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TOPOLOGICAL AUTOMATA

par Hartmut EHRIG et Wolfgang KÜHNEL⁽¹⁾

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Abstract. — Results concerning reduction, minimization and behavior characterization are given for several types of deterministic, partial and relational topological automata. Principal difficulties in the theory of topological automata can be overcome using the cartesian product with « bitopology » or restricting input, output and state spaces to be compactly generated. Thus we get results similar to corresponding ones for automata without topological structure. Background for all the considerations is a categorical theory for automata in closed and pseudoclosed monoidal categories given in earlier papers (cf. [8, 9, 12, 16]).

INTRODUCTION

Although automata have been studied extensively during the last ten years there are known up to now only a few number of results concerning topological and relational topological automata (cf. [3, 4, 19, 21]). In fact several results of reduction and minimization theory for finite automata don't remain true for topological automata, because the topological product

$$nat \times I : S \times I \rightarrow \bar{S} \times I$$

of a quotient $nat : S \rightarrow \bar{S}$ with an arbitrary space I , where $nat \times I$ denotes the cartesian product of nat with the identity on I , is no longer a quotient in general. Thus the induced next-state function $\bar{d} : \bar{S} \times I \rightarrow \bar{S}$ and output function $\bar{l} : \bar{S} \times I \rightarrow 0$ of the reduced automaton are not necessarily continuous (cf. [19]). Of course, things are even more difficult in the case of partial and relational topological automata which have been studied in [3, 4].

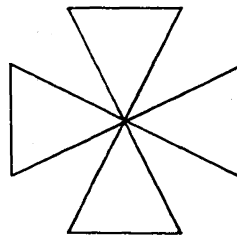
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One possibility to avoid this difficulty is to restrict the input space I to be locally compact (cf. [19]) or compact (cf. [3]). In fact it suffices to regard compactly generated Hausdorff-spaces, i.e. spaces in which a subset is open iff its intersection with all compact subspaces C is open in C (cf. [15, 20]), such that reduction, minimization and behavior characterization, well-known for deterministic automata, can be carried over to the topological case. These results are given in § 1 theorem 1.

Moreover it is possible to get around without restrictions for the spaces I , O and S . We take a slightly more general type of topological automata by replacing the product topology on $S \times I$ by the *bitopology*, i.e. the final topology on $S \times I$, written $S \otimes I$, such that the identity $S \times I \rightarrow S \otimes I$ is bicontinuous, i.e. continuous in each component. In fact this bitopology on $S \times I$ solves the universal problem corresponding to bicontinuous mappings, where for each topological space O a mapping $f : S \times I \rightarrow O$ is bicontinuous iff it is continuous considered as a mapping $\bar{f} : S \otimes I \rightarrow O$. In analogy to corresponding constructions in functional analysis the bitopology is also called *inductive topology* on $S \times I$. Explicitly a set O is open in $S \otimes I$ iff for all $s \in S$ and $x \in I$ the sections

$$O_s := \{ x \in I / (s, x) \in O \} \quad \text{and} \quad O_x := \{ s \in S / (s, x) \in O \}$$

are open in I and S respectively. Clearly each open set in the product topology is open in $S \otimes I$, but on the other hand the interior of the following crucifix together with the middle point is open in the bitopology but of course not in the product topology :



Consequently taking the « biproduct » $S \otimes I$ instead of the topological product we include more automata and in addition, the biproduct commutes with arbitrary colimits and especially with quotients which are needed for reduction. In fact it can be shown that the well-known results for finite deterministic automata can be carried over to this type of topological automata even in the case of partially defined functions. The corresponding results are given in theorem 2 of § 1.

Taking the same space $S \otimes I$ but replacing continuous mappings

$$d : S \otimes I \rightarrow S \quad \text{and} \quad l : S \otimes I \rightarrow O$$

by lower or upper semi-continuous relations we get relational topological automata (cf. [13]), where a relation $f \subseteq A \times B$, written $f : A \rightarrow B$, is called *lower* (resp. *upper*) *semi-continuous* if the inverse image of open (resp. closed) sets is open (resp. closed). In fact it is possible to give the construction of minimal and reduced automata and to characterize the behavior in this case, too. The results, similar to that of relational automata, are given in theorem 3 of § 1.

Of course, there are other interesting types of relational topological automata which will be studied in § 3.

In fact there exists a general theory of automata in monoidal categories (cf. [9, 10]) which can be applied to all of the above mentioned cases. It should be mentioned that the case of automata in closed categories, including the example of compactly generated Hausdorff-spaces, has been studied in [12] before. The categorical background and the corresponding categorical results are summarized in § 2 for the case of automata in closed and pseudoclosed categories respectively. These are generalizations of deterministic and nondeterministic automata in the classical case. Now, for the proofs of the theorems in § 1 it suffices to verify the categorical assumptions and to interpret the results in § 2.

Moreover the categorical theory of automata allows to consider several other types of topological automata, too, and to get similar results. This is done in § 3 for upper semi-continuous relations taking again the biproduct, for point-compact upper and lower semi-continuous relations (cf. [3, 4, 6]) between compactly generated spaces and for the case of locally compact input space corresponding to the treatment in [19]. Finally we discuss the applications to uniform, metric and tolerance automata, which are studied in [21].

§ 1. MAIN RESULTS

In this paragraph we give three theorems concerning minimization, reduction and behavior-characterization of deterministic, partial and relational topological automata respectively. The proofs will be given in § 2 and other types of topological automata will be studied in § 3.

Definition 1 : A *compactly generated automaton*, or shortly *CG-automaton*, consists of compactly generated Hausdorff-spaces I , O and S , called *input*, *output* and *state space* respectively, and continuous mappings

$$d : S\pi I \rightarrow S \quad \text{and} \quad l : S\pi I \rightarrow O,$$

called *next-state* and *output functions*, where by $S\pi I$ we denote the Kelleyfication of the topological product of the spaces S and I .

In order to get the Kelleyfication kX of a space X we take the same set and define a subset to be open in kX iff its intersection with each compact set C in X is open in C . Note that a function with the domain X is continuous on all compact subsets iff it is continuous regarded as a function with domain kX . Moreover X and kX have the same compact subsets.

