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"Towards a Schwarz list for Lamé's differential operators via division points on elliptic curves"

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by

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1. - Lamé's operator L_n .

It is the following (family of) second order linear differential operator(s) on P^1

$$(1.1) \qquad L_n = L_{n,B} = D^2 + \frac{1}{2}\frac{f'}{f} D - \frac{n(n+1)x + B}{f}$$

where $D = d/dx$, $y^2 = f(x) \equiv 4x^3 - g_2x - g_3 = 4 \prod_{i=1}^{3} (x-e_i)$ is an elliptic curve E , n is a rational number and B is a constant called "the accessory paremeter". Todays discussion will be mainly in characteristic zero, and g_2, g_3 , B will be algebraic numbers ; but we will occasionally refer to mod p properties of L_n , where p is a prime number s.t. $p \neq 2,3$, $n \in \mathbb{Z}_p$, B , g_2, g_3 are p-integral and E has good reduction mod p .

From the Riemann viewpoint (1 1) is a regular operator on $\mathbb{P}^1(\mathbb{C})$ with Riemann scheme :

$$(1.2) \qquad \begin{pmatrix} e_i & \infty & \\ 0 & -n/2 & ; \; x \\ 1/2 & (n+1)/2 & \end{pmatrix}$$

When pulled back to E, via $x : E \to \mathbb{P}^1$, it becomes

$$(1.3) \qquad (y\frac{d}{dx})^2 - [n(n+1)x + B]$$

so it only has one singular point on E ($x = \infty$, which we take as the zero point O_E for the addition law on E) with exponents $-n, n+1$.

The Lamé opérator has been the object of some attention in the past few years for several reasons.

I) (The problem of accessory parameters : very classical, but still open). L_n is the first case, in order of increasing complication, of an operator which is not determined by its Riemann scheme (1.2). Namely it is of order 2 and has 4 singularities on $\mathbf{P}^1$. Notice the independence of (1.2) from B . Some people are interested in the problem of "reading off the monodromy group of a d.e. from its coefficients" and in particular in understanding (for fixed (1.2) i.e. n and f) the dependence of the monodromy group up on the accessory parameter. With respect to this problem, this operator is the first example that should be considered.

II) (The Grothendieck conjecture) L_n is not (in general) an operator of Picard-Fuchs type, i.e. it does not, in general, admit an algebraic integral formula for the solutions. This was first noticed by Deligne, who exhibited e class of L_n non-globally nilpotent. So, the Katz proof of the Grothendieck conjecture does not apply to L_n . D.V. and G.V. Chudnovsky have published a proof of that conjecture that applies to L_n ("Applications of Padé approximations to the Grothendieck conjecture on linear differential equations" in "Number Theory, New York 1984, LNM 1135 pp. 52-100").
Unfortunately, they apply their result incorrectly and deduce that, for $n \in \mathbb{N}$, $L_{n,B}$ (any, fixed, f) is globally nilpotent iff B has one of the 2n+1 special values $\{B_n^m\}_{m=1,\dots,2n+1}$ that are classically known to give rise to the so called "Lamé functions" of degree n , $E_n^m(x)$, m = 1 ,..., 2n+1 (see Whittaker of Watson 23.42). The mistake they made is to assume, perhaps misunderstanding an assertion in Poole (Chap. IX § 39), that L_n , $n \in \mathbb{Z}$, can never have a full set of algebraic solutions. In fact no counter example to the former assumption was classically known, and in fact stronger conjectures have been formulated. But we can disprove them by exhibiting an equation with projective monodromy group dihedral of order 6 over $\mathbb{C}(x)$, namely $L_{1,0}$ with $f(x) = 4x^3 - g_3$.

III) (The problem of global nilpotence). It is extremely difficult to decide whether a given d.e. is globally nilpotent, unless an algebraic integral formula exists. So L_n is again the first non trivial example to look at.

Dwork ("Arithmetic theory of differential equations" INDAM, Symp. Math., Vol XXIV, 1981, pp. 225-243) showed that L_n , $n \in \mathbb{N}$, is globally nilpotent iff either B is one of the B_n^m , $m = 1, \ldots, 2n+1$, or L_n has in fact zero p-curvature for almost all primes p . He also conjectured that this second possibility could never occur. We therefore disproved this conjecture.

