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THE CONSTRUCTION OF SEMIGROUPS FROM GROUPS AND SEMILATTICES

par Gordon B. PRESTON

In the last few years, there has been a burst of activity among research workers investigating general structure theories for inverse semi-groups. Perhaps, the first such attempt was that of B. M. SCHEIN [3], who discussed how to construct an inverse semigroup from its trace, a set of disjoint Brandt groupoïds. SCHEIN characterised an inverse semigroup in terms of a partial order upon its trace. In [7] I discussed, being unaware at that time of Schein's work, various general strategies for constructing inverse semigroups from their semilattices, their $\mathcal{D}$-classes, and the relations between them. Since then various general theories have emerged. K. S. S. NAMBOORIPAD [9] developed a structure theorem for regular semigroups having a specialization to inverse semigroups similar to the theorems of John MEAKIN [7]. MEAKIN established a structure theorem for inverse semigroups in general showing that, A. H. CLIFFORD [1] structure theory for semilattices of groups could be extended to cover arbitrary inverse semigroups. P. A. GRILLET [11] showed how to construct an inverse semigroup from its semilattice of idempotents, and from a set of groups, one for each $\mathcal{D}$-class, this being directly in line with the programme proposed in [11]. And finally, R. Mc FADDEN and D. B. Mc ALISTER [12] constructed a kind of semigroup called a P-semigroup, in terms of which Mc ALLISTER[13], [14] has shown an arbitrary inverse semigroup may be constructed.

In this paper, I discuss various approaches to Mc Alister's results, and indicate another one.

1. Preliminary notions.

The prototype inverse semigroup $\mathfrak{I}_X$ is the semigroup of all bijections between subsets of X, combined under composition. Denote the domain and range of an element α in $\mathfrak{I}_X$ by Δ_α and ∇_α, respectively. Then

$$\Delta(\alpha\beta) = (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \text{ and } \nabla(\alpha\beta) = (\nabla\alpha \cap \Delta\beta)\beta .$$

The idempotents of $\mathfrak{I}_X$ are the identical mapping I_A upon subsets A of X. Since $I_A I_B = I_{A\cap B}$, any two idempotents of $\mathfrak{I}_X$ commute.

Denote by the $\alpha^{-1} : \nabla\alpha \to \Delta\alpha$, the converse, or inverse, of α. Then $\alpha\alpha^{-1}\alpha = \alpha$ and $\alpha^{-1}\alpha\alpha^{-1} = \alpha^{-1}$. Moreover, if $\alpha\beta\alpha = \alpha$ and $\beta\alpha\beta = \beta$ then $\beta = \alpha^{-1}$. An element β satisfying these two equations, in an arbitrary semigroup, is called an inverse of α. Thus in $\mathfrak{I}_X$, each element has a unique inverse.

By an inverse semigroup it means a semi group isomorphic to a subsemigroup of $\mathfrak{I}_X$ which, together with any element α, also contains α^{-1}. It may be shown that such semigroups may be equivalently defined as semigroups in which each element has a unique inverse. Because an inverse semigroup can be embedded in a semigroup $\mathfrak{I}_X$, for some X , it follows that any two idempotents of an inverse semigroup commute. Thus, the product of any two idempotents is, itself, an idempotent, and so the set of idempotents of an inverse semigroup forms a subsemigroup.

A semigroup of commuting idempotents is called a <u>semilattice</u> : for if we define $e \leqslant f$ is, and only is, $ef = e$, then the semigroup becomes a partially ordered set in which the greatest lower bound of any two elements is their product ef . Conversely, if we have a semilattice and define the product of any two elements to be their greatest lower bound, then we have a semigroup of commuting idempotents, in which in turn the order is determined in the above fashion.

A semilattice is a special case of an inverse semigroup, because in a semilattice

$$efe = e \quad \text{and} \quad fef = f$$

if, and only if, $e \leqslant f$ and $f \leqslant e$, i. e. if, and only if, $e = f$.

If S is an inverse semigroup and $a \in S$, then we denote the unique inverse of a in S by a^{-1} . We easily check that $(ab)^{-1} = b^{-1}a^{-1}$ and $(a^{-1})^{-1} = a$. Moreover, aa^{-1} and $a^{-1}a$ are idempotents of S . If E denotes the set of idempotents of S , then the partial order of the semilattice E may be extended to a partial order of S by agreeing that $a \leqslant b$ if, and only if, $a = aa^{-1}b$.