As usual each state $s \in S$ defines a continuous input-output-function $\mu_s : I^+ \rightarrow O$ assigning to each non-empty input-string the last output symbol. I^+ in our case is the free topological semigroup defined to be the topological sum of all n -fold iterated products I^n with $n \geq 1$. Considering the space $C(I^+, O)$ of all continuous mappings $f : I^+ \rightarrow O$ with the Kelleyfication of the compact-open topology the *behavior* $E(A)$ of a CG-automaton A is the subspace of $C(I^+, O)$ defined by all input-output-functions μ_s of the states of A . By a subspace in this context we always mean a subset furnished with the Kelleyfication of the relative topology. As usual A and A' are called *equivalent* if they have the same behavior, i.e. $E(A) = E(A')$. I and O are fixed in all our considerations.

Theorem 1 (Compactly Generated Automata)

a) For each CG-automaton A there is an equivalent CG-automaton A' which is minimal in the sense that different states have different input-output functions and S' is a subspace of $C(I^+, O)$ up to a canonical homeomorphism. Moreover A' is a homomorphic image of A .

More precisely A' is called *minimal* if the machine function

$$M(A') : S' \rightarrow C(I^+, O)$$

defined by $M(A')(s') = \mu_{s'}$ is a subspace inclusion up to homeomorphism.

b) Each minimal CG-automaton A' in the sense of a) has the property that for all CG-automata A with $E(A) \subseteq E(A')$ there is exactly one automata morphism $f : A \rightarrow A'$, i.e. a continuous function $f : S \rightarrow S'$ satisfying $f \circ d = d' \circ (f\pi I)$ and $l = l' \circ (f\pi I)$. Moreover equivalent minimal CG-automata are isomorphic.

c) A subspace B of $C(I^+, O)$ is the behavior of a suitable CG-automaton A iff B is closed under left shift L , i.e. for each $f \in B$ and $x \in I$ we have $fL_x \in B$ defined by $fL_x(w) = f(xw)$.

REMARK : The minimal realization in the sense of [13, 9] of such a behavior B is a CG-automaton A with states B , next-state and output function defined by $(f, x) \mapsto fL_x, f(x)$ respectively.

Now we consider the case of arbitrary topological spaces I, O and S and take the biproduct. Since this case is already studied in [10] we at once take the corresponding case of partial topological automata.

Definition 2 : A partial topological automaton A consists of topological spaces I , O and S and continuous partial functions $d : S \otimes I \rightarrow S$ and

$$l : S \otimes I \rightarrow O.$$

By $S \otimes I$ we mean the biproduct of S and I , i.e. the cartesian product with bitopology which has been defined in the introduction. A partial mapping is called *continuous* if the inverse image of open sets is open, especially its domain is supposed to be open.

Now we take I^+ to be the topological sum of all n -fold biproducts I^n of I for $n \geq 1$ leading to partial continuous input-output functions $\mu_s : I^+ \rightarrow O$ for each $s \in S$.

If we furnish the space $C_p(I^+, O)$ of all non-empty partial continuous mappings $f : I^+ \rightarrow O$ with the topology of pointwise convergence the behavior $E(A)$ of a partial automaton is the subspace of $C_p(I^+, O)$ defined by all non-empty input-output functions μ_s of the states of A .

Theorem 2 (Partial Topological Automata)

Partial topological automata exactly have the properties stated in theorem 1 for CG -automata if we replace CG -automata by partial topological automata, the space $C(I^+, O)$ by $C_p(I^+, O)$ and continuous by partial continuous functions. In particular B is closed under left shift in the partial case if for each $f \in B$ and $x \in I$ we have $f \circ L_x \in B$, provided that $f(xw)$ is defined for one string w in I^+ at least.

REMARK : The similarity of theorems 1 and 2 is due to the fact that both are specializations of a general theorem for automata in closed categories which will be given in § 2.

Now we come to the case of relational topological automata.

Definition 3 : A relational topological automaton A consists of topological spaces I , O and S and lower semi-continuous relations $d : S \otimes I \rightarrow S$ and $l : S \otimes I \rightarrow O$. The behavior $E(A)$ of A is the subspace of $C_l(I^+, O)$ defined by all input-output relations $\mu_s : I^+ \rightarrow O$ of the states of A . By $C_l(I^+, O)$ we mean the space of all lower semi-continuous relations which is homeomorphic to the space $C(I^+, p_1O)$ of all continuous mappings $f : I^+ \rightarrow p_1O$ to the powerset p_1O of O with the topology of pointwise convergence. The powerset topology of p_1X (non-empty subsets of X) has as subbasis-elements the sets of all subsets V of X such that $V \cap U$ is non-empty, where U ranges over all open sets of X . Then p_1X has the minimal topology such that the inclusion $p_1'X \subseteq p_1X$ is an open mapping (cf. [3]), called *lower finite topology*.

Theorem 3 (Relational Topological Automata)

a) For each relational topological automaton A there is an equivalent state-minimal automaton A' provided that A satisfies the following state-equivalence condition which implies that the natural mapping $\text{nat} : S \rightarrow S/\sim$, where \sim is the equivalence of states, is open :

For each open subset U of S and each $s' \in S$ which is equivalent to some $s \in U$ there exists an open neighborhood V of s' such that each state in V is equivalent to a state in U .

Moreover the state object S' of A' can be chosen to be a quotient space of S , and for each state-minimal automaton A'' there is a continuous bijection $b'' : S'' \xrightarrow{\cong} E(A'')$ from the states to the behavior. Hence for equivalent state-minimal automata A'' and A' there is a bijection $f : S'' \xrightarrow{\cong} S'$ such that $b' \circ f = b''$ and $b'' \circ f^{-1} = b'$ are continuous.

State-minimality of A' means that different states have different input-output relations.

b) Given a relational topological automaton A there is a reduction $u : A \rightarrow A_r$ to an equivalent reduced automaton A_r , and A_r is unique up to isomorphism with respect to this property.

By a *reduction* $f : A \rightarrow A'$ between relational topological automata A and A' we mean a surjective continuous mapping $f : S \rightarrow S'$ satisfying $f \circ d = d' \circ (f \otimes I)$ and $l = l' \circ (f \otimes I)$ as equalities of relations.

