DIAGRAMMES

RENÉ GUITART

Construction of an homology and a cohomology theory associated to a first order formula

Diagrammes, tome 23 (1990), p. 7-13
<http://www.numdam.org/item?id=DIA_1990__23__7_0>

DIAGRAMMES VOLUME 23, 1990

ACTES DES JOURNEES E.L.I.T.

(UNIV. PARIS 7, 27 JUIN-2 JUILLET 1988)

CONSTRUCTION OF AN HOMOLOGY AND A COHOMOLOGY THEORY ASSOCIATED TO A FIRST ORDER FORMULA

by René GUITART

RESUME - On montre comment chaque formule ϕ d'un langage $\mathcal{L}$ détermine une théorie d'homologie (et une théorie de cohomologie) sur la catégorie des interprétations de $\mathcal{L}$, dont la valeur sur chaque interprétation I de $\mathcal{L}$ est une obstruction à $I \models \phi$ "à des co-équations près" (et "à des équations près").

This paper is a sequel of [7].

0. Let $\mathcal{L}$ be a first order language, let I be an interpretation of $\mathcal{L}$, and let ϕ be a formula of $\mathcal{L}$. The aim of this note is to indicate a way in which it is possible to measure partially and to compute how I is far from the models of ϕ.

In the papers [7] and [8] it is shown how this question is connected with the possibility of a geometrical study of algorithms and ambiguïties.

1. PROPOSITION. *Let* $\mu(x_1,\dots,x_n)$ *be a first order formula of* $\mathcal{L}$, *and let* $\mathrm{Mod}_\mu\phi$ *be the category with objects the models of* ϕ, *and with morphisms from* M *to* M' *the morphisms (of models of* Φ*)* $m : M \longrightarrow M'$ *such that*

$$\forall\ x_1,\dots,x_n\ [\ \mu(m(x_1),\dots,m(x_n)) \longrightarrow \mu(x_1,\dots,x_n)\ ]$$

Then there is a small mixed sketch σ *such that* $\mathrm{Mod}_\mu\phi \cong \mathrm{Mod}\sigma$.

The existence of σ is proved by the juxtaposition of proposition 3 p.8 of [6], théorème 2.1 p.26 of [5], and proposition 3 p.301 of [7],II. In fact, this juxtaposition shows more than our proposition here.

2. For **C** a category, let $BC = |NC|$ be the geometric realization of the nerve of **C**. BC is a cw-complexe, and $\pi_1 BC \cong \mathbf{C}[\mathbf{C}^{-1}]$ (the category of fractions of **C**). Of course if **C** is a class, **BC** is a class too. But, if $\mathbf{C} = \mathrm{Mod}\sigma$ for a small sketch σ, then in BC we can construct a *set* $g\sigma$ such that the inclusion $g\sigma \longrightarrow \mathrm{BMod}\sigma$ is an equivalence of homotopy. In particular we get

PROPOSITION. $\mathrm{Mod}\sigma[(\mathrm{Mod}\sigma)^{-1}]$ *is a small groupoïd, up to equivalence. We call it the fundamental groupoïd of* σ, *and we denote it by* $\pi_1 g\sigma$.

The existence of the set $g\sigma$ comes from [7].

3. Let $\mathrm{Mod}_\mu\phi/I$ be the category with objects the morphisms (of interpretations of $\mathcal{L}$) $f : M \longrightarrow I$ where M is a model of ϕ, and with morphisms, from $f : M \longrightarrow I$ to $f' : M' \longrightarrow I$, the morphisms of models (morphisms of $\mathrm{Mod}_\mu\phi$) $g : M \longrightarrow M'$ such that $f'.g = f$.
Then

PROPOSITION. *There is a small sketch* $\sigma = \sigma(\mathcal{L}, I, \phi, \mu)$ *such that*

$$\mathrm{Mod}_\mu\phi/I \cong \mathrm{Mod}\sigma.$$

So we get a small cw-*complexe* $g\sigma(\mathcal{L}, I, \phi, \mu)$, *which is a geometric description of the position of* I *with respect to* $\mathrm{Mod}_\mu\phi$.

4. Let **Ab** be the category of small abelian groups, and let $F : \mathrm{Mod}_\mu\phi \longrightarrow \mathbf{Ab}$ be a functor. (In particular F could be the constant functor on a fixed abelian group A, or it could be a "canonical" functor if $\mathcal{L}$ is a language over the language of abelian groups, etc).
The André's homology measures "how I is far from $\mathrm{Mod}_\mu\phi$, from the point of view of F".In order to do that we consider the chain complexe

$$\longrightarrow C_2(I,F) \xrightarrow{d_2} C_1(I,F) \xrightarrow{d_1} C_0(I,F) \xrightarrow{d_0} 0$$

which is

$$\ldots \longrightarrow \sum_{M_2 \to M_1 \to M_0 \to I} FM_2 \xrightarrow{d_2} \sum_{M_1 \to M_0 \to I} FM_1 \xrightarrow{d_1} \sum_{M_0 \to I} FM_0 \xrightarrow{d_0} 0$$