Equivalently, $a \leqslant b$ if, and only if, $a = eb$ for some idempotent e of S . If S is regarded as a sub semigroup of $\mathfrak{I}_X$, then $a \leqslant b$ means that a is a submapping of b , aa^{-1} is the identity mapping on the domain Δa of a and $a^{-1}a$ is the identity mapping on the range ∇a of a . If $a \leqslant b$ then also, as is obvious from interpretation just given, $a^{-1} \leqslant b^{-1}$, $ca \leqslant cb$ and $ac \leqslant bc$ for all c in S .

An inverse semigroup with only one idempotent is a group. Then a^{-1} is the inverse of a in the group and $a \leqslant b$ if, and only if, $a = b$.

If S is inverse, then there exists a maximal group morphic image G(S) , say, of S , in the sense that there is a morphism $\eta : S \to G(S)$ such that, if H is any group and $\alpha : S \to H$ is any morphism, then there is a unique morphism $\vartheta : G(S) \to H$ such that $\eta\vartheta = \alpha$. The mapping assigning S to G(S) is the object mapping of a functor from the category of inverse semigroups to the category of groups. Thus, this functor has a left adjoint and G(S) may be regarded as the free group on S .

The group G(S) may be constructed as follows. Define

$$\sigma = \{(a , b) \in S ; \ ea = eb \text{ for some idempotent } e \text{ of } S\}.$$

Then σ is a congruence on S , S/σ is a group and we may take G(S) to be S/σ and the mapping η to be the natural mapping $\sigma^{\natural} : S \to S/\sigma$.

Clearly, each idempotent of S is mapped by $\sigma^{\natural}$ to the identity of S/σ . Thus, the set E of idempotents of S is contained in a σ-class. When E actually is a σ-class, i. e. when the idempotent of S/σ is as small as possible, then S will be said to be reduced [1].

If S is inverse and $\theta : S \to T$ is a multiplicative morphism of S upon T, then T is also an inverse semigroup and moreover, $(a\theta)^{-1} = a^{-1}\theta$ for all a in S, i. e. θ is morphic with respect to inversion also. Also each idempotent of T is the image of an idempotent of S. By an inverse subsemigroup of S, we mean a subsemigroup which is also inverse. The inverse of any element in such a subsemigroup is necessarily its inverse in S. The image, under θ, of an inverse subsemigroup of S is an inverse subsemigroup of T and, conversely, the inverse image under θ of an inverse subsemigroup of T is an inverse subsemigroup of S.

The class of inverse semigroups is closed under the operations of taking morphic images, inverse subsemigroups and direct products. Hence inverse semigroups form a variety. There are two operations, product and inversion, and the following identities suffice to define this vatiety

$$(ab)c = a(bc) ,$$

$$aa^{-1} a = a ,$$

$$aa^{-1} bb^{-1} = bb^{-1} aa^{-1} ,$$

$$(a^{-1})^{-1} = a ,$$

for all a , b , c.

If S is inverse, we define the equivalence relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ as follows

$$a \,\mathcal{L}\, b \Longleftrightarrow a^{-1} a = b^{-1} b$$

$$a \,\mathcal{R}\, b \Longleftrightarrow aa^{-1} = bb^{-1}$$

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}$$

$$\mathcal{D} = \mathcal{L}\,\mathcal{R} \;(= \mathcal{R}\,\mathcal{L})$$

where the latter product denotes composition of relations. If $a \in S$ then L_a, R_a, H_a, D_a will be used to denote the $\mathcal{L}$-class, $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class, respectively, to which a belongs.

If $e^2 = e$, then H_e is a subgroup of S and is indeed the maximal subgroup of S for which e is the identity

$$H_e = \{a \in S : aa^{-1} = a^{-1} a = e\} ,$$

and the inverse of a in the group H_e is its inverse in S. Hence, every subgroup of S is contained in an $\mathcal{H}$-class H_e, for some idempotent e of S.

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(1). Inverse semigroups with this property were first termed <u>proper</u> (in [4]), by T. SAITÔ. Then renamed as <u>reduced</u> by L. O'CARROL ([15]). They have since been called al also E-<u>unitary</u>. Reduced seems a natural and preferable term.

Since $a \,\mathcal{R}\, aa^{-1}$, for each a in S, each $\mathcal{R}$-class contains an idempotent. This idempotent is clearly unique. Hence, if E denotes the semilattice of idempotents of S, then

$$S = \{R_e \; ; \; e \in E\} .$$

If $e \geqslant f$, then we define

$$\varphi_{e,f} : R_e \longrightarrow R_f ,$$

by

$$a \longrightarrow fa .$$

The element fa is, then, the unique element b of R_f such that $b \leqslant a$.