A_r is called *reduced* if for arbitrary automata A' each reduction $f : A_r \rightarrow A'$ is already an isomorphism.

c) A subspace B of $C_l(I^+, O) \cong C(I^+, p_l O)$ is the behavior of a relational topological automaton A iff roughly spoken for each $f \in B$ and $x \in I$ the continuous mapping $f \circ L_x : I^+ \rightarrow p_l O$ can be regarded as a union of elements of B . More precisely B is a behavior provided that there is a continuous mapping $D : B \otimes I \rightarrow p_l B$ such that $f \circ L_x$ is the union of all elements in $D(f, x)$. Conversely, given A this D exists as a function and it is continuous if for each open subset U of S the set of all input-output relations μ_s with $s \in U$ is open in $E(A)$.

§ 2. CATEGORICAL BACKGROUND AND PROOFS

In this paragraph we give the basic notions of categorical automata theory leading to the formulation of two central theorems which have various applications on several types of automata (cf. [10]). We do not suppose that the reader is familiar with category theory, so we recall that

for example the class of all sets with functions between them define a category, and the same is true for all topological spaces with continuous functions and for the class of all linear spaces with linear functions between them. More precisely a *category* \mathbf{K} consists of a class of *objects* and a class of *morphisms* between the objects such that for each pair of morphisms $f : A \rightarrow B, g : B \rightarrow C$ there is defined a *composition* $g \circ f : A \rightarrow C$. It is assumed that this composition is associative and that for each object $K \in \mathbf{K}$ there is an *identity* morphism $id_K : K \rightarrow K$ which is right- and left-neutral under composition with an arbitrary morphism. A *functor* between two categories is a mapping defined on objects and morphisms which is compatible with the category structure.

Definition 4 (Automata in Monoidal Categories) :

A category \mathbf{K} together with a functor $\otimes : \mathbf{K} \times \mathbf{K} \rightarrow \mathbf{K}$ – called *tensor product* – and a *unitobject* $U \in \mathbf{K}$ is called *monoidal* if \otimes is associative up to isomorphism and there are natural isomorphisms $U \otimes A \cong A \cong A \otimes U$ for all $A \in \mathbf{K}$ compatible with associativity (cf. [17] for more details).

For example the class of all compactly generated Hausdorff-spaces and continuous functions between them define a category \mathbf{CG} , and similarly the class of all topological spaces with partial continuous functions define a category \mathbf{PDTop} and with lower semi-continuous relations a category \mathbf{RelTop}_l . \mathbf{PDTop} and \mathbf{RelTop}_l are monoidal if the tensor product is the biproduct \otimes defined in the introduction, and \mathbf{CG} is monoidal with respect to the Kelleyfication π of the topological product. In all these cases the unitobject is the one-point space.

A (*Mealy-*) *automaton* $A = (I, O, S, d, l)$ in a monoidal category (\mathbf{K}, \otimes) consists of objects I, O, S , called *input-, output- and state-object* respectively, and morphisms $S \xleftarrow{d} S \otimes I \xrightarrow{l} O$ called *next-state* resp. *output morphism*.

Given two automata A and A' with the same I and O , a \mathbf{K} -morphism $f : S \rightarrow S'$ is called *automata morphism* if it satisfies $d' \circ (f \otimes I) = f \circ d$ and $l' \circ (f \otimes I) = l$. It is clear from the definition that the automata in theorems 1, 2 and 3 can be regarded as automata in the monoidal categories (\mathbf{CG}, π) , $(\mathbf{PDTop}, \otimes)$ and $(\mathbf{RelTop}_l, \otimes)$ respectively.

The well-known extension of d and l to input strings for deterministic automata can be extended to automata in monoidal categories (\mathbf{K}, \otimes) provided that \mathbf{K} has countable coproducts which are distributive with \otimes up to isomorphism. These conditions are satisfied in (\mathbf{CG}, π) , $(\mathbf{PDTop}, \otimes)$ and $(\mathbf{RelTop}_l, \otimes)$. Hence we can construct a free semigroup $I^+ := \coprod_{n=1}^{\infty} I^n$ in (\mathbf{K}, \otimes) corresponding to the free topological semigroup in our examples and the family $[l_n : S \otimes I^n \rightarrow O]_{n \in \mathbf{N}}$, defined by $l_1 = l$ and $l_{n+1} = l_n \circ (d \otimes I^n)$, yields the *extended last-output function* $l^+ : S \otimes I^+ \rightarrow O$.

Definition 5 (Machine Function and Behavior of Automata):

In order to get the machine function of an automaton, assigning to each state $s \in S$ in our examples the corresponding input-output function

$$l^+(s, -) : I^+ \rightarrow O,$$

we have to make additional assumptions on the category (\mathbf{K}, \otimes) . We assume that there exists a monoidal closed subcategory $(\mathbf{K}', \otimes) \subseteq (\mathbf{K}, \otimes)$ with the same class of objects.

(\mathbf{K}', \otimes) is called *closed* if for each pair (I, O) of objects in \mathbf{K}' there is an object $\langle I, O \rangle$, called *internal hom object*, in \mathbf{K}' — which stands for the set of all morphisms $f : I \rightarrow O$ together with a suitable structure regarded as a \mathbf{K}' -object — such that for each $S \in \mathbf{K}'$ and each morphism $f : S \otimes I \rightarrow O$ there is one and only one morphism $f^* : S \rightarrow \langle I, O \rangle$ and vice versa. The morphisms f and f^* are called *adjoint* in this case, and this bijection is assumed to be a natural isomorphism, i.e. compatible with the category structure.

Now (\mathbf{K}, \otimes) is called *pseudoclosed* relative (\mathbf{K}', \otimes) iff for each object $O \in \mathbf{K}$ there exists a $\mathfrak{p}O \in \mathbf{K}'$ such that for each \mathbf{K} -morphism $g : S \rightarrow O$ there is one and only one \mathbf{K}' -morphism $g^* : S \rightarrow \mathfrak{p}O$ and vice versa. As above g and g^* are called *adjoint*. For example the categories (\mathbf{CG}, π) and $(\mathbf{PDTop}, \otimes)$ are closed and $(\mathbf{RelTop}_l, \otimes)$ is pseudoclosed relative (\mathbf{Top}, \otimes) . This will be proved later.

The following definitions we only give for the pseudoclosed case because a closed category is pseudoclosed relative itself if we define $\mathfrak{p}O := O$.

Now given an automaton A the \mathbf{K} -morphism $l^+ : S \otimes I^+ \rightarrow O$ is adjoint to the \mathbf{K}' -morphism $(l^+)^* : S \otimes I^+ \rightarrow \mathfrak{p}O$ and this is adjoint to a \mathbf{K}' -morphism $M(A) : S \rightarrow \langle I^+, \mathfrak{p}O \rangle$ called *machine morphism* of A .