NB. In fact, modulo the Grothendieck conjecture, the Dwork conjecture was equivalent to the statement that L_n should never have a full set of algebraic solutions. This is because when one solution is a lamé function, the other is automatically transcendal.

IV) (Schwarz list and torsion of elliptic curves) - In a paper of 1981 (J.D.E. 41, 1981, pp. 44-58), we tried to analyse all possible cases (any $n \in \mathbb{Q}$) in which a Lamé operator has a full set of algebraic solutions. For $n \notin \mathbb{Z}$ the results were quite complete and will be briefly reviewed later. But in the case $n \in \mathbb{Z}$ we were only able to show the close relation between algebraic solutions to L_n and torsion points on E . In particular we could bound the order of the projective monodromy group of L_n (when finite, hence necessarily dihedral) by twice the exponent of the group of division points on E , rational over a certain quadratic extension K_n of $\mathbb{Q}(g_2, g_3, B)$. We were unable at that time of exhibiting a single example of this situation. We now feel that we have a better understanding of this case. This will be the subject of the present talk.

I will now proceed to explain my results in completely classical terms (over $\mathbb{C}$) : from the above discussion it should be clear that this type of results give some insight also in the non-classical problems II and III.

2. - Finite monodromy for L_n , $n \in \mathbb{Q}$.

In general, if L is a second order linear d.o. defined over a Riemann surface C and $P_1, \ldots, P_2$ are the singular points of L or C and if τ denotes (as it will be the case all over this Talk) a ratio of independent solutions of L at an ordinary point P of L , continuation of τ along closed paths γ issuing from P on $C' = C \setminus \{P_1, \ldots, P_r\}$, produces a transformation

$$(2.1) \qquad \tau \to \varphi_\gamma(\tau) = \frac{a\tau + b}{c\tau + d}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2, \mathbb{C})$. The group of these transformations $G = \{\varphi_\gamma \mid \gamma \in \Pi_1(C', P)\}$ is what we call the projective monodromy group (or simply the group) of L (over C). If the wronskian of L is algebraic over C , as is the case for (1.1) over $\mathbb{P}^1$ (or (1.3) over E), the fact that G is finite is equivalent for L to having a full set of algebraic solutions. In that case, if K denotes the function field of C , the extension $K(\tau)/K$ is automatically Galois with group G . This can of course only happen if the singularities of L are all regular, the exponents are rational (and there are no logarithmic solutions). When $C = \mathbb{P}^1$ and $r = 3$ (e.g. if x is a coordinate on $\mathbb{P}^1$, the singular points are $0, 1, \infty$) an operator with exponent differences λ, μ, ν at $0, 1, \infty$ resp., is equivalent to

$$(2.2) \qquad L_{\lambda,\mu,\nu} = (d/dx)^2 + \frac{1 - \lambda^2}{4x^2} + \frac{1 - \mu^2}{4(x-1)^2} + \frac{\lambda^2 + \mu^2 - 1 - \nu^2}{4x(x-1)}$$

The group $G_{\lambda,\mu,\nu}$ of $L_{\lambda,\mu,\nu}$ over $\mathbb{P}^1$ is finite (Schwarz, 1879) iff (λ,μ,ν) (after suitable normalization e.g. $0 < \lambda , \mu , \nu < 1 \ldots$) is in the following list (the basic Schwarz list) :

	$(1/n , 1 , 1/n)$, $n \in \mathbb{N}$	cyclic of order	n
	$(1/2 , 1/n , 1/2)$, $n \in \mathbb{N}$	dihedral "	2n
(2.3)	$(1/2 , 1/3 , 1/3)$	Tetrahedral	
	$(1/2 , 1/3 , 1/4)$	octahedral	
	$(1/2 , 1/3 , 1/5)$	icosahedral	

In his book "Lectures on the Icosahedron", Klein showed that an L on C has finite monodromy iff it is (projectively equivalent to) a pullback of an operator $L_{\lambda,\mu,\nu}$ in the basic Schwarz list, via a suitable map $\xi : C \to \mathbb{P}^1$. In that case the group of L would be (a subgroup of) the groupe $G_{\lambda,\mu,\nu}$. We can also tell what the degree of the map ξ should be : if $\gamma(P)$ denotes the positive exponent difference of L at P , g is the genus of C , and

$$(2.4) \qquad \Delta(L) = \sum_{P \in C} (\gamma(P) - 1)$$

we have :

$$(2.5) \qquad \deg \xi = \frac{\Delta(L) - 2(g-1)}{\lambda + \mu + \nu - 1}$$

(this is an easy consequence of the Hurwitz formula, due to Dwork and myself).