The mappings $\varphi_{e,f}$, called the structure mappings of S, have the following properties

(1) $\varphi_{e,e}$ is the identical mapping on R_e,

(2) $\varphi_{e,f}\,\varphi_{f,g} = \varphi_{e,g}$, if $e \geqslant f \geqslant g$,

(3) $e\varphi_{e,f} = f$ if $e \geqslant f$,

(4) if $a \in L_e$, $b \in R_e$, $e \geqslant f$, and $\bar{f} = (a^{-1}\,\varphi_{e,f})^{-1}\,(a^{-1}\,\varphi_{e,f})$, then $aa^{-1} \geqslant \bar{f}$ (with strict inequality if $e > f$) and

$$(ab)\,\varphi_{aa^{-1},f} = (a^{-1}\,\varphi_{e,f})^{-1}\,(b\,\varphi_{e,f}) .$$

Indeed [17] MEAKIN has shown that once E and the partial groupoïds R_e are given, together with mappings $\varphi_{e,f}$ satisfying (1) to (4), then S is uniquely determined and the product in S is defined by

$$ab = (a^{-1}\,\varphi_{a^{-1}a,a^{-1}abb^{-1}})^{-1}\,(b\,\varphi_{bb^{-1},a^{-1}abb^{-1}}) .$$

We conclude this introducting section by giving two characterizations of reduced inverse semigroups. First, S is reduced, if and only if, $\mathcal{R} \cap \sigma = I_S$, the identity relation on S. Second, S is reduced, if and only if, each structure mapping $\varphi_{e,f}$ is injective.

2. Reduced semigroups and their representation.

Reduced semigroups were first considered, so far as I am aware, by T. SAITÔ in [4]. SAITÔ assumed, also, that his inverse semigroups had a compatible total order (in addition to the natural partial order on any inverse semigroup). He observed that $\mathcal{R} \cap \sigma = (I)$ (2) and hence that the mapping $\theta : S \longrightarrow E \times G$, where E is the semilattice of idempotents and G is the free group on S, defined by $a \longmapsto (aa^{-1}, a\sigma)$ $a \in S$, is injective. Moreover, as a subset of the cartesien set product $E \times G$, $S\theta$ is subdirect. Hence, as sets, S determines E and G and,

(2) Cf [4], p. 652, theorem 1.

conversely, E and G determine S. So there is the possibility that also the product in S can be determined from those of E and G.

SAITÔ solved this problem. The total order that he introduced on E was in fact just an embellishment and restriction of his results. For reduced inverse semigroups without such an order his arguments still applied. He obtained a structure theorem that we now describe.

If $\theta : S \longrightarrow E \times G$ is the mapping $s \longmapsto (ss^{-1}, s\sigma)$, then to each $(e, g) \in S\theta$ is associated an element e^g of E in such a fashion that the product in $S\theta$ is given by

$$(\alpha) \qquad (e, g)(f, h) = ((e^g f)^{g^{-1}}, gh).$$

Conversely, starting with E and G as an arbitrary semilattice and group, he gave a set of conditions on the e^g such that, with the above multiplication a subset of $E \times G$ formed a reduced inverse semigroup S with semilattice E and maximal morphic group image G.

Indeed, for each $e \in E$, let $G(e) \subseteq G$, and suppose, for each $g \in G(e)$, that e^g is defined and belongs to E. Suppose further that

(i) $\bigcup \{G(e) ; e \in E\} = G$,

(ii) $1 \in G(e)$, $\forall e \in E$ (where 1 is the identity of G), and $e^1 = e$,

(iii) $f \leqslant e$, $g \in G(e)$ imply that $g \in G(\ )$ and $f^g \leqslant e^g$,

(iv) $g \in G(e)$ and $h \in G(e^g)$ together imply that $gh \in G(e)$ and $e^{(gh)} = (e^g)^h$,

(v) $g \in G(e)$ implies that $g^{-1} \in G(e^g)$.

Then, if $S = \{(e, g) ; e \in E, g \in G(e)\}$, and we define a product on S by the rule (α), above, S is a reduced inverse semigroup with E as its semilattice and G is the free group $G(S)$ on S.

