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ON p-ADIC UNIFORMIZATION

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Consider a complete non-singular algebraic curve C or an abelian variety A over a p-adic field k (i. e., over a complete field containing $\mathbb{Q}_p$). To give a uniformization of C or A (in the category of rigid analytic spaces) means to give a good description of the universal covering $\hat{C}$ of C or $\hat{A}$ of A and of the group $\mathrm{Aut}(\hat{C}/C)$ or $\mathrm{Aut}(\hat{A}/A)$. That the situation is quite different from the classical complex case, can already be seen from Tate's work on elliptic curves. In particular, the definition of the universal covering presents problems. If the notion of covering maps is too general, universal coverings (satisfying the appropriate universal property) will not exist.

The uniformization problem was first solved by TATE for elliptic curves (see [10]). Then MUMFORD considered curves of higher genus, and he was able to solve the problem completely for a special class, the so-called Mumford curves ([8], see also [5]). Furthermore, the existence of the universal covering $\hat{C}$ can be derived, for any curve C , from the "stable reduction theorem" of DELIGNE-MUMFORD [4]. Similarly, RAYNAUD has shown ([9], see also [2]), how Grothendieck's semi-abelian reduction implied the existence of the universal covering $\hat{A}$ of an abelian variety A .

In the following, we will give a brief account on the known facts. Also we will indicate, how Grothendieck's result can be obtained by an analytic argument. For simplicity, the ground field k is supposed to be algebraically closed. (If k is not algebraically closed, everything which follows can be done at least over a suitable finite extension of k .)

1. Analytic spaces and reductions.

Let X be an analytic space over k , and consider an admissible open affinoid covering $X = \bigcup_{i\in I} X_i$. Denote by $\mathcal{O}$ the sheaf of analytic functions on X , and by $\mathring{\mathcal{O}}$ the subsheaf of functions having supremum norm ≤ 1 . If X is affinoid itself, the scheme $\tilde{X} := \mathrm{Spec}\, \mathring{\mathcal{O}}(X) \otimes_{\mathring{k}} \tilde{k}$ is called the reduction of X (where $\mathring{k}$ and $\tilde{k}$ denote the valuation ring and the residue field of k , respectively). The reduction $\tilde{X}$ is of finite type over $\tilde{k}$, and there is a canonical surjection $\pi : X \to \tilde{X}$ (between closed points) such that $\pi_*(\mathring{\mathcal{O}} \otimes_{\mathring{k}} \tilde{k})$ is the structure sheaf of $\tilde{X}$. A subset U is called a formal subdomain of X if U is the π-inverse

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of Zariski-open subset in $\tilde{X}$. Equivalently, we can say that U is required to be a finite union of sets of type $\{x \in X ; |f(x)| = 1\}$, where $f \in \mathring{\mathcal{O}}(X)$.

In order to extend the notion of the reduction to arbitrary analytic spaces, it is necessary to fix a formal covering of X, i. e. an admissible open affinoid covering $X = \bigcup_{i \in I} X_i$ such that $X_i \cap X_j$ is a formal subdomain in X_i for all $i, j \in I$. Then the reduction $\tilde{X}$ of X is constructed by pasting the affine schemes $\tilde{X}_i$ via the intersections $\widetilde{X_i \cap X_j}$, i. e. via the images of $X_i \cap X_j$ with respect to the maps $\pi_i : X_i \longrightarrow \tilde{X}_i$ and $\pi_j : X_j \longrightarrow \tilde{X}_j$. The maps π_i define a projection $\pi : X \longrightarrow \tilde{X}$ such that, as in the affinoid case, $\pi_*(\mathring{\mathcal{O}} \otimes_{\mathring{k}} \tilde{k})$ is the structure sheaf of $\tilde{X}$. For any (closed) point $\tilde{x} \in \tilde{X}$, we call $\pi^{-1}(\tilde{x})$ the <u>formal fibre</u> over $\tilde{x}$. Of course, the reduction $\tilde{X}$ is not intrinsically given by X. It depends on the formal covering which has been chosen.