So it is possible to examine whether a given L , for example (1.1) on $\mathbb{P}^1$, has a certain type of finite group : it is a problem of elimination theory. By this method one can analize in finite terms whether L is Tetrahedral, octahedral, or icosahedral : but the 2 sublists of cyclic and dihedral operators are infinite and in general we need some arithmetic considerations to establish whether, for example, a given operator L is not dihedral.

We now come back to L_n of (1.1) : G will be the group of L_n over $\mathbb{P}^1$. We could prove (in the 1981 paper) :

I. - If $n \notin \frac{1}{2}\mathbb{Z}$ and G is finite, then G must be either octahedral or icosahedral (of course this can only happen if $n + \frac{1}{2} \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z}$). We dispose here of explicit formulas. For example, if $n = 1/10$,

$$(2.6) \qquad D^2 + {}^{3x^2}/_{2(x^3-C)}\, D - {}^{11x}/_{400(x^3-C)}$$

is the only Lamé operator with finite (hence icosahedral) group. We leave aside this case.

II - If $n \in (\frac{1}{2}\mathbb{Z}) \setminus \mathbb{Z}$, then a classical trick due to Halphen and Brioschi shows that if G is finite, it is necessarily Klein's Vierer group (dihedral or order 4, abelian). The extension $\mathbb{C}(x,\tau)/\mathbb{C}(x)$ is isomorphic to $\mathbb{C}(\wp(u/2))/\mathbb{C}(\wp(u))$, where $\wp(u)$ denotes the Weierstrass elliptic function for E. We also leave aside this case.

III - If $n \in \mathbb{Z}$ and G is finite, then it is dihedral. This case, and its connection with the torsion of E, we analyze in detail.

3. - Finite monodromy for L_n , $n \in \mathbb{Z}$, and torsion points on E .

Since $L_{-n} = L_{n-1}$, we may assume $n \geq 0$. The symmetric square of L_n :

$$(3.1) \qquad M_n = f\,D^3 + \frac{3}{2} f'\,D^2 + \frac{1}{2} f''\,D - 4[\,n(n+1)x + B\,]\,D - 2n(n+1)$$

has Riemann scheme :

$$(3.2) \qquad \begin{pmatrix} e_i & \infty & \\ 0 & n+1 & \\ 1/2 & 1/2 & ;\ x \\ 1 & -n & \end{pmatrix}$$

It always admits a polynomial solution $F_n(x) = F_n(x,B)$ monic of degree n in x

For example

$$(3.3) \qquad \begin{aligned} F_0(x,B) &= 1 \\ F_1(x,B) &= x-B \\ F_2(x,B) &= x^2 - \frac{B}{3}x + \left(\frac{B^2}{9} - \frac{g_2}{4}\right) \end{aligned} .$$

On the other hand we have :

Prop. 3.4 - $\mathbb{C}(x,\tau)/\mathbb{C}(x)$ is never finite cyclic.

Proof - Otherwise it would be possible to express two independent solutions u,v of L_n as radicals of rational functions (if $\tau = u/v$, $\tau'/\tau = w/v^2 = c/yv^2$, c constant, w = wronskian of u,v).

So :

$$(3.4.1) \qquad u,v = \prod_{i=1}^{3} (x-e_i)^{\varepsilon_i}\, g(x)$$

with $\varepsilon_i = 0$ or $1/2$ and g a polynomial (not the same, in general, for u,v !). At ∞ this gives :

(3.4.2) $$\deg g(x) + \Sigma\, \varepsilon_i = -\text{ (an exponent of } L_n \text{ at } \infty) \ .$$

But the exponents of L_n at ∞ are $-n/2$ and $(n+1)/2$: so only one is non positive and u,v should belong to the same exponent at ∞ and should be dependent. Absurd. Q.E.D.

Prop. 3.5 - If L_n has a solution g whose square belongs to $\mathbb{C}(x,y)$, then τ is transcendental over $\mathbb{C}(x)$.