with $d_1 = s_1^0 - s_1^1$, where

$$s_1^0 : (FM_1)_{(M_1 \xrightarrow{\alpha} M_0 \xrightarrow{\beta} I)} \xrightarrow{Id} (FM_1)_{(M_1 \xrightarrow{\beta.\alpha} I)} \xrightarrow{Inc} \sum_{M \to I} FM$$

$$s_1^1 : (FM_1)_{(M_1 \xrightarrow{\alpha} M_0 \xrightarrow{\beta} I)} \xrightarrow{F(\alpha)} (FM_0)_{(M_0 \xrightarrow{\beta} I)} \xrightarrow{Inc} \sum_{M \to I} FM$$

and so on, and we define

$H_0(I, F) = \ker d_0/\mathrm{Im}\, d_1 = \mathrm{coker}\, d_1$, $H_1(I, F) = \ker d_1/\mathrm{Im}\, d_2$, and, for every $n \geq 0$, $H_n(I, F) = \ker d_n/\mathrm{Im}\, d_{n+1}$.

PROPOSITION. $H_n(I, F)$ *is a function of* F, I, μ, ϕ, *which in fact depends only of the homotopy type of* $\mathrm{Mod}_\mu\phi/I$ *and of* F *and could be denoted by* $H_n(\mathrm{Mod}_\mu\phi/I,F)$.

see [1], [2], [3] and [4].

Let $\mathrm{Int}\mathscr{L}$ be the category of interpretations of $\mathscr{L}$, let $J : \mathrm{Mod}\phi \longrightarrow \mathrm{Int}\mathscr{L}$ be the canonical inclusion. Then the inductive Kan extension of F along J is given by

$$[\underrightarrow{\mathrm{Ext}}_J F](I) = \lim_{M_0 \longrightarrow I} F(M_0)$$

and we have

$$H_0(I, F) = [\underrightarrow{\mathrm{Ext}}_J F](I).$$

If $I \models \phi$, then $H_n(\mathrm{Mod}_\mu\phi/I,F) = \begin{cases} F(I) & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$

5. Now, the point is that, because of the results hereover (§§ 1 to 4), we get

PROPOSITION. *The tools of* [1] *and of* [3], available in the situation where a full and small category **M** (called a category of "models") lives inside a big category of "spaces", *are also available in the situation where a* (possibly big and not necessarily full) *category* $Mod_\mu\phi$ *of models of a theory lives inside a big category of interpretations of a language* $\mathcal{L}$ (compare with the idea of "paires adéquates" p. 43 of [1]). *Precisely here we get the fact that the* $H_n(Mod_\mu\phi/I,F)$ *are small.*

6. After the existence of $g\sigma$ proved in [7], the theorem hereunder §9 is just a second stone for a work to be pursed. Theoretically the computation of our H_n is based on the effective construction of a "locally cofree diagram", and more precisely on the construction of a "relatively cofiltered locally cofree diagram" (r.cf.l.cf.d.)(see [5] and [6]) (in the category $Mod_\mu\phi$) generated by I. This r.cf.l.cf.d. contains all the information we need, and it will be the starting point of an absolute calculus. But for concrete situations we need a relative calculus, by the way of comparaisons between various H_n. For that it will be essential to go toward effective relative calculation of these small H_n, and especially we need a description of the link between these calculations and the theory of demonstrations. For example we need relations among $H_n(Mod_\mu\phi/I,F)$, $H_n(Mod_\mu\gamma/I,G)$, $H_n(Mod_\mu[\phi\wedge\gamma]/I,K)$, $H_n(Mod_\mu[\phi\Rightarrow\gamma]/I,L)$ (for conveniant K and L).
For that it will be necessary to describe the category **For**($\mathcal{L}$) of formulas of the language $\mathcal{L}$. At first this will be usefull to precise the functoriality of the $H_n(Mod_\mu\phi/I,F)$ with respect to ϕ and μ.

7.The first purpose of this paper was to show precisely how each classical first order formula ϕ of a language $\mathcal{L}$ determines a "small" homology theory on the category of interpretations of $\mathcal{L}$.
Now, the continuation of this research pass trough the description of **For**($\mathcal{L}$). With respect to that, I would like to make the following remark : what have to be morphisms between formulas ? it is not so clear a priori ; they have to be "demonstrations" or "proofs", but there is no

canonical idea of what is a demonstration.

But if we decide to stay in (or to come back to) the style of sketches, a first picture is easy to give. In fact $\mathcal{L}$ "is" a sketch σ_0 (i.e. the category of interpretations of $\mathcal{L}$ is isomorphic to Modσ_0), the formula ϕ (or $_\mu\phi$) is a sketch σ, and the inclusion of the category of models of ϕ (of $_\mu\phi$) in the category of interpretations of $\mathcal{L}$ is induced by a morphism of sketches P : $\sigma_0 \longrightarrow \sigma$. This P is the "proof" that a model of ϕ (of $_\mu\phi$) is an interpretation of $\mathcal{L}$. In fact P is not a general morphism of sketches, but determines σ as a σ_0-sketch (see [6] p.10 for the precise definition). So we choose to say now that a formula for σ_0 (in the place of a $\mathcal{L}$-formula) is nothing but such a P, a σ_0-sketch. In [6] the boolean calculus of σ_0-sketches (conjonctions, disjonctions, complements) is exposed as construction in the category of sketches. Then we can defined the category **For**(σ_0) as being the category of σ_0-sketches, as objects, with morphisms from P to P' the morphisms of sketches f : $\sigma \longrightarrow \sigma'$ which determine σ' as a σ-sketch, such that f.P = P'.

