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A NON-ARCHIMEDEAN ANALOGUE OF THE DISCRETE SERIES

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The purpose of this papers is to define a non-archimedean analogue of the discrete series of $SL_2(\mathbb{R})$. We will refer to what is conjectured, what can be proved, and what are the difficulties in studying our representations. For proofs and details, we quote MORITA-MURASE [5].

1. Classical case.

Let $\mathbb{C}$ and $\mathbb{R}$ be the complex number field and the real number field, respectively. For any field F, let $P^1(F) = F \cup \{\infty\}$ be the one dimensional projective space over F. Then $P^1(\mathbb{C}) - P^1(\mathbb{R})$ is the disjoint union of the upper half plane $H_+ = \{z \in \mathbb{C} ; \operatorname{Im}(z) > 0\}$ and the lower half plane $H_- = \{z \in \mathbb{C} ; \operatorname{Im}(z) < 0\}$. For any integer $s \leqslant -2$, put

$$V_s^+ = \{f : H_+ \dashrightarrow \mathbb{C} ; \text{ analytic}, \quad |f|_2^2 = \int_{H_+} |f(z)|^2 y^{-s-2}\, dx\, dy < \infty\} ,$$

$$\pi_s^+(g)\; f(z) = (bz + d)^s f\left(\frac{az + c}{bz + d}\right) \quad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})\right) ,$$

where $f \in V_s^+$ and $z = x + iy$ $(x , y \in \mathbb{R})$. Then V_s^+ becomes a Hilbert space with the norm $|\ |_2$ and $\pi_s^+(g)$ is a unitary operator on V_s^+. Hence π_s^+ defines a unitary representation of the locally compact group $SL_2(\mathbb{R})$ on V_s^+. We can also define π_{-1}^+ by modifying the norm suitably. Further, if we use H_- instead of H_+, then we obtain another unitary representation π_s^- for any integer $s \leqslant -1$. It is well known :

THEOREM C. - The $\pi_s^{\pm}$ are irreducible representations of $SL_2(\mathbb{R})$, and no two of them for various s are equivalent.

2. Definition of π_s in p-adic cases.

Now we are going to construct p-adic analogues of the $\pi_s^{\pm}$. We replace $\mathbb{R}$ by a finite extension L of the p-adic number field $\mathbb{Q}_p$, and replace $\mathbb{C}$ by a non-archimedean field $(k , |\ |)$ containing L such that k is complete with respect

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to $| \ |$ and algebraically closed. Hence we consider continuous representations of $G = SL_2(L)$ on linear topological k-vector spaces.

Put $D = P^1(k) - P^1(L)$. Since L is locally compact, $P^1(L)$ is compact. Hence, for any positive integer m, $P^1(L)$ is covered by $\{z \in P^1(k) ; |z| > |p^{-m}|\}$ and a finite number of mutually disjoint open balls

$$\{z \in P^1(k) ; |z - a_i| < |p^m|\} \quad (a_i \in L , \ i = 1 , \dots , N) .$$

Put

$$D_m = \{z \in P^1(k) ; |z| \leqslant |p^{-m}| , \ |z - a_i| \geqslant |p^m| \ (i = 1 , \dots , N)\} .$$

Then $\{D_m\}_{m=1}^{\infty}$ is an increasing sequence of subsets of D and $D = \bigcup_m D_m$. Let $\mathcal{O}(D_m)$ be the space of k-valued functions on D_m of the form

$$f(z) = \sum_{m=0}^{\infty} c_m z^m + \sum_{i=1}^{N} \sum_{m=-1}^{-\infty} c_m^{(i)} (z - a_i)^m ,$$

where c_m and $c_m^{(i)}$ are elements of k, and we assume that this limit converges (uniformly) on D_m. It is known that $\mathcal{O}(D_m)$ becomes a Banach space with the supremum norm $|f| = \sup_{D_m} |f(z)|$.