For deterministic automata the behavior $E(A)$ of A is defined to be the image of $M(A)$, i.e.

$E(A) = \{ M(A)(s) = l^+(s, -) : I^+ \rightarrow O / s \in S \}$. Thus we have to assume that \mathbf{K}' has an \mathfrak{E} - \mathfrak{M} -factorization, i.e. there is a class \mathfrak{E} of epimorphisms and a class \mathfrak{M} of monomorphisms — which are surjective resp. injective mappings in our examples — such that each \mathbf{K}' -morphism f has a representation $f = m \circ e$ with $e \in \mathfrak{E}$, $m \in \mathfrak{M}$, which is unique up to isomorphism.

Now the *behavior* $E(A)$ of an automaton A is defined to be the \mathfrak{M} -part of $M(A)$, or more precisely a canonical representative of the related isomorphism class in \mathfrak{M} . Thus the set of all behaviors of automata in (\mathbf{K}, \otimes) with fixed I and O is partially ordered.

REMARK : In most topological categories there are at least two, in general different, \mathfrak{E} - \mathfrak{M} -factorizations and therefore different behaviors which come from different topologies on the image-space (cf. [15]). To make the

behavior independent of the topology on the state-space we choose the epixtremmono-factorization, i.e. the image-space is furnished with subspace topology relative to the codomain.

Now we review two central theorems concerning minimization, reduction and behavior characterization of automata in monoidal categories corresponding to theorems 4.8 and 5.5 resp. 6.7, 7.4 and 7.7 in [10].

Assumptions : Let (\mathbf{K}, \otimes) be pseudoclosed relative (\mathbf{K}', \otimes) with countable coproducts distributive with \otimes and let \mathbf{K}' have an $\mathfrak{E}\text{-}\mathfrak{M}$ -factorization such that $\mathfrak{E} \otimes K, K \otimes \mathfrak{E} \subseteq \mathfrak{E}$ for each $K \in \mathbf{K}'$.

Theorem A :

a) For each automaton A' in (\mathbf{K}', \otimes) there exists an observable automaton A and a morphism $f : A' \rightarrow A$ with $f \in \mathfrak{E}$. A is called *observable*, if the machine morphism of A is in \mathfrak{M} (intuitively spoken : two different states in A are inequivalent).

b) Given an observable automaton A in (\mathbf{K}', \otimes) and an automaton A' with $E(A) \subseteq E(A')$ there exists a unique automata morphism $f : A' \rightarrow A$ (intuitively f assigns to each state in A' the only equivalent state in A). Moreover two observable automata with the same behavior are isomorphic.

c) There exists a left shift $L : \langle I^+, O \rangle \otimes I \rightarrow \langle I^+, O \rangle$ (which intuitively means $L(f, s)(w) = f \circ L_x(w) = f(xw)$) such that an \mathfrak{M} -morphism $m : B \rightarrow \langle I^+, O \rangle$ is the behavior of a certain automaton A iff B is closed under L , i.e. there is a morphism $L' : B \otimes I \rightarrow B$ such that

$$m \circ L' = L \circ (m \otimes I).$$

Theorem B :

a) For each automaton A in (\mathbf{K}, \otimes) with the property that the \mathfrak{E} -part $e : S \rightarrow B$ of its machine morphism is a retraction in \mathbf{K} , i.e. that there is a \mathbf{K} -morphism $c : B \rightarrow S$ with $e \circ c = id_B$, there exists an observable automaton A_c in (\mathbf{K}, \otimes) with the same behavior as A . Moreover two observable automata with the same behavior are state-isomorphic but in general not isomorphic.

b) For each automaton A in (\mathbf{K}, \otimes) there exists up to isomorphism a unique reduced automaton A_r and an automata morphism $f : A \rightarrow A_r$ with $f \in \mathfrak{E} \subseteq \mathbf{K}'$ provided that \mathbf{K}' has all colimits and is cowellpowered. A_r is called *reduced* if each automata morphism $f : A_r \rightarrow A'$ with $f \in \mathfrak{E} \subseteq \mathbf{K}'$ is in fact an isomorphism in \mathbf{K}' .

c) There exists a \mathbf{K}' -morphism $s : \mathfrak{p} \langle I^+, \mathfrak{p}O \rangle \rightarrow \langle I^+, \mathfrak{p}O \rangle$ — which means intuitively the union of relations — such that an \mathfrak{M} -mor-

phism $m : B \rightarrow \langle I^+, \text{p}O \rangle$ is the behavior of a certain automaton A in (\mathbf{K}, \otimes) if there exists a \mathbf{K}' -morphism $D : B \otimes I \rightarrow \text{p}B$ satisfying

$$L \circ (m \otimes I) = s \circ \text{p}m \circ D.$$

Vice versa for each automaton A in (\mathbf{K}, \otimes) there exists such a morphism D provided that the \mathfrak{E} -part of the machine morphism $M(A)$ is a retraction in \mathbf{K} .

Proof of Theorem 1 :

We will apply theorem A and so we have to verify the assumptions concerning the category (\mathbf{CG}, π) .

It is well-known that the category (\mathbf{CG}, π) of all compactly generated Hausdorff-spaces is closed (cf. [20]). In this category the internal hom object $\langle I, O \rangle$ is given by the Kelleyfication of $C(I, O)$ which means the space of all continuous functions from I to O with the compact-open topology. Furthermore there is a suitable \mathfrak{E} - \mathfrak{M} -factorization if we take for \mathfrak{E} the class of all surjective continuous functions and for \mathfrak{M} the class of all injective continuous functions with the property that the domain space carries the Kelleyfication of the initial topology (such functions are called « inclusive » in [20]).

Let us remark that the topological sum of compactly generated spaces is also compactly generated and therefore the coproducts in \mathbf{CG} are the same as in \mathbf{Top} .

Now we can apply theorem A to automata in (\mathbf{CG}, π) leading to theorem 1. ■

Proof of Theorem 2 :

First we prove that $(\mathbf{PDTop}, \otimes)$ is closed. The internal hom object $\langle I, O \rangle$ is given by $C_p(I, O)$, being the set of all non-empty continuous partial functions from I to O with the topology of pointwise convergence. For a given $f : A \otimes B \rightarrow C$ define $f^* : A \rightarrow \langle B, C \rangle$ by $f^*(a)(b) := f(a, b)$ provided that $f(a, b')$ is defined for at least one $b' \in B$, otherwise f^* is undefined. Conversely a given non-empty f^* yields an f by $f(a, b) := f^*(a)(b)$.