Proof - Let h be another solution independent of g and put $\tau = h/g$. Then :

(3.5.1) $$\tau^1 = c/g^2 y$$

with c constant. If τ were algebraic over $\mathbb{C}(x)$, let $\sigma \in \mathrm{Gal}(\mathbb{C}(x,y,\tau)/\mathbb{C}(x,y))$. We would have

$$(\sigma\, \tau)' = \sigma(\tau') = c/g^2 y = \tau'$$

and therefore

$$\sigma\, \tau = \tau + A_\sigma$$

with A_σ constant. So either $A_\sigma = 0$ and $\tau \in \mathbb{C}(x,y)$ or G is infinite. But the first case is impossible by Prop. 3.4 and the second is absurd. Q.E.D.

We conclude that if G is finite there exist 2 independent solutions u_1, u_2 of L such that

(3.6) $$u_1\, u_2 = F_n(x,B) \ .$$

We then have, for $\tau = u_1/u_2$:

(3.7) $$\tau' = \frac{c_n}{y\, u_2^2}$$

for some constant c_n , and, intrinsically

$$(3.8) \qquad \frac{d\tau}{\tau} = \frac{c_n}{F_n(x,B)} \frac{dx}{y} \underset{def}{=} : \omega \quad .$$

Since τ is, locally on E , a projective coordinate at each point $P \neq 0_E$, and since ω has a zero (of order $2n$) at 0_E , we conclude a priori, that ω has only simple poles on E with residues ± 1 . More precisely if for $P(x,y)$ $\overline{P}$ denotes the point $P(x, -y)$, and if we introduce the divisor

$$(3.9) \qquad D = \sum_{Q \in E} \mathrm{res}_Q(\omega)\, Q$$

we conclude that D is of the form

$$(3.10) \qquad \sum_{i=1}^{n} P_i - \sum_{i=1}^{n} \overline{P}_i$$

with $P_1, \ldots, P_n$ n distinct points in $E \setminus \{0_E\}$.

Our problem is then to determine in which cases a multiple of ω is logarithmic on E . This consists of 2 problems :

A) Is a multiple ND of D principal ?

B) If $H \in \mathbb{C}(x,y)$ (a priori regular and non-zero at 0_E) is such that

$$ND = (H)$$

can we conclude that the differential

$$\eta = N\,\omega - \frac{dH}{H} \quad ,$$

a priori of the 1^{st} Kind, is in fact zero ?

Of course, E being elliptic, η is zero iff it has a zero at 0_E or in other words.

B') Can the function H of B) be chosen in

$1 + \mathfrak{m}^2$

where $\mathfrak{m}$ is the maximal ideal of O_E on E ?

$$0 \longrightarrow 0 \quad .$$

Notice first of all that the constant c_n appearing in (3.8) is determined by $F_n(x,B)$ and the condition that the residues of ω are ± 1 .
For $n = 0$, ω is of the 1^{st} kind so, unless it is 0 , no multiple of it can be logarithmic.

Hence :

Prop. 3.11 - L_o never has a full set of algebraic solutions.

We now discuss L_1 . Here :

$$F_1 = x - B$$

$$c_1 = f(B)^{1/2}$$

$$\omega = \frac{f(B)^{1/2}}{x-B} \frac{dx}{y} \tag{3.12}$$

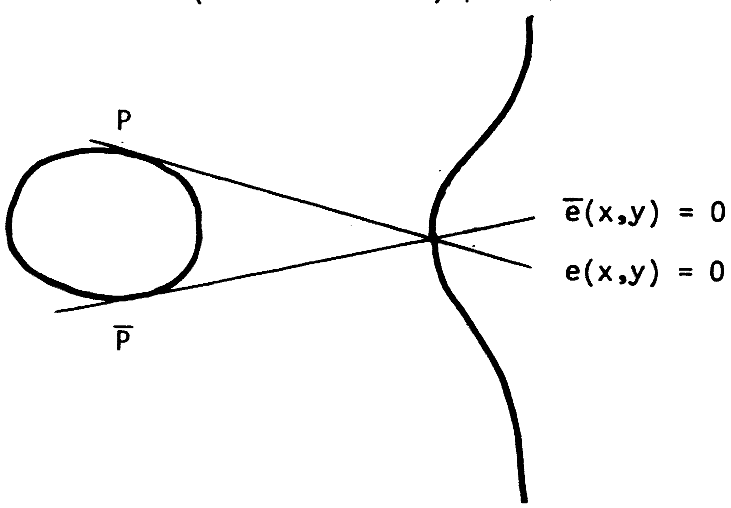
$$P = (B, f(B)^{1/2})$$

$$D = P - \overline{P} \quad .$$

Also $N(P-\overline{P}) \sim 0$ iff P is a 2N-division point.

