$;
(ii) it is of the form $1_A : A \to A$;
(iii) it is obtained by a cut:

$$\frac{f : \Lambda \to A \qquad g : \Gamma A \Delta \to C}{g < f >: \Gamma \Lambda \Delta \to C};$$

(iv) it is of the form (2a):

$$\mathbf{e}_{AB} : (A/B)B \to A;$$

(v) it is of the form (2b):

$$\frac{f : \Theta B \to A}{f^* : \Theta \to A/B}.$$

Given Θ and C, it is easy to find all sequents $\Theta \to C$ (if any) of the form (i), (ii) and (iv). If $C = A/B$, we may reduce the problem of finding all sequents $\Theta \to C$ of the form (v) to that of finding all sequents $\Theta B \to A$. The difficulty is with the cut (iii): suppose we have decomposed Θ as $\Gamma\Lambda\Delta$, how do we know which A to try? The way out, following an idea of Gentzen, is to replace (iv) (that is (2b)) by the following more powerful rule:

(iv')
$$\frac{f : \Lambda \to B \qquad g : \Gamma A \Delta \to C}{g[f] : \Gamma(A/B)\Lambda\Delta \to C},$$

where $g[f] = g < \mathbf{e} < f >>$ is given by

$$g[f]utwv = gu\mathbf{e}tfwv,$$

t being a variable of type A/B. While (iv') was constructed with the help of a cut, it is a harmless rule, inasmuch as the premises contain nothing which does not already occur in the conclusion.

One byproduct of replacing (iv) by (iv') is that we may dispense with (ii). For, unless (ii) is already an arrow in $\mathcal{G}$, hence a special case of (i), it is of the form $1_{A/B} : A/B \to A/B$, in which case it may easily be proved, with the help of the internal language, that

$$1_{A/B} = (1_A[1_B])^*.$$

Thus $1_{A/B}$ can be constructed without using (ii), provided 1_A and 1_B can be so constructed, which we may assume by an appropriate inductional assumption.

More importantly, if we use (iv') in place of (iv), any sequent constructed with cut is *equal* to one without cut. This is Gentzen's *cut elimination theorem* in the absence of structural rules, although Gentzen himself did not bother about "equality".

Limitations of space do not permit me to include the proof of the cut elimination theorem here. On the face of it, the demonstration that a sequent constructed with cut equals one constructed without makes use of both equations $e < f^* >= f$ and $(e < g >)^* = g$. When translated into λ-notation however, it appears that, while $\lambda_{x\in A}\varphi(x)$ '$x = \varphi(x)$ is used, the equation $\lambda_{x\in A}(f$ '$x) = f$ is not used in full generality, but only a consequence of it, namely the implication:

$$\text{if } \varphi(x) =_x \psi(x) \text{ then } \lambda_{x\in A}\varphi(x) = \lambda_{x\in A}\psi(x).$$

Anyway, both the proof of cut elimination and the proof that $1_{A/B} = (1_A[1_B])^*$ make use of the following technique, which seems to be useful for attacking Problem II: to compare two arrows $f, g : \Gamma \to A/B$, compare the expressions $efuy$ and $eguy$ in the internal language. Details of these arguments will be found in the proceedings of the forthcoming conference "Kleene '90" in Bulgaria.

5. Some linguistic applications.

Although I now believe that production grammars are better suited to natural languages then categorial ones, I am willing to ask the question: of what possible significance are the connectives $\otimes, /, \backslash, \times$ etc. in linguistics? While we only discussed one of the connectives in some detail here, it appears that they may be freely added to a contextfree grammar without loss in generality. More precisely, when we look at the contextfree grammar $\mathcal{G}$ and the enhanced grammar $UF(\mathcal{G})$ as multicategories, the multifunctor $\mathcal{G} \to UF(\mathcal{G})$ is a full embedding. In particular, it is a conservative extension.

For the purpose of illustration, let $\mathcal{G}$ be the contextfree component of the small fragment of English grammar discussed in "Grammar as mathematics" [L1989b]. In this contextfree grammar one may, for example, generate the pseudo-sentence

(†) *John may have* Perf *be* Part *come often*.

Here Perf (= perfect participle of) and Part (= present participle of) are certain grammatical terms, which I have called "inflectors". Of course Perf *be* should be converted to *been* and Part *come* to *coming*, but the rules permitting this are not contextfree.