We give a <u>basic example</u>. Let c be a constant in k, $0 < |c| < 1$. Provide the annulus $\{x \in k ; |c| \leq |x| \leq 1\}$ with its canonical analytic structure. Then we get an affinoid space X, which satisfies

$$\mathcal{O}(X) = k\langle \zeta , \eta\rangle/(c - \zeta\eta) ,$$

$$\mathring{\mathcal{O}}(X) = \mathring{k}\langle \zeta , \eta\rangle/(c - \zeta\eta) ,$$

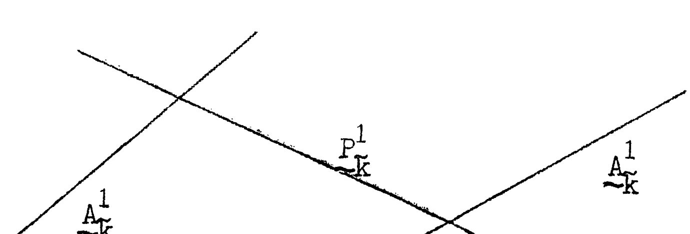
$$\mathring{\mathcal{O}}(X) \otimes_{\mathring{k}} \tilde{k} = \tilde{k}[\zeta , \eta]/(\zeta\eta) ,$$

($k\langle \zeta , \eta\rangle$ is the algebra of strictly converging power series in the variables ζ and η over k.) In particular, the reduction $\tilde{X} = \operatorname{Spec} \tilde{k}[\zeta , \eta]/(\zeta\eta)$ consists of two lines $\mathbb{A}^1_{\tilde{k}}$ intersecting each other at an ordinary double point. The formal fibre over this singular point is the "open" annulus with radii $|c|$ and 1. For any non-singular point $\tilde{x} \in \tilde{X}$, the formal fibre $\pi^{-1}(\tilde{x})$ equals an "open" disc (either of radius $|c|$ or of radius 1).

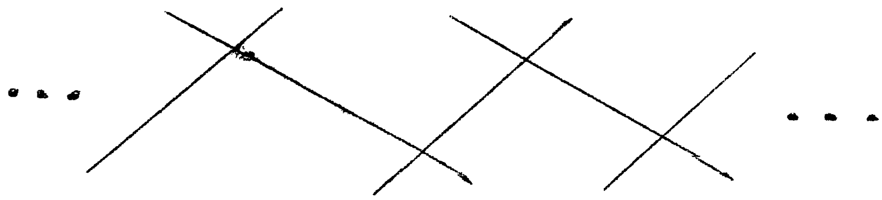
The phenomenon encountered here is a general one. Namely,

PROPOSITION. - <u>Let</u> X <u>be a non-singular purely</u> 1-<u>dimensional analytic space over</u> k. <u>Consider a reduction</u> $\tilde{X}$ <u>of</u> X (<u>with respect to a given formal covering of</u> X). <u>Let</u> $\tilde{x}$ <u>be a</u> (<u>closed</u>) <u>point in</u> $\tilde{X}$. <u>Then</u> $\tilde{x}$ <u>is non-singular</u> $\Longleftrightarrow$ <u>the formal fibre</u> $\pi^{-1}(\tilde{x})$ <u>is a disc</u>. $\tilde{x}$ <u>is an ordinary double point</u> $\Longleftrightarrow$ <u>the formal fibre</u> $\pi^{-1}(\tilde{x})$ <u>is an annulus</u>.

If in the above example one considers a formal covering of the annulus X by two (non-degenerated) annuli X_1 and X_2, the corresponding reduction of X is as follows :



It is obtained from the previous one by blowing up the intersection point (and killing nilpotent functions). Likewise, a formal covering of the affine space $\underset{\sim}{A}_k^1 - \{0\}$ by annuli leads to a reduction, which is a chain of projective spaces $\underset{\sim}{P}_k^1$:



2. Analytic spaces of standard type and covering maps.

In the following, let X be a connected 1-dimensional non-singular analytic space. We are mainly interested in the case where X is the analytification of a non-singular complete algebraic curve.

Definition. - A formal covering of X is called of standard type if the singular points of the associated reduction $\tilde{X}$ are ordinary double points and if all components of $\tilde{X}$ are complete. Considering such a covering on X, we simply say that X is of standard type.