Of course we may restrict to $N = 1,2,3,\ldots$ and in fact to $N > 1$ because $N = 1$ would give $\tau \in \mathbb{C}(x,y)$, absurd by Prop. 3.4. Consider $N = 2$.

Here P is a 4-division (not 2-division) point, so :



$$e(x,y) \equiv y - y'(P)X + x(P)\, y'(P) - y(P)$$

(3.13)

$$\bar{e}(x,y) \equiv - e(x, -y) \quad .$$

So

(3.14) $$H = - \frac{e(x,y)}{e(x, -y)} = \frac{1 - y'(P)\, \frac{x}{y}}{1 + y'(P)\, \frac{x}{y}} + o(\tfrac{x}{y}) = 1 - 2\, y'(P)\, \frac{x}{y} + o(\tfrac{x}{y}) \quad .$$

But Fig. 1 shows that $y'(P) \neq 0$, so $H \notin 1 + \mathfrak{m}^2$. In other words.

<u>Prop</u>. 3.15 - L_1 <u>never admits a group</u> G <u>of order</u> 4 .

Now consider $N = 3$. We can repeat the previous considerations to conclude that : $H \in 1 + \mathfrak{m}^2$ iff $y'(P) = y''(P) = 0$. This happens iff $B = g_2 = 0$. So

<u>Prop</u>. 3.16 - $L_{1,0}$ <u>with</u> $f(x) \equiv 4X^3 - g_3$ <u>is the only Lamé operator of</u> L_1 <u>type with group of order</u> 6.

To explain the uniqueness statement we must exclude that a 6-division point P which is neither a 3-nor a 2-division point may give rive to an $H \in 1 + \mathfrak{m}^2$. But first let me show that in fact, if $y^2 = 4x^3 - 1$, $L_{1,0}$ has algebraic τ .
Here

$$\omega = \frac{i\, d\, x}{x\, y}$$

$$P = (0,i) \text{ , tangent line } \quad y = \tau$$

and

$$\tau = H^{1/3} = \left(\frac{y - i}{y + i}\right)^{1/3}$$

gives

$$\frac{d\tau}{\tau} = \frac{1}{3} \left(\frac{dy}{y-i} - \frac{dy}{y+i}\right) = \frac{1}{3}\, \frac{2\, i dy}{y^2 + 1} =$$

$$\text{(since } \; 2y\, dy = 12\, X^2\, dx) = \frac{1}{3}\, \frac{12\, i\, X^2\, dX}{4X^3\, Y} = \frac{i\, dX}{XY} = \omega \quad .$$

To illustrate how to proceed to treat higher division points, we conclude the proof of (3.16) by considering a 6-division point P which is neither a 3-nor a 2-division point.

We consider the embedding Φ_6 of E into $\mathbb{A}^5 \subset \mathbb{P}^5$ associated to the very ample divisor $6\,0_E$. In affine coordinates $(Y_1, \ldots, Y_5)$ this is :

$$
\begin{aligned}
&Y_1(x,y) = x \\
&Y_2(x,y) = y \\
(3.17) \qquad &Y_3(x,y) = x^2 \\
&Y_4(x,y) = xy \\
&Y_5(x,y) = x^3 \qquad .
\end{aligned}
$$

The image curve E_6 is a curve of degree 6 in $\mathbb{P}^5$, isomorphic to E via Φ_6. We take $\Phi_6(0_E)$ = the unique point at ∞ of E_6 as the zero point 0_6 of E_6 and denote by $\boxplus$ the addition law (and by $\boxed{N}\,P$ the point $\overbrace{P \boxplus \ldots \boxplus P}^{N\text{times}}$). The osculating hyperplane Π_Q to E_6 at any Q intersects E_6 in $5Q + (\boxed{-5}\,Q)$. So P is 6-torsion iff the osculating hyperplane at P to E_6 is in fact hyperosculating.