One can of course handle strings like (†) in a categorial grammar. Let us introduce the following types, except for subscripts the same as in [L1959]:

$$\begin{aligned} \mathbf{s}_1 &= \text{statement in present tense,} \\ \mathbf{n}_3 &= \text{name (= third person noun phrase),} \\ \mathbf{i} &= \text{infinitive,} \\ \mathbf{p} &= \text{present participle,} \\ \mathbf{q} &= \text{past participle.} \end{aligned}$$

If the different words in (†) are assigned types as follows, (†) is easily calculated to be of type $\mathbf{s}_1$:

John	*may*	*have*	Perf	*be*	Part	*come*	*often*
$\mathbf{n}_3$	$(\mathbf{n}_3\backslash\mathbf{s}_1)/\mathbf{i}$	$\mathbf{i}/\mathbf{q}$	$\mathbf{q}/\mathbf{i}$	$\mathbf{i}/\mathbf{p}$	$\mathbf{p}/\mathbf{i}$	$\mathbf{i}$	$\mathbf{i}\backslash\mathbf{i}$

To convert (†) into an actual sentence, the production grammar of [L1989b] requires two non-contextfree rules. As it is customary to reverse arrows in categorial grammars, these two rules will take the forbidden form:

$$\textit{coming} \to \text{Part}\ \textit{come}, \quad \textit{been} \to \text{Perf}\ \textit{be}.$$

But all is not lost. Let us incorporate the tensor product $\otimes$ to mimic juxtaposition, then these rules take the permitted form:

$$\textit{coming} \to \text{Part} \otimes \textit{come}, \quad \textit{been} \to \text{Perf} \otimes \textit{be}.$$

We may infer that *coming* has type $(\mathbf{p}/\mathbf{i}) \otimes \mathbf{i} \to \mathbf{p}$ and *been* has type $(\mathbf{q}/\mathbf{i}) \otimes (\mathbf{i}/\mathbf{p}) \to \mathbf{q}/\mathbf{p}$. We need the more complex types to the left of the arrows, for example, to check that

coming	*often*
$(\mathbf{p}/\mathbf{i}) \otimes \mathbf{i}$	$\mathbf{i}\backslash\mathbf{i}$

has type $\mathbf{p}$, as is shown in the following Gentzen-style argument:

$$\dfrac{\dfrac{\mathbf{i} \to \mathbf{i} \qquad \dfrac{\mathbf{i} \to \mathbf{i} \quad \mathbf{p} \to \mathbf{p}}{(\mathbf{p}/\mathbf{i})\mathbf{i} \to \mathbf{p}}}{(\mathbf{p}/\mathbf{i})\mathbf{i}(\mathbf{i}\backslash\mathbf{i}) \to \mathbf{p}}}{((\mathbf{p}/\mathbf{i}) \otimes \mathbf{i})(\mathbf{i}\backslash\mathbf{i}) \to \mathbf{p}}.$$

In a similar manner we find that *has* and *is* have types

$$((\mathbf{n}_3\backslash\mathbf{s}_1)/\mathbf{i}) \otimes (\mathbf{i}/\mathbf{q}) \to (\mathbf{n}_3\backslash\mathbf{s}_1)/\mathbf{q}$$

and

$$((\mathbf{n}_3\backslash\mathbf{s}_1)/\mathbf{i}) \otimes (\mathbf{i}/\mathbf{p}) \to (\mathbf{n}_3\backslash\mathbf{s}_1)/\mathbf{p}$$

respectively, but *being* does not have type

$$(\mathbf{p}/\mathbf{i}) \otimes (\mathbf{i}/\mathbf{p}) \to \mathbf{p}/\mathbf{p},$$

plausible as this might be, since the following is not an English sentence:

* *John is being coming.*

The connectives $\otimes$, / and \ have been used before; in fact, the above account closely follows [L1959]. With a bit of imagination, it is not difficult to find linguistic applications for some of the other connectives. For example, if

$\mathbf{n}_1 =$ first person noun phrase,
$\mathbf{n}_2 =$ plural noun phrase,

then *have* in *they have come* will have the following type:

$$((\mathbf{n}_1\backslash\mathbf{s}_1)/\mathbf{i}) \otimes (\mathbf{i}/\mathbf{q}) \to (\mathbf{n}_1\backslash\mathbf{s})/\mathbf{q}$$

and the same with $\mathbf{n}_1$ replaced by $\mathbf{n}_2$. Ignoring the more elaborate type on the left of the arrow, for the purpose of illustration only, we may infer that *have* has type

$$(\mathbf{i}/\mathbf{q}) \times ((\mathbf{n}_1\backslash\mathbf{s}_1)/\mathbf{q}) \times ((\mathbf{n}_2\backslash\mathbf{s}_1)/\mathbf{q}),$$

which is isomorphic to

$$(\mathbf{i}/\mathbf{q}) \times (((\mathbf{n}_1 + \mathbf{n}_2)\backslash\mathbf{s})/\mathbf{q}).$$