Reduced semigroup next appeared in a paper by R. Mc FADDEN and L. O'CARROL [8] this time dealing with the special kind of reduced inverse semigroup, F-inverse semigroups, in which each σ-class has a unique maximal element (F for "fermé"). The existence of such maximum elements simplifies considerably the problem of defining a multiplication in $E \times G$: the maximum elements can be used to provide a natural cross-section of σ that can be identified with G.

In fact, let G be the set of maximum elements of σ-classes. We make G into a group by defining

$$g * h = \text{maximum element of } \sigma\text{-class to which } gh \text{ belongs},$$
$$= \text{maximum element of } (gh)\sigma .$$

If $g \in G$, then it is easily checked that g^{-1} (the inverse of g in S) also belongs to G and that : $g * g^{-1} = g^{-1} * g = 1$, where 1 is the maximum element of E (which, since S is reduced, is a σ-class). If $g \in G$, then $e \longmapsto e^g$, where $e^g = geg^{-1}$, is a morphism of E into E such that

(i) $(1^{g^{-1}})^g = 1^g$,

(ii) $e = e^1$, $\forall\, e \in E$,

(iii) $e^{(gh)} = (1^{gh})e^{h*g}$.

Set $S^* = \{(f1^g , g) ; f \in E , g \in E\}$ and if (e , g) , $(f , h) \in S$, define a product by

$$(\epsilon , g)(f , h) = (ef^g , gh) .$$

Then S^* is an inverse semigroup isomorphic to S , and the properties we have listed of the morphisms $e \longmapsto e^g$ serve to characterize reduced inverse semigroups.

The result, but also the situation, is a considerable simplification of that of SAITÔ. We almost have reduced F-inverse semigroups characterized as semidirect products of E dans G . The mapping $G \longrightarrow \mathrm{End}(E)$ which maps g to the endomorphism $e \longrightarrow e^g$, of E , fails to be an antimorphism solely because of the factor $1^{(gh)}$ in (iii).

The next appearance of reduced inverse semigroups was also a specialization, and one in which the fact that the inverse semigroups involved were reduced was not observed at the time. In [10], in a paper widely circulated before it was finally published (3), H. E. SCHEIBLICH gave a construction of the free inverse semigroup on a set, from which it was easy to verify that a free inverse semigroup was reduced. Reduced semigroups, thus, assumed a more central position in the theory of inverse semigroups.

More important however, was the fact that SCHEIBLICH exhibited free inverse semigroups in a way that suggested that a similar representation might be possible for other reduced inverse semigroups. Let us be explicit. The free inverse semigroup I_X on a set X was exhibites by SCHEIBLICH as a set of ordered pairs (A , g) , where g was an element of the free group G_X , on X , and A is a subset of G to which g belongs. The permissible such subsets of G are in fact the finite subsets A of G such that, if $a \in A$, then each initial segment of a , when a is written in reduced form, belongs to A . Here 1 is regarded as initial segment of each element of G . Let us call such a subset of G a <u>finite closed subset</u> of G and denote the set of all such sets by $\mathcal{Y}$. Note that $A \cap B \in \mathcal{Y}$ if $A , B \in \mathcal{Y}$; $\mathcal{Y}$ is, thus, an (intersection) semilattice of subsets of G . Then Scheiblich's representation of I_X is

$$I_X \simeq \{(A , g) : g \in G_X , A \in \mathcal{Y} , g \in A\} ,$$

with a product defined by

(3) For example, in 1972, the paper "Free generators in free inverse semigroups" (by N. R. REILLY, Bull. Austral. math. Soc., t. 7, 1972, p. 407-424), based on Scheiblich's paper, was published.

$$(A\ ,\ g)(B\ ,\ h) = (A \cup gB\ ,\ gh)\ ,$$

where $gB = \{gb\ ;\ b \in B\}$. It is easily checked that, although gB is not necessarily an element of $\mathcal{Y}$, we do have $A \cup gB \in \mathcal{Y}$ and $gh \in A \cup gB$. Thus, the product is well-defined and, as asserted, the semigroup thereby constructed is I_X.

The semilattice of idempotents of I_X is isomorphic to $\mathcal{Y}$, consisting in fact of all element $(A\ ,\ 1)$, where $A \in \mathcal{Y}$. The maximal group morphic image of I_X is the free group on X, viz. G_X. Thus, I_X is constructed from its semilattice and maximal morphic group image G_X as is also the case for the general reduced inverse semigroups of Saitô, and for the special F-inverse reduced semigroups of Mc Fadden and O'Carrol. But there is an additional feature of Scheiblich's construction, namely the sets $gA : A \in \mathcal{Y}$, $g \in G_X$, needed to perform the multiplication in I_X.