Now for a given f , $f^*(a)$ is continuous for each $a \in A$ because for U open in C $[f^*(a)]^{-1}(U) = \{ b \in B / (a, b) \in \text{def} \wedge f(a, b) \in U \} =]\hat{f}^{-1}(U)]_a$ is open in B . f^* is continuous, because for an open subbasis element $\langle b, V \rangle$ of the space $\langle B, C \rangle$,

$$f^{*-1}(\langle b, V \rangle) = \{ a \in A / a \in \text{def}_{f^*} \wedge f(a, b) \in V \} = [f^{-1}(V)]_b$$

is open in A . Now let f^* be continuous and V open in C . Then for each $a \in A$, $b \in B$ $[f^{-1}(V)] = f^{*-1}(\langle b, V \rangle)$ and $[f^{-1}(V)]_a = [f^*(a)]^{-1}(V)$ both are open in A and B respectively and therefore $f^{-1}(V)$ is open in $A \otimes B$.

Furthermore we have an \mathfrak{E} - \mathfrak{M} -factorization for a given f if we take as the \mathfrak{E} -part of f the surjective partial function f onto the image of f and as the \mathfrak{M} -part the totally defined inclusion from the image, furnished with the subspace topology, into the codomain. The coproducts in \mathbf{PDTop} are the topological sums.

Now we can apply theorem A to automata in $(\mathbf{PDTop}, \otimes)$ which leads to theorem 2. ■

Proof of Theorem 3 :

It is easy to verify that the biproduct of two lower semicontinuous relations is lower semicontinuous, too.

This implies that $(\mathbf{RelTop}_l, \otimes)$ is monoidal. In the proof of theorem 2 it is already shown that (\mathbf{Top}, \otimes) is closed with respect to $\langle I, O \rangle = C(I, O)$ carrying the topology of pointwise convergence. This is well-known, cf. [5], Remark 1.15. Furthermore in [3] it is shown that for a given topological space O the powerset $p_l O$ with the lower finite topology (cf. § 1) has the property that a relation $f : S \rightarrow O$ is lower semi-continuous iff the induced $f^* : S \rightarrow p_l O$ is a continuous function. This completes the proof that $(\mathbf{RelTop}_l, \otimes)$ is pseudoclosed relative (\mathbf{Top}, \otimes) .

The \mathfrak{E} - \mathfrak{M} -factorization we choose is the same as in \mathbf{PDTop} and the condition on the coproducts is satisfied in the same way.

Proof of a) : Given an automaton A in $(\mathbf{RelTop}_l, \otimes)$ satisfying the state-equivalence condition let $S \xrightarrow{e} S/\sim \xrightarrow{m} \langle I^+, p_l O \rangle$ be the extremepi-mono-factorization of the machine morphism $M(A)$. This means that S/\sim carries final topology relative $e = nat$ and \sim denotes the equivalence of states. Define $c : S/\sim \rightarrow S$ by $c([s]) = \{s'/s' \sim s\}$ and for a given U open in S , $c^{-1}(U)$ is open because $e^{-1}c^{-1}(U) = e^{-1}e(U)$ is open by the state-equivalence condition. Therefore c is a lower semi-continuous relation satisfying $e \circ c([s]) = [s]$ which implies that e is a retraction in \mathbf{RelTop}_l . Now by theorem B c) there exists a state-minimal automaton A_c taking in this construction an extremepi-mono-factorization in \mathbf{K}' . Moreover A and A_c have the same behavior and for each automaton A' with injective machine function there exists a continuous bijection from the state space onto its behavior which is not a homeomorphism in general. Such a continuous bijection for example is the \mathfrak{E} -part of $M(A')$.

b) Follows directly from theorem B b) by the fact that \mathbf{Top} is cocomplete and cowellpowered (cf. [15]).

Proof of c) : Given an \mathfrak{M} -morphism $m : B \rightarrow \langle I^+, p_l O \rangle$ and a \mathbf{K}' -morphism, i.e. a continuous function, $D : B \otimes I \rightarrow p_l B$ satisfying

$$L \circ (m \otimes I) = s \circ p_l m \circ D$$

– this means exactly that $L(f, x) = f \circ L_S$ is the union of some elements of B – by theorem B c) m is the behavior of a topological automaton. Conversely a given topological automaton A can be regarded as a relational automaton without topology and then the \mathfrak{E} -part of its machine function is a surjective mapping and hence a retraction in the category of relations. Now by theorem B c) D exists as a function, and it is continuous if the \mathfrak{E} -part of $M(A)$ is an open mapping and hence a retraction in \mathbf{RelTop}_1 . Of course, this property is much stronger than the state-equivalence condition in a). ■

§ 3. FURTHER APPLICATIONS AND RESULTS

In § 2 we have shown that the theorems of § 1 are specializations of theorems A and B known for automata in closed resp. pseudoclosed monoidal categories. Of course it is possible to apply these categorical theorems to several other kinds of automata, too, e.g. the corresponding classical types of deterministic, partial and relational automata and moreover to nondeterministic, stochastic, linear and bilinear automata. All these examples are discussed in [10].

In fact there are even more interesting topological examples and some of them will be studied in this section. In particular we get other kinds of relational topological automata regarding upper and lower semi-continuous relations and corresponding automata with compactly generated spaces I, O, S . The discussion of metric and tolerance automata given in [1], [13] and [21] will be extended to the relational case, and we will sketch how the case of automata based on uniform spaces, which is considered in [21], can be embedded into the general theory. Some types of linear topological automata are mentioned in [10].

Let us recall that a relation $f : X \rightarrow Y$ between two topological spaces is called *upper semi-continuous* iff, for each A closed in Y , $f^{-1}(A)$ is closed in X . Clearly upper semi-continuous relations are closed under composition and the biproduct, such that topological spaces and upper semi-continuous relations define a monoidal category $(\mathbf{RelTop}_u, \otimes)$. We know that a relation $f : X \rightarrow Y$ is upper semi-continuous iff def_f is closed in X and for each $x \in X$ and each V open in Y which contains $f(x)$ there exists an open neighborhood U of x such that $f(U) \subseteq V$ (cf. [14], III 21.2 or [3]).