At this level of language, we can change our notations, replacing Mod$_\mu\phi$ by Modσ, or even, more precisely, by P, and the H_n(Mod$_\mu\phi$/I,F) will be denoted by H_n(P/I,F). Of course for general mixed sketches (and not only for those associated to first order formulas) the result in §5 works, and the abelian groups H_n(P/I,F) are smalls. Now

PROPOSITION. *The functoriality of these* H_n, *with respect to* P, I *and* F *are trivial facts.*

8. In a dual way, given a functor F : Mod$_\mu\phi \longrightarrow$ **Ab** and an interpretation I of $\mathcal{L}$, the cohomology of I with coefficient in F is defined by considering the cochain complexe

$$\longleftarrow C^2(I,F) \overset{d^1}{\longleftarrow} C^1(I,F) \overset{d^0}{\longleftarrow} C^0(I,F)$$

which is

$$\longleftarrow \prod_{M_2\leftarrow M_1\leftarrow M_0\leftarrow I} FM_2 \xleftarrow{d^1} \prod_{M_1\leftarrow M_0\leftarrow I} FM_1 \xleftarrow{d^0} \prod_{M_0\leftarrow I} FM_0$$

with

$$d^1(x)_{(I\xrightarrow{\gamma} M_0\xrightarrow{\lambda} M_1)} = F(\lambda)(x_{(I\xrightarrow{\gamma} M_0)}) - x_{(I\xrightarrow{\lambda.\gamma} M_1)}$$

and so on, and we define $H^n(I, F) = \ker d^n/\mathrm{Im}\, d^{n-1}$.
For these cohomology groups, the same result is true, that is to say that they are small. But now, the computation is based on the effective construction of a "relatively filtered locally free diagram" (r.f.l.f.d.) (in the category $\mathrm{Mod}_\mu\phi$) generated by I. These cohomolgy groups will be denoted by $H^n((I/\mathrm{Mod}_\mu\phi)^{op},F)$.

9.Collecting the results of §5, §7 and §8, we get :

THEOREM :*The abelian groups* $H_n(\mathrm{Mod}_\mu\phi/I,F)$ *and* $H^n((I/\mathrm{Mod}_\mu\phi)^{op},F)$ *are small, i.e. they are elements of the category* **Ab**, *they are functorial with respect to* I, F, μ *and* ϕ, *and if* $I\models\phi$, *then* $H_n(\mathrm{Mod}_\mu\phi/I,F) = 0$, *for every* $n > 0$, *and* $H^n((I/\mathrm{Mod}_\mu\phi)^{op},F) = 0$, *for* $n > 0$, *In fact, more precisely, we have* $H_n(\mathrm{Mod}_\mu\phi/I,F) = 0$, *for every* $n > 0$, *if there is a cofree model generated by* I, *and we have* $H^n((I/\mathrm{Mod}_\mu\phi)^{op},F) = 0$, *for* $n > 0$, *if there is a free model generated by* I. *So they are small obstructions to the satisfaction of* ϕ *in* I *"up to co-equations" and "up to "equations".*

REFERENCES

[1] M. André, *Méthode simpliciale en algèbre homologique et algèbre commutative*, S.L.N. 32, 1967.

[2] M. Evrard, *Homotopie d'un espace topologique relativement à un recouvrement. Applications à l'homotopie des préschémas*, Thèse d'Etat, Université Paris 7, 1973.

[3] M. Evrard, *Fibrations de petites catégories,* Bull. Soc. Math. France 103, p.241-265, 1975.

[4] D. Quillen, *Higher algebraic K-theory,I,* S.L.N. 341, p.85-147, 1973.

[5] R. Guitart et C. Lair, *Calcul syntaxique des modèles et calcul des formules internes,* Diagrammes, vol. 4, 106 p., 1980.

[6] R. Guitart et C. Lair, *Limites et co-limites pour représenter les formules,* Diagrammes, vol. 7, 24 p., 1982.

[7] R. Guitart, *On the geometry of computations, I and II*, Cahiers Top. Géo. Diff. Cat., vol. XXVII-4 (1986), p. 107-136, et vol. XXIX-4 (1988), p. 297-326.

[8] R. Guitart, *L'unité de la théorie des modèles et de l'algèbre homologique* (conf. at the Summer meeting on Category Theory, Louvain-la-Neuve, July 26 - Aug 1, 1987), in Cahiers Top. Géo. Diff. Cat. XXX-3 (1989).

U.F.R. Maths, Tours 45-55, 5ème ét., Université. PARIS 7,
2 place Jussieu, 75005, FRANCE.