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ABOUT  $p$ -ADIC INTERPOLATION OF CONTINUOUS AND DIFFERENTIABLE FUNCTIONS

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0. Introduction.

In 1958, MAHLER proved that  $\left\{ \binom{x}{n} ; n \in \mathbb{N} \right\}$  form a normal base for  $C(\mathbb{Z}_p, \mathbb{Q}_p)$ . Since then, a number of different proofs of this theorem were given (cf. [1], [2], [4], [5], [8]).

In section 1, we show that the method used by Yvette AMICE [1] can be generalised to prove that  $\left\{ \binom{x}{n}^s ; n \in \mathbb{N} \right\}$  form a normal base, for each  $s \in \mathbb{N}^*$ . This leads to a generalisation of Mahler's formula (1.2). It is a remarkable fact that some polynomials (e. g.  $x$ ) get an infinite expansion. So the linear space spanned by the  $\binom{x}{n}^2$  lays dense in  $C(\mathbb{Z}_p, \mathbb{Q}_p)$ ; however, it does not lay dense in  $C^1(\mathbb{Z}_p, \mathbb{Q}_p)$ .

In section 2, we prove that there exist polynomials  $\tilde{R}_n$ , with  $\deg \tilde{R}_n = 2n + 1$ , such that the polynomials  $\gamma_n \binom{x}{n}^2$  together with the  $\tilde{R}_n$  form a normal base of  $C^1(\mathbb{Z}_p, \mathbb{Q}_p)$ . A close relation with Van der Put's base, consisting of locally constant and locally linear functions should be noted.

1. Normal bases for  $C(\mathbb{Z}_p, \mathbb{Q}_p)$ .

1.1 THEOREM. - For each  $s \in \mathbb{N}^*$ ,  $\{q_n = \binom{x}{n}^s ; n \in \mathbb{N}\}$  form a normal base of  $E = C(\mathbb{Z}_p, \mathbb{Q}_p)$ .

Proof. - In view of [2] n° 3.1.5, or [7] lemme 1, it is sufficient to prove that  $\{\bar{q}_n ; n \in \mathbb{N}\}$  form a vectorial base of  $\bar{E} = C(\mathbb{Z}_p, \mathbb{F}_p)$ . Let  $\bar{E}_h$  be the space of  $\mathbb{F}_p$ -valued functions constant on each ball

$$B'_{p^{-h}}(a) = \{x \in \mathbb{Z}_p ; |x - a| \leq p^{-h}\}.$$

Since  $\bar{E} = \bigcup \bar{E}_h$ , our proof will be finished if we can show that  $\{\bar{q}_i ; i < p^h\}$  form a base of  $\bar{E}_h$ .

For  $i < p^h$  and  $|x - y| < p^{-h}$ , we have

$$\left| \binom{x}{i} - \binom{y}{i} \right| < 1 \quad ([2], 3.2.2.3),$$

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hence

$$\left| \binom{x}{i}^s - \binom{y}{i}^s \right| = \left| \binom{x}{i} - \binom{y}{i} \right| \left| \sum_{m=0}^{s-1} \binom{x}{i}^m \binom{y}{i}^{s-m-1} \right| < 1,$$

so  $\bar{q}_i(x) = \bar{q}_i(y)$ . It follows that  $\bar{q}_i \in \bar{E}_h$ , and

$$\bar{q}_i = \sum_{j=i}^{p^h-1} \bar{q}_i(j) \chi_j.$$

So the transition matrix from  $\{\chi_j; i < p^h\}$  to  $\{\bar{q}_i; i < p^h\}$  is triangular; the desired result follows.

1.2 COROLLARY. - Let  $s \in \mathbb{N}^*$ . Each continuous  $f: \mathbb{Z}_p \rightarrow \mathbb{Q}_p$  can be written as a uniformly convergent series

$$f(x) = \sum_{n=0}^{\infty} a_n^{(s)} \binom{x}{n}^s$$

where

$$a_n^{(s)} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k}^s \beta_{n-k}^{(s)} f(k)$$

and

$$\beta_0^{(s)} = 1, \quad \beta_m^{(s)} = \sum_{\substack{\ell_1, \dots, \ell_r \\ \sum \ell_i = m \\ 1 \leq \ell_i \leq m}} (-1)^{r+m} \binom{m}{\ell_1, \dots, \ell_r}^s$$