Theorem 4 : Theorem 3 remains true for automata with upper semi-continuous next-state and output functions with respect to the following slight modifications :

1) The state-equivalence condition is modified : For each subset C closed in S and each $s \in S$ not equivalent to any state in C there is an open neighborhood U of s such that no state in U is equivalent to any state in C .

2) The last sentence in c) must be replaced by : Conversely for a given A this D exists as a function and is continuous if for each closed subset C of S the set of all input-output relations μ_s with $s \in C$ is closed in B .

Proof : ($\mathbf{RelTop}_u, \otimes$) is pseudoclosed relative (\mathbf{Top}, \otimes) if the powerset of a given space O is furnished with the upper finite topology, denoted by $p_u O$. An element of a subbasis of $p'_u O$ ($p_u O$ without the empty set) is the powerset of an open set U in O without the empty set, and we obtain a subbasis of $p'_u O$ if U ranges over all open sets in O . Finally $p_u O$ is furnished with the final topology such that the inclusion $p'_u O \subseteq p_u O$ is a closed mapping (cf. [3]). The conditions on the \mathfrak{C} - \mathfrak{M} -factorization and the coproducts are the same as in theorem 3.

Now we apply theorem B in the same way as in theorem 3. In particular the modifications concerning the state-equivalence condition guarantee that c becomes upper semi-continuous and hence e is a retraction in \mathbf{RelTop}_u . ■

Of course theorems 3 and 4 together lead to the analogous results for automata which next-state and output relations are lower and upper semi-continuous and hence H -continuous in the sense of [3].

The same is valid for automata with compactly generated Hausdorff-spaces I, O and S under the additional assumption of point-compactness.

A relation $f : X \rightarrow Y$ is called *point-compact* iff, for all $x \in X$, $f(x)$ is compact in Y .

Similar to [6] we define a relation to be *continuous* iff it is both lower and upper semi-continuous and moreover point-compact.

Theorem 5 : Theorem 3 remains true for automata with compactly generated I, O, S and continuous relations $d : S\pi I \rightarrow O$ with the slight modification, that both additional properties formulated in theorem 3 and 4 together with the assumption, that for each state $s \in S$ the set of all states s' equivalent to s is compact, give the new additional properties in theorem 5.

In order to give the proof of the theorem, we have to verify four lemmata!

Lemma 1 : A point-compact upper semi-continuous relation (and hence a continuous one) maps compact subsets onto compact subsets, i.e. $f(K)$ is compact for each K compact in the domain. Hence continuous relations are closed under composition.

Proof : Let $f : X \rightarrow Y$ be point-compact and upper semi-continuous and K compact in X together with an open covering of $f(K)$. Now $f(x)$ is compact for each $x \in K$ and this yields a finite subcovering of each $f(x)$. Because f is upper semi-continuous there exists for each $x \in K$ an open neighborhood U of x such that the finite subcovering of $f(x)$ covers $f(U)$, too.

By compactness of K there exists a finite covering $U(x_1), \dots, U(x_n)$ of K and the collection of the corresponding subcoverings of $f(x_1), \dots, f(x_n)$ is a finite subcovering of $f(K)$. ■

Lemma 2 : Let be X a CG-space and $f : X \rightarrow Y$ a relation. Then f is continuous iff it is continuous considered as a relation from X to kY (cf. Def. 1).

Proof : Since the identity from kY to Y is a continuous function it suffices to show that f being continuous implies $f^* : X \rightarrow kY$ being continuous. Now, since Y and kY have the same compact subsets, f^* is point-compact.

Furthermore f^* is upper semi-continuous iff the induced function $f^{**} : X \rightarrow p_u kY$ is continuous, and because X is compactly generated f^{**} is continuous iff it is continuous on each compact subset of S . So we firstly prove that f^* is upper semi-continuous on an arbitrary subset K compact in X . Let $x \in K$ and V open in kY containing $f^*(x)$. By Lemma 1 $f(K)$ is compact in Y and hence $V \cap f(K)$ is open in $f(K)$ and this implies that there exists V' open in Y such that $V \cap f(K) = V' \cap f(K)$. Since f is upper semi-continuous there exists an open neighborhood U' of x such that $f(U') \subseteq V'$. It follows that $U' \cap K$ is a neighborhood of x open in K satisfying

$$\begin{aligned} f^*(U' \cap K) &\subseteq f^*(U') \cap f^*(K) = f(U') \cap f(K) \\ &\subseteq V' \cap f(K) = V \cap f(K) \subseteq V \end{aligned}$$

which yields that f^* is upper semi-continuous.

To prove that f^* is lower semi-continuous we use an equivalent characterization of lower semi-continuity (cf. [3]) :

Let be $x \in K$ and V open in kY with $f(x) \cap V \neq \emptyset$. As above there exists V' open in Y such that $V \cap f(K) = V' \cap f(K)$ and hence $V' \cap f(x) \neq \emptyset$. Since f is lower semi-continuous there exists an open neighborhood U' of x such that for each $y \in U'$ $f(y) \cap V' \neq \emptyset$ and hence for each $y \in U' \cap K$

$$\emptyset \neq f(y) \cap V' = f(y) \cap V' \cap f(K) = f(y) \cap V \cap f(K) \subseteq f(y) \cap V.$$

This completes the proof. ■

Lemma 3 : The topological product of two continuous relations $f : X \rightarrow Y$, $g : X' \rightarrow Y'$ is continuous, too. Then by lemma 2 the same is valid for the Kelleyfication π of the topological product.

Proof : The point-compactness of $f \times g$ follows from the Tychonoff theorem (cf. [7]) and it is easy to prove that $f \times g$ is lower semi-continuous. We only prove that $f \times g$ is upper semi-continuous.

Let $f(x) \times g(y)$ be contained in U open in $Y \times Y'$. Each point

$$(a, b) \in f(x) \times g(y)$$

has an open neighborhood of the form $U_a \times V_b$ which is contained in U , and because of the compactness we find a finite covering $U_1 \times V_1, \dots, U_n \times V_n$ of $f(x) \times g(y)$. Now for each $p \in f(x)$ and $q \in g(y)$ the sets

$$U'_p := \bigcap_{\substack{i=1 \\ p \in U_i}}^n U_i \quad \text{and} \quad V'_q := \bigcap_{\substack{i=1 \\ q \in V_i}}^n V_i$$

are open neighborhoods of p and q respectively. It follows that

$$U'' := \bigcup_{p \in f(x)} U'_p \quad \text{and} \quad V'' := \bigcup_{q \in g(y)} V'_q$$

both are open and satisfy $f(x) \times g(y) \subseteq U'' \times V'' \subseteq U$.