Similarly, a modal verb such as *may* has type $\mathbf{n}\backslash(\mathbf{s}_1/\mathbf{i})$, where $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3$. However, we must distinguish between *have* of type $((\mathbf{n}_1 + \mathbf{n}_2)\backslash\mathbf{s}_1)/\mathbf{q}$ and *has* of type $(\mathbf{n}_3\backslash\mathbf{s}_1)/\mathbf{q}$.

A case can also be made for assigning to the word *only* the polymorphic type $\prod_X(X/X)$, where X is a variable type, although some provision has to be made to ensure that *only* modifies only expressions which are stressed. We may also try to ascribe to the word *and* the polymorphic type $\prod_X (X\backslash X)/X$; but there is a difficulty with this: when *and* is placed between any two noun phrases, whether of type $\mathbf{n}_1, \mathbf{n}_2$ or $\mathbf{n}_3$, the resulting expression always has type $\mathbf{n}_2$.

There seems to be no doubt that contextfree grammars can be meaningfully enhanced by introducing operations on types such as $/$, $\backslash$, $\otimes$, $\times$ and $+$, and perhaps also $\prod_X$ and others. More controversial is the claim implicit in many articles on categorial grammar, including early papers of the present author, that the entire grammar of English can be squeezed into the dictionary by assigning appropriate types to the words of the language. In view of the decision procedure provided by cut elimination, this would imply that the set of English sentences is not only recursively enumerable, which almost no one denies, but even recursive.

6. Postscript.

At a recent workshop and conference on categorial grammar, the author was surprised by the number of people seriously engaged in research in this area. In a parallel course of lectures by Moortgat and Oehrle, the above decision procedure was discussed in great detail. Following van Benthem [1983, 1988], they associated λ-expressions to derivations, particularly in connection with cut elimination, although apparently their λ-calculus was the usual one and neither the linear variant discussed here nor the variant permitting interchange by van Benthem.

Not all the speakers at the conference, however, were aware of the existence of a decision procedure. Thus, one speaker proposed the rule

$$(X/(Y/Z))\ (Y/W) \rightarrow X/(W/Z)$$

on empirical grounds, although it is in fact provable. Another speaker proposed the rule

$$((Z/Y)/X)\ (Y/X) \rightarrow Z/X,$$

which is not provable, in fact, not valid syntactically, although it holds of course in the semantic calculus one obtains when Gentzen's structural rules are admitted.

Several people suggested applications for some of the not so traditional connectives. Thus Oehrle had an interesting use for $\times$, allowing one to account for a phonological component in addition to the syntactic one. Morrill

[1989] also proposed uses for + and even Girard's [1987] "why not". Actually, Girard's linear logic also allows a negation and consequently the De Morgan duals of $\otimes$ and "why not", but these seem to have no obvious applications in linguistics.

There is however another kind of duality, in view of the fact that the dual of a production grammar is also a production grammar. Thus one might consider the connectives corresponding to "over" and "under" in the dual grammar. They could be viewed as two kinds of subtraction, say "less" and "from", even though the connective corresponding to $\otimes$ is still $\otimes$. As far as I know, no one has explored applications of these last mentioned connectives.

Corrections added in proof.

To Section 1, penultimate paragraph, add:

More generally, if a_i is any term of type A_i, $fa_1 \ldots a_n$ is a term of type A_i. We stipulate that $1_A x = x$ and that $g < f > u_1 \ldots u_m w_1 \ldots w_n v_1 \ldots v_k = g u_1 \ldots u_m f w_1 \ldots w_n v_1 \ldots v_k$.

In Section 3, (2), replace $f^*\mathrm{e}$ by $\mathrm{e}f^*$.

To Section 4, add: Alternatively, we could have constructed $1_{A/B} = \mathrm{e}^*_{AB}$.

To Section 5, last paragraph, add:

The above claim amounts to saying that the grammar of English has the form $F(\mathcal{G})$, where $\mathcal{G}$ has only sequents of the form 1_A.

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