Proof. - We have to calculate the interpolation coefficients  $a_n^{(s)}$ . They are determined by the formulas

$$a_n^{(s)} = f(0), \quad a_n^{(s)} = f(n) - \sum_{i=0}^{n-1} a_i^{(s)} q_i(n).$$

We prove the formula using induction on  $n$ . Suppose true for  $n \leq N$ , then we have:

$$\begin{aligned} a_{N+1}^{(s)} &= f(N+1) - \sum_{n=0}^N a_n^{(s)} \binom{N+1}{n}^s \\ &= f(N+1) - \sum_{n=0}^N \sum_{k=0}^n (-1)^{n-k} \binom{n}{k}^s \beta_{n-k}^{(s)} f(k) \binom{N+1}{n}^s \\ &= f(N+1) - \sum_{k=0}^N \sum_{n=k}^N (-1)^{n-k} f(k) \sum_{i=1}^r \sum_{\ell_i=n-k} \binom{n-k}{\ell_1, \dots, \ell_r}^s \binom{n}{k}^s \binom{N+1}{n}^s \\ &= f(N+1) + \sum_{k=0}^N f(k) \sum_{n=k}^N \sum_{i=1}^r \sum_{\ell_i=n-k} (-1)^{r+1} \frac{(N+1)!}{\ell_1! \dots \ell_r! k! (N+1-n)!}. \end{aligned}$$

Putting  $\ell_{r+1} = N+1-n$ , we get

$$\begin{aligned}
 a_{N+1}^{(s)} &= f(N+1) + \sum_{k=0}^N f(k) \sum_{\substack{\sum_{i=1}^{r+1} \ell_i = N+1-k \\ 1 \leq \ell_i \leq m}} (-1)^{r+1} \binom{N+1-k}{\ell_1 \dots \ell_r}^s \binom{N+1}{k}^s \\
 &= f(N+1) + \sum_{k=0}^N f(k) (-1)^{N+1-k} \beta_{N+1-k}^{(s)} \binom{N+1}{k}^s,
 \end{aligned}$$

this finishes the proof.

1.3 Note. - We can write down explicit formulas for the  $\beta_m^{(s)}$  :

$$\beta_0^{(s)} = \beta_1^{(s)} = 1,$$

$$\beta_2^{(s)} = 2^5 - 1,$$

$$\beta_3^{(s)} = 6^5 - 2 \cdot 3^5 + 1,$$

$$\beta_4^{(s)} = 24^5 - 3 \cdot 12^5 + 6^5 + 2 \cdot 4^5 - 1.$$

It is easy to tabulate the  $\beta_m^{(s)}$  :

m \ s	1	2	3	4
0	1	1	1	1
1	1	1	1	1
2	1	3	7	15
3	1	19	163	1135
4	1	211	8993	271375

1.4 Note. - Comparing the case  $s = 1$  with Mahler's formula, we get the following arithmetic formula :

$$\beta_m^{(1)} = \sum_{\substack{\sum_{i=1}^m \ell_i = m \\ 1 \leq \ell_i \leq m}} (-1)^{r+m} \binom{m}{\ell_1 \dots \ell_p} = 1.$$

1.5 Note (due to L. VAN HAMME). - One can determine the  $\beta_m^{(s)}$  also, by using generating functions. One has the following identity between formal power series :

$$\left( \sum_{n=0}^{\infty} (-1)^n a_n \frac{z^n}{(n!)^s} \right) \left( \sum_{n=0}^{\infty} (-1)^n b_n \frac{z^n}{(n!)^s} \right) = \sum_{n=0}^{\infty} (-1)^n c_n \frac{z^n}{(n!)^s},$$

if, and only if,  $c_n = \sum_{k=0}^n \binom{n}{k}^s a_k b_{n-k}$ . Put  $b_n = 1$ ,  $c_n = f(n)$ ,  $a_n = a_n^{(s)}$ .