As f and g are upper semi-continuous there are open neighborhoods $\tilde{U}(x)$ and $\tilde{V}(y)$ with $f(\tilde{U}) \subseteq U''$, $g(\tilde{V}) \subseteq V''$ and hence

$$(f \times g)(\tilde{U} \times \tilde{V}) = f(\tilde{U}) \times g(\tilde{V}) \subseteq U'' \times V'' \subseteq U. \blacksquare$$

Lemma 4 : For a given Hausdorff space Y the space of all compact subsets of Y furnished with the relative finite topology, which is defined below, is a Hausdorff space, too.

Proof : A subsbasis of the relative finite topology is given by

$$\{ \{ A / \emptyset \neq A \subseteq U \} / U \text{ open in } Y \} \cup \{ \{ A / A \cap U \neq \emptyset \} / U \text{ open in } Y \} \cup \{ \{ \emptyset \} \}$$

where A stands for compact subsets of Y (cf. [3,6]).

Now let $B \neq C$ be two compact subsets of Y .

If $B = \emptyset$ then $\{ \emptyset \}$ resp. $\{ A / A \cap Y \neq \emptyset \}$ are two disjoint open neighborhoods of B resp. C .

If $B \neq \emptyset$, let $x \in B \setminus C$ (otherwise $x \in C \setminus B$).

Since $B \cup C$ is compact and hence normal, for each $x \in C$ there exist neighborhoods U of x and V of C open in $B \cup C$ such that $\bar{U} \cap \bar{V} = \emptyset$. Let U' and V' be open in Y such that

$$U = U' \cap (B \cup C) \quad \text{and} \quad V = V' \cap (B \cup C).$$

Now $\{ A / \emptyset \neq A \subseteq V' \}$ resp. $\{ A / A \cap (U' \setminus \bar{V}') \neq \emptyset \}$ are disjoint neighborhoods of C resp. B which are open in the relative finite topology. \blacksquare

Proof of Theorem 5 : By lemma 1-3 CG -spaces and continuous relations constitute a monoidal category, written $(\mathbf{Rel}CG_c, \pi)$. By lemma 4 for a given CG -space Y the space $p_c Y$ of all compact subsets of Y with the relative finite topology is a Hausdorff space and hence its Kelleyfication is a CG -space.

Furthermore a relation $f : X \rightarrow Y$ is continuous iff the induced function $f^* : X \rightarrow \mathfrak{p}_c Y$ is continuous, and this is true iff f^* is continuous considered as a function from X to $k(\mathfrak{p}_c Y)$. Hence (\mathbf{RelCG}_c, π) is pseudoclosed relative (\mathbf{CG}, π) .

Now we can apply theorem *B* in the same way as in theorem 3 and 4. The additional compactness assumption implies that c in the proof of theorem 3 becomes point-compact and hence e is a retraction in \mathbf{RelCG}_c . Note that in *b*) we make use of the fact that \mathbf{CG} has all colimits and is cowellpowered (cf. [15]). ■

All constructions concerning reduction and minimization of automata were starting from automata with next-state and output functions or relations which are continuous on the biproduct resp. the Kelleyfication of the topological product. Of course, there is the following question : If we start from an automaton with d and l continuous on the topological product, does the reduced and minimal automaton have the same property?

In fact this is true for the minimal automaton in the case of theorems 1 and 2 and the reduced automaton in the case of theorem 3, 5 provided that I is locally compact (cf. [3, 4, 19]). Moreover if I is locally compact we can omit all restrictions on O and S , and nevertheless we have the same results as in the case of compactly generated spaces.

Theorem 6 :

(i) Theorem 1 remains true for automata with arbitrary topological spaces O and S and continuous functions $d : S \times I \rightarrow S$, $l : S \times I \rightarrow O$, where \times denotes the topological product, provided that I is locally compact.

(ii) Theorem 5 remains true for automata with arbitrary topological spaces O and S and continuous relations $d : S \times I \rightarrow S$, $l : S \times I \rightarrow O$ provided that I is locally compact.

Outline of a proof :

The fact that I is locally compact implies that I^+ is locally compact, and the proof – analogue to theorems 1 and 5 – is based on the fact that for I locally compact the continuous functions of the form $f : S \times I \rightarrow O$ are in natural bijection to the continuous functions of the form $f^* : S \rightarrow C(I, O)$, where $C(I, O)$ is furnished with the compact-open topology (cf. [7]). ■

Similar to topological spaces with topological product the category \mathbf{Unif} of uniform spaces with uniformly continuous mappings and the uniform product \times is not closed but theorem 1 remains true for automata in (\mathbf{Unif}, \times) if the set $\langle I^+, O \rangle$ of all uniformly continuous mappings from I^+ to O is equicontinuous, because the proof of theorem 6 (i) can be extended to this

case using § 2.1 of [2]. The above condition is satisfied especially for a discrete input space I which is considered in [21].

The above condition is satisfied especially for a discrete input space I which is considered in [21].

On the other hand there is also a biproduct \otimes in **Unif** such that (\mathbf{Unif}, \otimes) becomes a closed category. Similar to the case of topological spaces, $S \otimes I$ carries final uniform structure on $S \times I$ such that the identity $S \times I \rightarrow S \otimes I$ is uniformly bicontinuous.

More general such a construction of a biproduct \otimes is possible in categories which are initial and cointial over the category of sets in the sense of [22] including topological, uniform, measurable and limit spaces. In all these cases (\mathbf{K}, \otimes) is a closed category with $\mathfrak{C}\text{-}\mathfrak{M}$ -factorization such that theorem *A* can be applied to the corresponding automata in (\mathbf{K}, \otimes) .