If X is of standard type, the components of genus > 0 of the reduction $\tilde{X}$ are uniquely determined by X up to birational isomorphism. Any two reductions (with respect to formal coverings of standard type) of X can be transformed into each other by blowing up certain double points and blowing down certain components isomorphic to $\underset{\sim}{P}_k^1$ (similarly as in the case of an annulus X; which has been considered above). Thus, the reduction of X is essentially unique. Also it gives a good geometric description of X itself. Namely, consider the projection $\pi : X \longrightarrow \tilde{X}$, and denote by $S(\tilde{X})$ the set of singular points of the reduction $\tilde{X}$. Then $\tilde{X} - S(\tilde{X})$ is a disjoint union of non-singular quasi-projective varieties. Likewise, its π-inverse $X - \pi^{-1}(\tilde{S})$ decomposes into disjoint open connected subspaces U_j, $j \in J$, which have non-singular reduction. Then X is derived from $X - \pi^{-1}(\tilde{S}) = \bigcup_{j \in J} U_j$ by adding all formal fibres $\pi^{-1}(\tilde{x})$, $\tilde{x} \in S(\tilde{X})$ (which have the structure of "open" annuli). Hence X is constructed by connecting the components U_j by means of suitable annuli, according to the double points in $\tilde{X}$. It is this fact which leads to the construction of the universal covering $\hat{X}$ of X. Just resolve all loops which are generated when X is derived from $X - \pi^{-1}(\tilde{S})$.

This can be made more precise. Consider the intersection graph G of $\tilde{X}$. Its vertices are the components of $\tilde{X}$, and its edges are the singularities of $\tilde{X}$. Two vertices (corresponding to the components $\tilde{X}'$ and $\tilde{X}''$ of $\tilde{X}$) are joined by an edge if the corresponding double point is an intersection point of $\tilde{X}'$ and $\tilde{X}''$. Furthermore, if a singular point $\tilde{x} \in \tilde{X}$ belongs to just one component $\tilde{X}'$ of $\tilde{X}$ (i. e., if $\tilde{X}'$ has selfintersection), a loop is attached to the corresponding vertex. It is clear that G contains all data of $\tilde{X}$ which were needed to give the geometric description of X we have discussed above. Thus, in order to obtain the universal covering $\hat{X}$ of X, one looks at the universal covering $\hat{G}$ of G, and one constructs $\hat{X}$ by pasting suitable open subspaces of X according to the geometric data furnished by $\hat{G}$. Il can be verified that $\hat{X}$ satisfies the usual universal property of a universal covering if the notion of covering maps is chosen as follows:

Definition. - A morphism $\varphi : Y \longrightarrow X$ of analytic spaces is called a covering map if there exists an admissible open covering $\{X_i\}_{i\in I}$ of X such that, for each $i \in I$, the inverse image $\varphi^{-1}(X_i)$ admits an admissible open covering $\{Y_{ij}\}_{j\in J_i}$ by disjoint subspaces Y_{ij} which are isomorphic to X_i via φ.

A covering map between analytic spaces of standard type induces always a covering map between the associated graphs. This fact is used in showing that the space $\hat{X}$ constructed above is indeed the universal covering of X. On the other hand, note that the map $\underset{\sim}{A}^1_k - \{0\} \longrightarrow \underset{\sim}{A}^1_k - \{0\}$, $x \longmapsto x^2$, is not a covering map in the sense of our definition (although it is a finite map without ramification points).

3. The universal covering of curves.

As an example, we consider an elliptic curve C over a field k satisfying char $\tilde{k} \neq 2$. Let C be defined in $\underset{\sim}{P}^2_k$ by the equation

$$Y^2 Z - X(X - Z)(\lambda X - Z) = 0 ,$$

where $\lambda \in k$, $\lambda \neq 0, 1$. We may assume $|\lambda| \leqslant 1$ and $|\lambda - 1| = 1$. The formal covering $\underset{\sim}{P}^2_k = U_0 \cup U_1 \cup U_2$, where

$$U_i := \{(x_0 , x_1 , x_2) \in \underset{\sim}{P}^2_k ;\quad |x_j| \leqslant |x_i| \quad \text{for } j = 0 , 1 , 2\} ,$$

induces a formal covering of C, and we claim that C is of standard type.