Then it follows that

$$f(n) = \sum_{k=0}^n \binom{n}{k}^s a_k$$

if, and only if,

$$\sum_{n=0}^{\infty} (-1)^n a_n^{(s)} \frac{Z^n}{(n!)^s} = \left( \sum_{n=0}^{\infty} \frac{\beta_n^{(s)} Z^n}{(n!)^s} \right) \left( \sum_{n=0}^{\infty} (-1)^n f(n) \frac{Z^n}{(n!)^s} \right)$$

if, and only if,

$$a_n^{(s)} = \sum_{k=0}^n \binom{n}{k}^s (-1)^{n-k} \beta_{n-k}^{(s)} f(k),$$

where

$$\sum_{n=0}^{\infty} \frac{\beta_n^{(s)} Z^n}{(n!)^s} = \frac{1}{\sum_{n=0}^{\infty} (-1)^n Z^n / (n!)^s}$$

this last condition determines the  $\beta_n^{(s)}$ .

**1.6 Note.** - Applying corollary 1.2, we can obtain a lot of  $p$ -adically convergent series, e. g.

$$\begin{aligned} x &= x^2 - \frac{1}{2} x^2 (x-1)^2 + \frac{1}{3} x^2 (x-1)^2 (x-2)^2 + \dots \\ &= x^3 - \frac{3}{4} x^3 (x-1)^3 + \frac{23}{36} x^3 (x-1)^3 (x-2)^3 + \dots \end{aligned}$$

It is a remarkable fact that these series converge  $p$ -adically for each prime number  $p$ . Also note that derivation of the series yields an apparent contradiction after putting  $x = 0$ ; this shows that the series do not converge in  $C^1(\mathbb{Z}_p, \mathbb{Q}_p)$ -norm. The same phenomenon happens with Van der Put's base. We return to this problem in section 2.

**1.7 Note.** - The proof of theorem 1.1 is merely based on the proof of Mahler's theorem as given in [2].

One could try to adapt the proof given by BOJANIC [4], MAHLER [5] or VAN ROOÿ [8] to prove the theorem; however, it seems that these kinds of argument do not work here.

We can generalise theorem 1.1 if we replace  $\mathbb{Z}_p$  by regular compact part  $M$  of a local field  $K$  and the interpolation sequence  $\mathbb{N}$  by a very well distributed sequence  $u : \mathbb{N} \rightarrow M$ . For more details about very well distributed sequences, we refer to the work of Yvette AMICE [1]. We denote, following the notations in [1],

$$P_n(X) = (X - u_0)(X - u_1) \dots (X - u_{n-1}), \quad Q_n(X) = P_n(X) / P_n(u_n).$$

Given  $\alpha_1, \alpha_2, \dots$  in  $K$ , with  $|\alpha_i| \leq 1$  and  $\sum_{i=1}^{\infty} \alpha_i = 1$ , we define  $q_n = \sum_{i=1}^{\infty} \alpha_i Q_n^i$ . We omit the proof of the following theorem, since it is merely the same as the proof of theorem 1 in [1], up to one modification as in theorem 1.1.

1.8 PROPOSITION. - If  $u : \underline{\mathbb{N}} \rightarrow M$  is a very well distributed sequence in a regular compact part  $M$  of the local field  $K$ , and the  $q_n$  are defined as above, then  $\{q_n ; n \in \underline{\mathbb{N}}\}$  form a normal base of  $C(M, K)$ .

2. A normal base for  $C^1(\underline{\mathbb{Z}}_p, \underline{\mathbb{Q}}_p)$ .

For details about  $p$ -adic differentiability, we refer to [6]. Recall that a function  $f : \underline{\mathbb{Z}}_p \rightarrow \underline{\mathbb{Q}}_p$  is called  $C^1$  (or continuously differentiable) if the difference quotient  $\delta_1 f$  defined by

$$\delta_1 f(x, y) = (f(x) - f(y))/(x - y)$$

can be extended to a continuous function  $\delta_1 f$  on  $\underline{\mathbb{Z}}_p^2$ . The space of  $C^1$ -functions becomes the Banachspace  $C^1 = C^1(\underline{\mathbb{Z}}_p, \underline{\mathbb{Q}}_p)$  under the norm

$$\|f\|_1 = \max\{|f(0)|, \sup\{|\delta_1 f(x, y)| ; x \neq y\}\}.$$

It is known ([3], [6], [9]) that the following sets form normal bases for  $C^1$ :

$$\{\gamma_n \binom{x}{n} ; n \in \underline{\mathbb{N}}\} \quad (\text{Mahler's base})$$

$$\{\gamma_n \chi_n(x) ; n \in \underline{\mathbb{N}}\} \cup \{\chi_n(x)(x - n) ; n \in \underline{\mathbb{N}}\} \quad (\text{Van der Put's bases}).$$