Finally let us consider the case of metric and tolerance automata which are studied in [21] and have been mentioned in [1] resp. [13] as examples of automata in closed categories. In fact the corresponding category (\mathbf{Metr}, \otimes) of metric spaces with distance $d \leq 1$ and distance decreasing mappings f , i.e. $d'(f(x), f(y)) \leq d(x, y)$, is closed if the tensor product $(A, d) \otimes (A', d')$ is defined to be $(A \times A', \min(d + d', 1))$ and the internal hom object $\langle (A, d), (A', d') \rangle$ is the set of all decreasing mappings $f : A \rightarrow A'$ furnished with the metric $d_{\langle \rangle}$ defined by $d_{\langle \rangle}(f, g) = \sup \{ d(f(x), g(x)) / x \in A \}$. Moreover the category of metric spaces with $d \leq 1$ and point-compact decreasing relations is pseudoclosed relative (\mathbf{Metr}, \otimes) if the distance for non-empty compact subsets A and B is defined by

$$d(A, B) := \min \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b) + \sup_{b \in B} \inf_{a \in A} d(a, b), 1 \right\}$$

$$\text{and } d(\emptyset, C) := \begin{cases} 0 & C = \emptyset \\ 1 & C \neq \emptyset. \end{cases}$$

We don't want to give a proof for this property here, but let us remark that point-compact decreasing relations are continuous relations in the sense defined above. Theorem *A* and at least part *a*) and *c*) of theorem *B* are applicable to metric and relational metric automata respectively.

Taking pairs (A, r) of a set A and a reflexive symmetric relation r on A as objects and tolerance mappings $f : A \rightarrow A'$ as morphisms, i.e. xry implies $f(x)r'f(y)$, we obtain the category **Tol** of tolerance spaces. (\mathbf{Tol}, \otimes) is again closed if $(A, r) \otimes (A', r')$ is $(A \times A', r \times r')$ and $\langle (A, r), (A', r') \rangle$ is the set of all tolerance mappings $f : A \rightarrow A'$ with $f r_{\langle \rangle} g$ iff for all $x, y \in A$ xry implies $f(x)r'g(y)$. As before we can also consider tolerance relations, defining a relation between subsets U and V of (A, r) by UrV iff for each $x \in U$ there exists $y \in V$ with xry and vice versa. In fact the category of tolerance relations is again pseudoclosed relative (\mathbf{Tol}, \otimes) which can be shown straight forward.

Moreover **Tol** is cocomplete and cowellpowered such that theorem *B* can be applied to tolerance and relational tolerance automata respectively.

In all our considerations we only have applied the theorems *A* and *B* in § 2 to topological examples. In fact there are several other results known for automata in closed and pseudoclosed categories (cf. [8, 9, 10, 12, 16]) which can be applied to our examples. But we think the only difficulty was to verify the corresponding categorical properties for all the examples. Thus it should be left to the reader to apply the other results concerning some more special problems and the case of power, kernel and initial automata, too.

REFERENCES

- [1] ARBIB M. A. and MANES E. G., *Adjoint Machines, State-behavior and Duality*, Technical Report 73 B-1 COINS, University of Massachusetts at Amherst (1973), (to appear in *J. Pure Appl. Alg.*).
- [2] BOURBAKI N., *Topologie Générale*, Ch. 10 : Espaces Fonctionnels, Hermann, Paris, 1961.
- [3] BRAUER W., *Zu den Grundlagen einer Theorie topologischer sequentieller Systeme und Automaten*, Berichte GMD Bonn Nr. 31, 1970.
- [4] BRAUER W., *Automates topologiques et ensembles reconnaissables*, Séminaire Schützenberger-Lentin-Nivat, année 1969/1970, n° 18.
- [5] BROWN R., *Function Spaces and Product Topologies*, *Quart. J. Math.*, Oxford, Ser. 2, 15, 2964, 238-250.
- [6] DAY J. M. and FRANKLIN S. P., *Spaces of Continuous Relations*, *Math. Ann.*, 169, 1967, 289-293.
- [7] DUGUNDJI J., *Topology*, Allyn and Bacon, Boston, 1966.
- [8] EHRIG H. and KREOWSKI H.-J., *Power and Initial Automata in Pseudoclosed Categories*, *Proc. Intern. Symp. : Category Theory Applied to Computation and Control*, San Francisco, 1974, 162-169 (to appear in *Lecture Notes in Computer Science*).
- [9] EHRIG H. and KREOWSKI H.-J., *Systematic Approach of Reduction and Minimization in Automata and System Theory*, TU Berlin, 1973, FB 20 Bericht 73-16 (to appear in *J. Comp. Syst. Sc.*).
- [10] EHRIG H., KIERMEIER K. D., KREOWSKI H.-J. and KÜHNEL W., *Universal Theory of Automata*, Teubner, Stuttgart, 1974.
- [11] EHRIG H. and PFENDER M. u. a., *Kategorien und Automaten*, de Gruyter, Berlin-New York, 1972.
- [12] GOGUEN J. A., *Discrete-Time-Machines in Closed Monoidal Categories*, I. *Quart. Report no. 30*, *Inst. f. Comp. Res.*, Univ. of Chicago (1971), condensed version in : *Bull. American Math. Soc.* 78, 1972, 777-783.
- [13] GOGUEN J. A., *Systems and Minimal Realization*, *Proc. of the 1971 IEEE Conf. on Decision and Control*, Miami Beach, 1971, 42-46.
- [14] HAHN H., *Reelle Funktionen*, I., Akademische Verlagsges. Leipzig, 1932.
- [15] HERRLICH H., *Topologische Reflexionen und Coreflexionen*, Berlin, Heidelberg, New York, 1968 (*Lecture Notes in Mathematics* 78).

- [16] KREOWSKI H.-J., *Automaten in pseudoabgeschlossenen Kategorien*, Diplomarbeit TU Berlin, 1974.
- [17] MAC LANE S., *Categories for the Working Mathematician*, Springer, Berlin, Heidelberg, New York, 1972.
- [18] MICHAEL E., *Topologies on Spaces of Subsets*, Trans. American Math. Soc., 81, 1951, 152-182.
- [19] POHL H.-J., *Ein Ansatz zur Theorie topologischer Automaten*, EIK, 1974.
- [20] STEENROD N. E., *A Convenient Category of Topological Spaces*, Michigan Math. J., 14, 1967, 133-152.
- [21] VALK R., *The Use of Metric and Uniform Spaces for the Formalization of Behavioral Proximity of States*, in : 1. GI-Fachtagung über Automatentheorie und Formale Sprachen, 1973, 116-122, (Lecture Notes in Computer Science 2).
- [22] WISCHNEWSKI M., *Initialkategorien*, Diss. München, 1972.