This gives for Π_P the equation :

$$
(3.18) \qquad \begin{vmatrix}
Y_1 - x & Y_3 - x^2 & Y_5 - x^3 & Y_2 - y & Y_4 - xy \\
1 & 2x & 3x^2 & y' & xy' + y \\
0 & 2 & 6x & y'' & xy'' + 2y' \\
0 & 0 & 6 & y''' & xy''' + 3y'' \\
0 & 0 & 0 & y^{(iv)} & xy^{(iv)} + 4y'''
\end{vmatrix} = 0
$$

with

$$\begin{vmatrix} y^{(iv)} & 4y''' \\ y^{(v)} & 5y^{(iv)} \end{vmatrix} = 0 \qquad \text{(P of 6-division)}$$

(3.19) $$\begin{vmatrix} y''' & 3y'' \\ y^{(iv)} & 4y''' \end{vmatrix} \neq 0 \qquad \text{(P not of 5-division)}$$

$$y''' \neq 0 \qquad \text{(P not of 4-division)}$$

$$y'' \neq 0 \qquad \text{(P not of 3-division)}$$

NB. To obtain the conditions of 4 and 5 division, one may consider ϕ_4 and ϕ_5 .

On the other hand, if we had $y^{(iv)} = 0$, we would deduce from (3.19) that also $y^{(v)} = 0$, hence, from the identity :

(3.20) $$20\, y''\, y''' + 10\, y'\, y^{(iv)} + 2\, y\, y^{(v)} = 0$$

(obtained by differentiating $y^2 = f(x)$) we would have $y''\, y''' = 0$, a contradiction to (3.19).

On the other hand if we write

(3.21) $$\sum_{i=1}^{5} a_i(P)\ (Y_i - Y_i(P))$$

for the determinant (3.18) we conclude, as before, that the function $H(x,y) \in 1 + \mathfrak{m}$ such that

$$6(P - \overline{P}) = (H)$$

is

$$H(x,y) = \frac{\sum_{i=1}^{5} a_i(P)\ (Y_i(x,y) - Y_i(P))}{\sum_{i=1}^{5} (-1)^{i+1}\ a_i(P)\ (Y_i(x,y) - Y_i(\bar{P}))} =$$

(3.22)

$$= 1 + 2\,\frac{a_4(P)}{a_5(P)}\ \frac{y}{x^2} + o\left(\frac{x}{y}\right) =$$

$$= 1 + \left(12\, y^{(iv)} \Big/ \begin{vmatrix} y''' & 3y'' \\ y^{(iv)} & 4y''' \end{vmatrix} (P)\right) \frac{y}{x^2} + o\left(\frac{x}{y}\right)$$

and, by the previous considerations we conclude that $H \notin 1 + \mathfrak{m}^2$.

We can reach the same conclusions via Klein's theory of pullbacks. We know that $L_{1,B}$ has a group of order 6 iff it is a pullback of $L_{1/2,\,1/3,\,1/2}$ via a rational map $\xi(x)$.

The Riemann scheme of $L_{1,B}$ is

$$\begin{pmatrix} e_i & \infty & & \\ 0 & -1/2 & ; & X \\ 1/2 & 1 & & \end{pmatrix}$$

so the positive exponent difference at e_i is $1/2$ while at ∞ it is $3/2$. We conclude by (2.5) that $\deg \xi(x) = 3$. So we must have one of the two situations :

$$\begin{cases} \xi(\infty) = 3\,.\,\infty & \\ \xi(e_i) = 0 & i = 1,2,3 \\ \xi(\gamma) = 3.1 & \end{cases} \qquad \text{or} \qquad \begin{cases} \xi(\infty) = 3\,.\,0 & \\ \xi(e_i) = \infty & ,\quad i = 1,2,3 \\ \xi(\gamma) = 3\,.\,1 & \end{cases}$$

for some $\gamma \in \mathbb{C}$.

Therefore, in the first case :

$$\xi(x) = a\ f(x) = 1 + b(x-\gamma)^3$$

for some $a,b \in \mathbb{C}^\times$. So

$$4x^3 - g_2x - g_3 = 4(x-\gamma)^3 + c$$

for some $c, \gamma \in \mathbb{C}$ and

$$f(x) = 4x^3 - g_3 \quad , \quad \xi(x) = 1 - \frac{4}{g_3} x^3 \quad .$$

In the second case we get

$$f(x) = 4x^3 - \rho_3 \quad , \quad \xi(x) = (1 - \frac{4}{g_3} x^3)^{-1} \quad .$$

to determine the value of B , we must replace by $\xi(x)$ the independent variable of $L_{1/2,\ 1/3,\ 1/2}$ and normalize the resulting operator by making its wronskian equal to $f(x)^{-1/2}$, while keeping τ unchanged. We then obtain an operator of the form $L_{1,B}$, with $B = 0$.