We remind of the fact that  $\gamma_n$  is defined by

$$\gamma_0 = 1$$

$$\gamma_n = a_s p^s \quad \text{if } n = a_s p^s + a_{s-1} p^{s-1} + \dots + a_0, \quad a_s \neq 0.$$

$$\text{So } v(\gamma_n) = s, \quad \text{and } |\gamma_n|^{-1} = \max\{|m|^{-1} ; 0 < m \leq n\}.$$

$\chi_n$  is the characteristic function of  $\{x ; |x - n| < |\gamma_n|\}$ .

Define  $R_n = \gamma_n \binom{x}{n}^2$ ; it will then follow from lemma 2.2 that  $\|R_n\|_1 = 1$ ; however, the  $R_n$  do not form a base for  $C^1$ , as we already know from 1.6. Can we choose polynomials  $\tilde{R}_n$  such that  $\deg \tilde{R}_n = 2n + 1$  and the  $R_{n-1} \tilde{R}_n$  form a normal base for  $C^1$ ? Inspired by Van der Put's base, we could try  $\tilde{R}_n \sim R_n(x - n)$ .

After normalisation, we get  $\tilde{R}_n = \gamma_{n+1} \binom{x}{n} \binom{x}{n+1}$ . Unfortunately, it turns out that  $\{R_n, \tilde{R}_n ; n \in \underline{\mathbb{N}}\}$  are not orthogonal in  $C^1$ . This comes from the fact that

$$|\tilde{R}'_n(n)| = \left| \frac{\gamma_{n+1}}{n+1} \right| < 1 \quad \text{for some } n.$$

An answer to our question is furnished by following theorem.

2.1 THEOREM. - Let

$$R_n = \gamma_n \binom{x}{n}^2,$$

$$\tilde{R}_n = \gamma_{n+1} \binom{x}{n} \binom{x - (n+1 - \gamma_{n+1})}{\gamma_{n+1}} \binom{x+1 - \gamma_{n+1}}{n+1 - \gamma_{n+1}}$$

then  $\{R_n; \tilde{R}_n; n \in \mathbb{N}\}$  form a normal base for  $C^1(\mathbb{Z}_{\sim p}, \mathbb{Q}_{\sim p})$ .

Note that for  $n = a_s p^s - 1$ ,  $0 < a_s < p$ ,  $\tilde{R}_n = \gamma_{n+1} \binom{x}{n} \binom{x}{n+1}$ . We need some lemmas.

2.2 LEMMA. -  $\|R_n\|_1 = \|\tilde{R}_n\|_1 = 1$ .

Proof. - For all  $x \neq y$ , we have

$$\left| \frac{R_n(x) - R_n(y)}{x - y} \right| \leq \frac{|\gamma_n|}{|x - y|} \left| \binom{x}{n} - \binom{y}{n} \right| \max(|\binom{x}{n}|, |\binom{y}{n}|) \leq 1$$

because  $\|\gamma_n \binom{x}{n}\|_1 = 1$  and  $\|\binom{x}{n}\| = 1$ .

Furthermore

$$\left| \frac{R_n(n) - R_n(n - \gamma_n)}{n - \gamma_n} \right| = 1$$

In quite a similar way, we prove that  $\|\tilde{R}_n\|_1 \leq 1$ ; finally

$$\|\tilde{R}_n\|_1 \geq |R'_n(n)| = \left| \frac{\gamma_{n+1}}{\gamma_{n+1}} \right| = 1.$$

2.3 LEMMA. - If  $0 \leq m < n$ , then  $|\tilde{R}'_n(m)| < 1$ .

Proof.

$$\tilde{R}'_n(m) = \gamma_{n+1} \frac{d}{dx} \binom{x}{n} \Big|_{x=m} \binom{m - (n+1 - \gamma_{n+1})}{(n+1) - (n+1 - \gamma_{n+1})} \binom{m - (\gamma_{n+1} - 1)}{n+1 - \gamma_{n+1}}.$$

If  $m \geq n+1 - \gamma_{n+1}$ , then  $\tilde{R}_n(m) = 0$ . Suppose  $m < n+1 - \gamma_{n+1}$ . If  $|\gamma_{n+1}| < |\gamma_n|$ , the result follows easily from the fact that  $\|\binom{x}{n}\|_1 = |\gamma_n|^{-1}$ . So we can suppose that  $\gamma_n = \gamma_{n+1} = a_s p^s$ .