Set $\mathcal{X} = G_X\,\mathcal{Y}$. Then $G_X\,\mathcal{X} = \mathcal{X}$ and $\mathcal{X}$ is a set partially ordered by inclusion on which G_X acts, on the left, by order automorphisms. Moreover $\mathcal{Y}$, with operation $\cup$, is a <u>subsemilattice</u> of $\mathcal{X}$, i. e. not only does $A \cup B \in \mathcal{Y}$ when $A\ ,\ B \in \mathcal{Y}$, but also $A \cup B$ is then the least upper-bound of A dans B in $\mathcal{X}$. In addition, if $A \in \mathcal{Y}$, $B \in \mathcal{X}$, and $A \subseteq B$, then $B \in \mathcal{Y}$, in other words $\mathcal{Y}$ is <u>an order ideal</u> (with respect to the operation $\bigcup$) of $\mathcal{X}$. Finally, $\mathcal{X}$ and $\mathcal{Y}$ have the property that, if $B \in \mathcal{X}$, then there exists $A \in \mathcal{Y}$ such that $A \subseteq B$.

Thus, in summary, we have a group G, a partially ordered set $\mathcal{X}$ and a subset $\mathcal{Y}$ of $\mathcal{X}$ such that

(i) $\mathcal{Y}$ is a subsemilattice of $\mathcal{X}$,

(ii) $\mathcal{Y}$ is an order ideal of $\mathcal{X}$,

(iii) G acts on $\mathcal{X}$ by order automorphisms,,

(iv) $\mathcal{X} = G\,\mathcal{Y}$,

(v) if $A \in \mathcal{X}$, then there exists $B \in \mathcal{Y}$ such that $B \leqslant A$ (for I_X, in Scheiblich's construction, we write $B \leqslant A$ if, and only if, $A \subseteq B$).

An ordered triple $(G\ ,\ \mathcal{X}\ ,\ \mathcal{Y})$ satifying the above conditions will be called a <u>reduced triple</u>. If two elements, $A\ ,\ B$ of $\mathcal{X}$ have a greatest lower bound in $\mathcal{X}$, then we write $A \wedge B$ for their greatest lower bound. Thus, $\mathcal{Y}$ is a semilattice with operation $\wedge$.

From a reduced triple $(G\ ,\ \mathcal{X}\ ,\ \mathcal{Y})$, we may form a semigroup $P(G\ ,\ \mathcal{X}\ ,\ \mathcal{Y})$, as follows

$$P = P(G\ ,\ \mathcal{X}\ ,\ \mathcal{Y}) = \{(A\ ,\ g) \in \mathcal{Y} \times G\ ;\ g^{-1}A \in \mathcal{Y}\}\ ,$$

with a product defined by

$$(A\ ,\ g)(B\ ,\ h) = (A \wedge gB\ ,\ gh)\ .$$

P is then a reduced inverse semigroup.

This generalization of Scheiblich's representation of I_X (in which $g^{-1}A \in \mathcal{Y}$

if, and only if, $g \in A$, for $A \in \mathcal{Y}$) appeared only gradually. A special case appeared first in Mc FADDEN and Mc ALISTER's paper [12]. In the above form it first appeared in Mc ALISTER [14] W. D. MUNN and N. R. REILLY [18] have called triples $(G, \mathcal{X}, \mathcal{Y})$ satisfying conditions (i) - (v) above Mc Alister triples.

3. Mc Alister's theorems.

Two remarkable theorems of Mc Alister, the first in [13], and the second in [14], gave a new importance to reduced inverse semigroups. We describe the theorems in the reverse order to that of their discovery.

In [14] Mc ALISTER showed that any reduced inverse semigroup S is isomorphic to a semigroup $P(G, \mathcal{X}, \mathcal{Y})$. He also showed that this representation is essentially unique : G is (isomorphic to) the free group $G(S)$ on S, $\mathcal{Y}$ is (isomorphic to) the semilattice of S, $\mathcal{X}$ is determined to within order automorphism, and the action of G on $\mathcal{X}$ is determined to within isomorphism of actions. If $S \simeq P(G, \mathcal{X}, \mathcal{Y})$, we shall call $P(G, \mathcal{X}, \mathcal{Y})$ a standart representation of S.

We now describe McAlister's construction of $\mathcal{X}$ and the action of $G = G(S)$ upon it.