4. - <u>Open problems</u>

4.1 - Fix a finite group G and a rational number n . It seems that the conditions on g_2, g_3, B ensuring that $L_{n,B}$ has projective monodromy group G have the form of two homogeneous equations in $g_2^{1/2}, g_3^{1/3}, B$, possibly with an open condition.

For example, if $n=1$ and G is dihedral of order 10, the conditions are :

$$\begin{cases} -2^5 . 3^4 . 13\,(4B^6 - g_2 B^4 - g_3 B^3) + 2^{14}\,(4B^3 - g_2 B - g_3)^2 + 3^6 . 5^2 . B^6 = 0 \\ -24\,(4B^4 - g_2 B^2 - g_3 B) = 12^2 . B^4 - 24 . g_2 B^2 + g_2^2 \end{cases} .$$

If $n \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ and G is the Vierergruppe, the two equations reduce to one and, for any choice of g_2, g_3 , there are B's that fullfill the requirement (<u>Brioschi solutions</u>). Aside from that case we ask whether the set of isomorphism classes of elliptic curves $E : y^2 = f(x)$, such that there exists an $L_{n,B}$ associated to f with group G , is finite.

Can one characterize those elliptic curves otherwise ? E.g ., do they all admit complex multiplications ?

(4.2) If $n \in \mathbb{Z}$, $n \geqslant 0$, $\frac{d\tau}{\tau} = \frac{C_n}{F_n(x)} \frac{dx}{y}$

guaranitees that, if G is finite :

(4.2.1) $\mathbb{C}(x,y) \subset \mathbb{C}(x,\tau)$.

In fact the field extension $\mathbb{C}(x,\tau)/\mathbb{C}(x,y)$ is unramified and cyclic. So $\mathbb{C}(x,\tau)$ is an elliptic function field associated to an elliptic curve E_1 and E is isomorphic to a quotient of E_1 modulo the subgroup generated by an N-division (if order $G = 2N$) point P_1 . Which unramified cyclic coverings of elliptic curves $E_1 \to E$ are related to a Lamé equation ?

4.3 - Let us reexamine the case of an L_1 with a group of order $2N$, by transcendental methods. Let ω_1, ω_2 be the semiperiods of the differentiel $\frac{dx}{y}$ on $E \cong \mathbb{C}/\Lambda$

$$\Lambda = 2\,\omega_1\,\mathbb{Z} + 2\,\omega_2\,\mathbb{Z} \quad .$$

For $\alpha \in \mathbb{C}$ we denote by $[\alpha]$ the image in $\mathbb{C}/\Lambda$. We consider a 2N-division point $[\alpha]$, and write explicitly the elliptic function $H(u)$ such that

$$(4.3.1) \qquad (H) = N\,[\alpha] - N\,[-\alpha] \quad .$$

We have :

$$(4.3.2) \qquad 2N\alpha = 2n_1\,\omega_1 + 2n_2\,\omega_2 = \lambda\,, \quad n_i \in \mathbb{Z}$$

and

$$(4.3.3) \qquad H(u) = \text{cost}\,.\,\frac{\sigma(u-\alpha)^N}{\sigma(u+\alpha)^{N-1}\,\sigma(u+\alpha-\lambda)}$$

(since $N\alpha = (N-1)(-\alpha) + (\lambda - \alpha)$).

Therefore :

$$(4.3.4) \qquad \frac{H'}{H}(u) = N\,\zeta(u-\alpha) - (N-1)\,\zeta(u+\alpha) - \zeta(u+\alpha-\lambda) \quad .$$

We must impose the condition (on ω_1, ω_2) that $\frac{H'}{H}(0) = 0$; therefore we get :

$$(4.3.5) \qquad \begin{aligned} N_{\alpha} &= n_1\,\omega_1 + n_2\,\omega_2 \qquad n_i \in \mathbb{Z} \\ N\,\zeta(\alpha) &= n_1\,\eta_1 + n_2\,\eta_2 \end{aligned}$$

where $\eta_i = \zeta(\omega_i)$.

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