We introduce the notation

$$\text{Schiff}(a_s p^s + a_{s-1} p^{s-1} + \dots + a_0) = a_s + a_{s-1} + \dots + a_0.$$

We remind of the fact that

$$\text{Schiff } m + \text{Schiff}(n - m - 1) + 1 - \text{Schiff } n \leq (p-1) v(\gamma_n), \text{ for } 0 \leq m < n.$$

This follows from the fact that  $\|\binom{x}{n}\|_1 = 1$ , but it can also be proved directly.

Now, let  $n = a_s p^s + \dots + a_0$ , then  $m < a_{s-1} p^{s-1} + \dots + a_0$ , and  $n-m-1 \geq a_s p^s$ .

We have

$$\begin{aligned}
v\left(\left|\frac{d}{dx} \binom{x}{n}\right|_{x=m}\right) &= v\left(\frac{m! (n-m-1)!}{n!}\right) \\
&= (p-1)^{-1} (\text{Schiff}(n) - \text{Schiff}(m) - \text{Schiff}(n-m-1) + 1) \\
&= (p-1)^{-1} (a_s + \text{Schiff}(n - a_s p^s) - \text{Schiff}(m) - \text{Schiff}(n-m-1 - a_s p^s) - a_s + 1) \\
&\geq -v(\gamma_n - a_s p^s) > -v(\gamma_n),
\end{aligned}$$

hence

$$\left|\frac{d}{dx} \binom{x}{n}\right|_{x=m} < \frac{1}{|\gamma_n|} = \frac{1}{|\gamma_{n+1}|};$$

the result follows.

Proof of theorem 2.1. → The polynomials form a dense subspace of  $C^1$  (cf. Mahler's base). Since the  $R_n$  and  $\tilde{R}_n$  generate the polynomials, it only remains to show that  $\{R_n, \tilde{R}_n; n \in \mathbb{N}\}$  form an orthogonal system.

Using [8], 5.1.(e), it is sufficient to show that for each  $m \in \mathbb{N}$ :

$R_n$  is orthogonal to the linear hull of  $\{\tilde{R}_n, R_{n+1}, \tilde{R}_{n+1}, \dots\}$

$\tilde{R}_n$  is orthogonal to the linear hull of  $\{R_{n+1}, \tilde{R}_{n+1}, R_{n+2}, \dots\}$ .

This follows from the fact that for all  $\alpha_j, \beta_j \in K$ , we have

$$\begin{aligned}
&\|R_n - \sum'_{j>n} \alpha_j R_j - \sum'_{j>n} \beta_j \tilde{R}_j\|_1 \\
&\geq |\varphi_1 (R_n - \sum'_{j>n} \alpha_j R_j - \sum'_{j>n} \beta_j \tilde{R}_j)(n, n - \gamma_n)| = 1 = \|R_n\|_1,
\end{aligned}$$

and

$$\begin{aligned}
&\|\tilde{R}_n - \sum'_{j>n} \alpha_j R_j - \sum'_{j>n} \beta_j \tilde{R}_j\|_1 \\
&> |\tilde{R}'_n(n) - \sum'_{j>n} \alpha_j R'_j(n) - \sum'_{j>n} \beta_j \tilde{R}'_j(n)| = |\tilde{R}'_n(n)| = 1 = \|\tilde{R}_n\|_1,
\end{aligned}$$

using the fact that  $R'_j(n) = 0$ ,

$$|R'_j(n)| < 1 \text{ for } j > n.$$

**2.4 Note.** - Our proof is merely inspired by Van Rooÿ's proof of Mahler's theorem ([8], 5.27). It is also possible to give a proof using the residue class space (as in 1.1), which is, however, considerably longer.



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