Case 1 : $|\lambda| = 1$. - Then the reduction $\tilde{C}$ is the curve in $\underset{\sim}{P}^2_{\tilde{k}}$ which is given by the equation

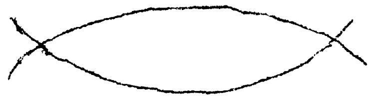
$$Y^2 Z - X(X - Z)(\tilde{\lambda} X - Z) = 0$$

($\tilde{\lambda}$ is the residue class of λ in $\tilde{k}$). Since $\tilde{\lambda} \neq 0 , 1$, we see that $\tilde{C}$ is non-singular and that C is of standard type. Furthermore, the intersection graph G of $\tilde{C}$ consists of just one vertex. Therefore G is simply connected, and the universal covering of C is trivial.

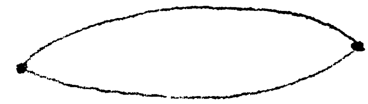
Case 2 : $|\lambda| < 1$. - The reduction $\tilde{C}$ is given in $\underset{\sim}{P}^2_{\tilde{k}}$ by the equation

$$(Y^2 + X(X - Z))\ Z = 0 .$$

Thus, $\tilde{C}$ consists of two components isomorphic to $\underset{\sim}{P}^1_{\tilde{k}}$ which intersect each other at two ordinary double points :



$\tilde{C}$



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In particular, C is of standard type. The non-singular locus of $\tilde{C}$ consists of two components isomorphic to $\underset{\sim}{A}_k^1 - \{0\}$. The inverse image in C is a disjoint union of two degenerated annuli, and one obtains C if both are joined by two "open" annuli. Thereby the picture of a torus occurs. Likewise, the universal covering $\hat{C}$ of C is constructed by considering a collection of degenerated annuli U_υ , $\upsilon \in \underset{\sim}{Z}$ and by connecting U_υ with $U_{\upsilon+1}$ for all υ via an "open" annulus. The resulting space admits a formal covering by (closed) annuli and is seen to be isomorphic to $\underset{\sim}{A}_k^1 - \{0\}$. Thus C is a Tate curve in this case.

Similarly as in the case of elliptic curves, explicit calculations can be made for hyperelliptic curves ([3], [5]). For general curves C , one uses the "stable reduction theorem" of DELIGNE-MUMFORD [4], which has recently been generalized by van der PUT to the case of arbitrary ground fields. We give an analytic version of this result.

THEOREM. - Each non-singular complete algebraic curve C is of standard type.

Thus, the universal covering $\hat{C}$ exists for each such curve C . If $\hat{C}$ can be embedded into $\underset{\sim}{P}_k^1$, one calls C a Mumford curve. The Mumford curves are characterized by the fact that all components of $\tilde{C}$ have genus 0 or, equivalently, by the fact that $\mathrm{rg}\, H^1(C , \underset{\sim}{Z}) = \mathrm{rg}\, H_1(G)$ equals the genus of C .

4. The universal covering of abelian varieties.

The construction of the universal covering is more complicated in dimensions > 1 than in dimension 1 . However, when dealing with abelian varieties, there is substantial simplification due to the fact that these varieties are group varieties. We start with a result on multiplicative subgroups of abelian varieties which may be viewed as a key-lemma for the construction of the universal covering. Denote by $\underset{\sim}{G}_m$ the 1-dimensional multiplicative group over k , and by $\bar{\underset{\sim}{G}}_m$ its open affinoid subgroup of elements of value 1 .

THEOREM ([2], [9]). - Let A be an abelian variety over k . Consider an analytic homomorphism $\bar{\sigma} : \bar{\underset{\sim}{G}}_m \hookrightarrow A$ defining $\bar{\underset{\sim}{G}}_m^r$ as a locally closed subgroup of A . Then $\bar{\sigma}$ extends to an analytic homomorphism $\sigma : \underset{\sim}{G}_m^r \longrightarrow A$.