Let $\{D_i ; i \in I\}$ be the set of $\mathcal{D}$-classes of S. For each $i \in I$ choose a maximal subgroup H_i, say, of S contained in D_i. Take G to be S/σ and write G_i for $H_i \sigma^{\natural}$. Let f_i be the idempotent of H_i and R_i the $\mathcal{R}$-classe containing H_i. From each $\mathcal{H}$-class contained in R_i select a representative r_{iu}, choosing f_i as the representative from H_i. Set $k_{iu} = r_{iu}\sigma^{\natural}$, and, for all i, j in I, define

$$B_{ij} = \{k_{ju} \mid r_{ju}^{-1}, \ r_{ju} \leqslant f_i\} .$$

Denote by G/G_i the set of right cosets of G_i in G.

We now define

$$\mathcal{X} = \cup \{\{i\} \times G/G_i ; \ i \in I\} ,$$

and turn $\mathcal{X}$ into a partially ordered set by agreeing that

$$(i, G_i x) \geqslant (j, G_j y)$$

if, and only if,

$$G_j y = G_j k_{ju} x \quad \text{for some } k_{ju} \in B_{ij} .$$

The action of G on $\mathcal{X}$ is given by

$$g(i, G_i x) \overset{\text{def}}{=} (i, G_i xy^{-1}) .$$

The sub-semilattice $\mathcal{Y}$ is

$$\mathcal{Y} = \{(i, G_i k_{iu}) ; \ i \in I\} .$$

Starting from an arbitrary group G, and a semilattice E, Mc ALISTER also gives an abstract characterisation of such an $\mathcal{X}$ and $\mathcal{Y}$, reminiscent of the theorem of

Saitô with which we began.

The second theorem of Mc Alister [13] showed that, any inverse semigroup is an idempotent-separating morphic image of a reduced inverse semigroup. A morphism is said to be idempotent-separating if any two idempotents have distinct image. If θ is an idempotent-separating morphism of S, then the corresponding congruence $\theta \circ \theta^{-1}$ on S is determined a set $\{N_e \leq H_e \; ; \; e \in E\}$ of normal subgroups of the maximal subgroups H_e of S. In fact N_e is the kernel of the restriction of θ to H_e [2]. S may then be constructed from $S/\theta \circ \theta^{-1}$ by the method of A. COUDRON [5] or of H. d'ALARCAO [6].

4. Other approaches to Mc Alister's theorems.

Several alternative proofs of one or the other of Mc Alister's theorems have been given. A short construction of $\mathfrak{X}$ was given by W. D. MUNN in [19] in which he obtained $\mathfrak{X}$ as a quotient set of $E \times G$. B. M. SCHEIN in [16] exhibited $\mathcal{Y}$ effectively as the set of right ideals of S, and obtained $\mathfrak{X}$ as $G\mathcal{Y}$ by describing a suitable action of G on $\mathcal{Y}$.

In [18] MUNN and REILLY find constructions for all idempotent-separating and all idempotent-determined (a congruence is idempotent-determined if the congruence classes to which idempotents belong consist solely of idempotents) congruence on a reduced inverse semigroup $P(G, \mathfrak{X}, \mathcal{Y})$ in standard form. As a result of their analysis they are able to retrieve both Mc Alister's theorems by the following procedure. Let S be an inverse semigroup and let I_X be a free inverse semigroup of which S is a morphic image, $S \cong I_X/\rho$, say. Then there exists an idempotent-determined congruence τ, say, on I_X, and an idempotent separating congruence τ, say, on I_X/π, such that $I_X/\rho \cong (I_X/\pi)/\tau$. Scheiblich's result gives a standard form for I_X, MUNN and REILLY show that I_X/π is reduced and construct a standard representation for it from that of I_X. This gives a proof of Mc Alister's first theorem in [13]. If S is also reduced, then in turn they show how to construct a standard representation of $(I_X/\pi)/\tau$ from that of I_X/π. This gives Mc Alister's second theorem [14].

Other approaches may be found in the papers of L. O'CARROLL (see, for exemple, [15]).

Some results of P. R. JONES [19] may be used to provide another approach. An inverse subsemigroup of a reduced semigroup is reduced. JONES finds standard representations of inverse subsemigroups, in terms of a standard representation of the reduced semigroup containing it. In [13] Mc ALISTER showed that each inverse semigroup S is the idempotent separating morphic image of an inverse subsemigroup of a reduced inverse semigroup. Jones' results can then be used to give this inverse subsemigroup in standard form and, when S is reduced, to obtain an isomorphism.

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