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SIEGFRIED BOSCH

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# ON THE REDUCTION OF ABELIAN VARIETIES

by Siegfried BOSCH (*)
[Universität Münster]

Let $A$ be an abelian variety of dimension $n$ over a **complete non-Archimedean** field $k$. Viewing $A$ as a rigid analytic group variety over $k$, we say that an open analytic subgroup $N \subset A$ has semi-abelian reduction if it is smooth over the valuation ring $\overset{\circ}{k}$ of $k$ (see below) and if the analytic reduction $\tilde{N}$ of $N$ is an extension of an abelian variety $\tilde{B}$ by an affine torus $\tilde{T}$ (everything defined over the residue field $\tilde{k}$ of $k$), i. e., if there is an exact sequence

$$0 \longrightarrow \tilde{T} \longrightarrow \tilde{N} \longrightarrow \tilde{B} \longrightarrow 0 .$$

The existence of a subgroup $N \subset A$ having semi-abelian reduction is of indispensible value for the construction of the universal covering $\hat{A}$ of $A$, see [2] and [3].

If the valuation on $k$ is discrete, one can obtain a subgroup $N \subset A$ of the above type as follows. One considers the formal completion $\hat{\mathcal{N}}$ of the Néron model $\mathcal{N}$ of $A$. Then $\hat{\mathcal{N}}$ can be viewed as an open analytic subgroup of $A$, and its identity component $N := \hat{\mathcal{N}}_0$ has potential semi-abelian reduction (meaning that $N \otimes k'$ has semi-abelian reduction for some finite extension field $k'$ of $k$). It is not yet known how to construct such a group $N$ by analytic means. In this article, we want to discuss this question. In particular, we will characterize the semi-abelian reduction in terms of analytic properties.

Let $H$ be an analytic group variety over $k$. Then $H$ is called <u>formal</u> if it carries a formal analytic structure (given by some formal affinoid covering) (see [1], § 1), such that the structure is compatible with the group operations. The formal structure of $H$ is unique if it exists. In particular, it gives rise to a well-defined analytic reduction $\tilde{H}$ of $H$. (The reduction is a scheme of locally finite type over $\tilde{k}$.) A formal analytic group $H$ is called <u>smooth over</u> $\overset{\circ}{k}$ if the reduction $\tilde{H}$ is geometrically regular and if $H$ is <u>distinguished</u>. The latter means that, for all formal affinoid parts $\mathrm{Sp}\, C$ of $H$, the supremum norm on $C$ is a residue norm with respect to some epimorphism $T_m \longrightarrow C$ (where $T_m$ is a free Tate algebra) (see [1], § 2). One knows that $\tilde{H}$ is a group scheme over $\tilde{k}$ if $H$ is smooth over $\overset{\circ}{k}$.

(*) Siegfried BOSCH, Mathematisches Institut der Universität, 64 Roxeler Strasse, D-4400 MÜNSTER (Allemagne fédérale).

In the following we always assume that $k$ is a p-adic field (i. e., a complete field containing $\mathbb{Q}_p$ ). The valuation of $k$ can be discrete or dense. We denote by $[p] : A \to A$ the homomorphism of the abelian variety $A$ obtained by multiplying elements with $p$ .

PROPOSITION 1. - Let $N$ be a connected open analytic subgroup of $A$ , smooth over $\overset{\circ}{k}$ . Then the following conditions are equivalent :

(i) $N$ is maximal among all connected open formal analytic subgroups of $A$ .

(ii) $[p] : N \to N$ is surjective.

(iii) $N$ has potential semi-abelian reduction.

The difficult part of the proof is to show that condition (iii) of the proposition implies condition (i). One shows more generally that $N$ contains all connected open formal analytic subgroups of $A$ if it has potential semi-abelian reduction. Consequently, a subgroup $N \subset A$ satisfying the equivalent conditions of the proposition is unique.

Each commutative analytic group of dimension $n$ over a field of characteristic 0 is locally isomorphic to the n-dimensional additive group $\mathbb{G}_a^n$ . Therefore one can find an open analytic subgroup $I \subset A$ which is isomorphic to the unit ball in $\mathbb{G}_a^n$ . For $\nu \geq 1$ , we denote by $I_\nu$ the identity component of the affinoid group $[p^\nu]^{-1}(I)$ . Let $A_+ := \bigcup_{\nu=1}^{\infty} I_\nu$ .

PROPOSITION 2. - Let $N$ be a connected open analytic subgroup of $A$ , smooth over $\overset{\circ}{k}$ . Let $N_+$ be the kernel of the reduction map $N \to \tilde{N}$ . Then $N$ has potential semi-abelian reduction if, and only if $N_+ = A_+$ .

This result suggests how to proceed with an analytic construction of subgroups $N \subset A$ having semi-abelian reduction. All one has to do is to construct some "quasi-compact closure" of the Stein group $A_+ \subset A$ . Two steps are necessary. The first one is established by the following result :

THEOREM. - Modulo extension of the ground field $k$ , the analytic variety $A_+$ is isomorphic to the "open" unit ball in affine n-space.

The second step is still in the stage of a conjecture.

PROBLEM. - Find a distinguished open affinoid subspace $U \subset A$ containing the unit element $e \in A$ such that $U_+(e) = A_+$ (where $U_+(e) := \pi^{-1}(\pi(e))$ with $\pi : U \to \tilde{U}$ denoting the canonical reduction map).

It is expected that the problem can be solved, at least if the ground field $k$ is replaced by a finite extension. If $U$ is an open affinoid subspace of $A$ satisfying the stated properties, then it follows from the theorem that $U$ is smooth over

$\overset{\circ}{k}$ in some formal neighborhood U' of e . We may assume U' is connected. Let N be the subgroup of A generated by U' . Then N is a connected open analytic subgroup, smooth over $\overset{\circ}{k}$ , such that $N_+ = U'_+(e) = U_+(e) = A_+$ . Hence by proposition 2, N has potential semi-abelian reduction. Thereby we see that, modulo finite field extension, an open subgroup $N \subset A$ having semi-abelian reduction can be obtained by an analytic construction, provided the problem stated above can be solved.

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