In particular, if $r = n := \dim A$, it follows that $\sigma : \underset{\sim}{G}_m^n \longrightarrow A$ is the universal covering of A , and that A is a quotient of $\underset{\sim}{G}_m^n$ by a discrete subgroup of rank n . Hence A is an analytic torus in this case. Although A will not in general contain a subgroup isomorphic to $\bar{\underset{\sim}{G}}_m^n$, there is precise information on the multiplicative subgroups of A by the "semi-abelian reduction theorem" of GROTHENDIECK

[6]. Let N^0 be the identity-component of the formal completion of the Néron model of A. Then N^0 is an open analytic subgroup of A having a canonical reduction $\tilde{N}^0$. (Since the Néron model exists only for abelian varieties over fields k' carrying a discrete valuation, we have to assume at this point that A is derived from an abelian variety A' over such a field k' by extending the group field. Likewise N^0 has to be interpreted as the extension of the corresponding open subgroup of A'.) Writing $\tilde{\underset{\sim}{G}}_m$ for the 1-dimensional multiplicative group over $\tilde{k}$, the result of Grothendieck reads as follows :

THEOREM. – The group $\tilde{N}^0$ is an extension of an abelian variety $\tilde{B}$ over $\tilde{k}$ by an affine torus $\tilde{\underset{\sim}{G}}_m^r$, i. e., there is an exact sequence

$$0 \longrightarrow \tilde{\underset{\sim}{G}}_m^r \longrightarrow \tilde{N}^0 \longrightarrow \tilde{B} \longrightarrow 0 .$$

The above exact sequence lifts to an exact sequence

$$0 \longrightarrow \bar{\underset{\sim}{G}}_m^r \longrightarrow N^0 \longrightarrow B \longrightarrow 0 ,$$

where B is an abelian variety over k having $\tilde{B}$ as reduction. Furthermore, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underset{\sim}{G}_m^r & \longrightarrow & \hat{A} & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & \searrow^{\varphi} & \| & & \\
0 & \longrightarrow & \bar{\underset{\sim}{G}}_m^r & \longrightarrow & N^0 & \longrightarrow & B & \longrightarrow & 0 \\
 & & & & & \searrow & & & \\
 & & & & & & A & &
\end{array}$$

The group $\hat{A}$ is defined as the quotient of $\underset{\sim}{G}_m^r \times N^0$ by the diagonal in $\bar{\underset{\sim}{G}}_m^r \times \bar{\underset{\sim}{G}}_m^r$ and the map $\varphi : \hat{A} \longrightarrow A$ is constructed by extending the homomorphism $\bar{G}_m^r \longrightarrow N^0 \hookrightarrow A$ to a homomorphism of the full multiplicative group $\underset{\sim}{G}_m^r$ into A. It follows that $\varphi : \hat{A} \longrightarrow A$ is the universal covering of A, and that A is a quotient of $\hat{A}$ by a discrete subgroup of rank r.

The crucial step in the above considerations is, of course, the construction of the group N^0. One would like to avoid the use of the Néron model and also to remove the assumption that A can be defined over a field k' carrying a discrete valuation. Therefore we give a simple analytic characterization of N^0 which, as is expected, can lead to a direct construction of this group.

By the Theorem of MATTUCK ([7], see also [1]), the group A is locally isomorphic to the additive group $\underset{\sim}{G}_a^n$. Thus A contains an open subgroup A_0 which is isomorphic to the unit ball centered at 0 in $\underset{\sim}{G}_a^n$. We denote by $[p] : A \longrightarrow A$ the multiplication by $p = \text{char}\, \tilde{k}$, and by A_ν the identity component of $[p]^{-\nu}(A_0)$. Then $A_+ := \bigcup_{\nu=0}^{\infty} A_\nu$ is an open subgroup of A. In fact, it is the formal fibre $\pi^{-1}(0)$ over the identity element $0 \in \tilde{N}^0$ with respect to the projection

$\pi : N^0 \longrightarrow \tilde{N}^0$. In order to derive N^0 from A_+ , one considers a group chunk in A having A_+ as formal fibre. The associated group is N^0 . That N^0 has semi-abelian reduction, can then be seen by using standard arguments involving the relationship between N^0 and $\tilde{N}^0$.

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