We say that a k-valued function f on D is <u>analytic</u> if, and only if, the restriction of f to each D_m belongs to $\mathcal{O}(D_m)$. (This is the definition of analytic functions in the theory of rigid analytic spaces (cf. MORITA [4]).) Let V be the space of all k-valued functions on D. Hence V is the projective limit of the $\mathcal{O}(D_m)$ with respect to the restriction map $\mathcal{O}(D_{m+1}) \longrightarrow \mathcal{O}(D_m)$. Therefore V has a natural Fréchet topology. In particular, V is a complete linear topological space. We consider this space V as the analogue of $V_s^{\pm} \oplus V_s^{\pm}$ for any s.

Let s be a negative integer. For any $f \in V$, put

$$\pi_s(g) \ f(z) = (bz + d)^s \ f\left(\frac{az + c}{bz + d}\right) \quad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\right) .$$

Then π_s defines a continuous representation of the locally compact group G on the Fréchet space V. This is our analogue of the classical discrete series.

3. Conjecture.

Let U be the space of rational functions of z (with coefficients in k) which have no poles in D. Then U is a dense subspace of V. Further the subspace U of V is G-invariant because $(bz + d)^s$ belongs to U for any $g \in G$.

Since k is algebraically closed, any element f of U can be expressed as a partial fractional series of the form

$$f(z) = \sum_{m=0}^{\infty} d_m z^m + \sum_{j=1}^{n} \sum_{m=-1}^{-\infty} d_m^{(j)} (z - b_j)^m \quad \text{(a finite sum)}$$

$(d_m , d_m^{(j)} \in k , \ n \geqslant 1 , \ b_j \in L)$. We define a subspace U_s of U by

$$U_s = \{f \in U \; ; \; d_m^{(j)} = 0 \text{ for any } j \text{ and } m \text{ with } 0 > m > s\} .$$

Then we can prove that U_s is a closed G-invariant subspace of U. Let V_s be the closure in V of the subspace U_s. It is obvious that V_s is a closed G-invariant subspace of V. Hence we obtain representations of G on V_s and V/V_s.

Now we have the following conjecture (cf. Theorem C) :

CONJECTURE V.

(i) V_s and V/V_s are (topologically) irreducible G-modules.

(ii) No two of them for various s are G-equivalent.

4. The result and remarks.

Though we can not prove the conjecture now, we can prove the corresponding assertion for the dense subspace U of V.

Let $\mathfrak{g} = \{X \in M_2(L) \; ; \; tr(X) = 0\}$, and put

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!} \quad \text{for any } X \in \mathfrak{g} .$$

Then this series converges in $M_2(L)$ to an element of G if the eigenvalues $\pm \lambda$ of X satisfies $|\lambda| < |p^{1/(p-1)}|$. It follows that

$$(d\pi_s)(X) \; f(z) = \lim_{t \to 0} \frac{1}{t} \{\pi_s(\exp(tX)) \; f(z) - f(z)\}$$

is well-defined for any $X \in \mathfrak{g}$ and $f \in V$, and $d\pi_s$ is a representation of the Lie algebra $\mathfrak{g}$ on the space V. It is obvious that any closed G-invariant subspace of U is $\mathfrak{g}$-invariant.

Let O be the integer ring of L, and put $K = SL_2(O)$. Then K is a maximal compact subgroup of G. Since K is an open subgroup of G, a closed K-invariant subspace of U is $(\mathfrak{g}, K)$-invariant.

Now our main result can be stated as :

THEOREM U.

(i) U_s and U/U_s are algebraically irreducible $(\mathfrak{g}, K)$-modules (i. e. they have no nontrivial $(\mathfrak{g}, K)$-invariant k-subspaces).

(ii) No two of them for various s are isomorphic as $\mathfrak{g}$-modules.

Obviously this theorem implies topological irreduciblities of U_s and U/U_s, and the non-equivalence of them.

Remark.

(i). The difficulties in proving the conjecture lie in the fact that Shur's lemma does not hold in our case. For example, we can prove that the only intertwining operators of V are the scalar operators, though V has the closed G-invariant subspace V_s. We prove our theorem by constructing the total space from any non-zero element (cf. MORITA-MURASE [5], 3-3).

(ii). It is remarkable that U_s is infinite dimensional and irreducible as a K-module though K is a compact group. This phenomenon causes a difficulty in defining the admissible representations.

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