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## EACH REGULAR CODE IS INCLUDED IN A MAXIMAL REGULAR CODE (\*)

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*Abstract.* — It is proved that each regular code is included in a maximal regular code. A corollary of this result settles an open question from [R].

*Résumé.* — On prouve que tout code rationnel est contenu dans un code rationnel maximal. Un corollaire de ce résultat répond à une question ouverte posée dans [R].

### INTRODUCTION

A language  $C \subseteq \Sigma^+$  is called a *code* if  $C^*$  is a free submonoid of  $\Sigma^*$  with base  $C$ . The theory of codes initiated by M. Schützenberger [Sch] forms an interesting fragment of formal language theory. A code  $C \subseteq \Sigma^+$  is called *maximal* if, for any  $x \in \Sigma^* - C$ ,  $C \cup \{x\}$  is not a code. All codes are subsets of maximal codes and the investigation of maximal codes forms an active research area within the theory of codes (see, e. g., [BPS], [P1], [R] and [SM]). In particular one is often interested in the problem of the following kind: given a code  $C$  of type  $X$  (e. g. finite or regular) is it possible to find a maximal code  $D$  of type  $X$  such that  $C \subseteq D$ ?

It was shown in [R] that for finite codes this question gets a negative answer. Since then the following question remained open: is every finite code included in a maximal regular code? Obviously any finite (resp. regular) prefix code is included in a finite (resp. regular) maximal prefix code. Recently it was shown in [P2] that every *finite biprefix* code is included in a maximal biprefix regular code.

In this paper we provide a positive answer to the above question. As a matter of fact we prove a more general result (theorem 5): each *regular* code is included in a regular maximal code. We would like to emphasize the

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following: *the new result presented in this paper is theorem 5*; most of the other results is in one form or the other (and perhaps in a different terminology) retrievable from the literature. However we have decided to make this paper rather self-contained and to provide all the needed results with their (sometimes different from the literature) proofs carried out in a “uniform manner”.

We assume the reader to be familiar with basic formal language theory—in particular with rudimentary theory of regular languages (*see, e. g., [S]*).

## PRELIMINARIES

We use mostly standard language theoretic notation and terminology.

For a set  $A$ ,  $\#A$  denotes the cardinality of  $A$ .

For sets  $A, B$ ,  $A - B$  denotes the set theoretic difference of  $A$  and  $B$ .

For a word  $x$ ,  $|x|$  denotes its length and  $first(x)$  denotes the first letter of  $x$ ; if  $x = x_1 y x_2$  then  $y$  is called a *subword* of  $x$  (also referred to as a *segment* or a *factor* of  $x$ ). The set of all subwords of  $x$  is denoted by  $sub(x)$  and for a language  $K$ ,  $sub(K) = \bigcup_{x \in K} sub(x)$ .

A nonempty word  $x$  is called *bordered* if  $x = yzy$  for a nonempty word  $y$ ; otherwise  $x$  is called *unbordered*.

A language  $C \subseteq \Sigma^+$  is called a *code* if every word  $y \in C^+$  satisfies the following condition:

if  $y = u_1 \dots u_n$  and  $y = x_1 \dots x_m$  for  $n, m \geq 1$  and  $u_1, \dots, u_n, x_1, \dots, x_m \in C$  then  $n = m$  and  $u_i = x_i$  for  $1 \leq i \leq n$ . (In other words,  $y$  has a unique representation in  $C$ ; subwords  $u_1, \dots, u_n$  of this representation are referred to as *C-blocks* of  $y$ ).

A code  $C \subseteq \Sigma^+$  is called *maximal* if, for each  $x \in \Sigma^* - C$ ,  $C \cup \{x\}$  is not a code.

*In the sequel of this paper we consider an arbitrary but fixed alphabet  $\Sigma$  where  $\sigma = \#\Sigma > 1$ ; all languages we will consider are over  $\Sigma$ .*

For a language  $K$  and a positive integer  $n$ ,  $L_n(K) = \{w \in K : |w| = n\}$  and  $\alpha_n(K) = \#L_n(K)$ .

We will define now and recall a number of notions concerning languages—they will be central to our paper.

Let  $K \subseteq \Sigma^+$ .

(1)  $K$  is *dense* if  $x \in sub(K^*)$  for each  $x \in \Sigma^*$ .

(2)  $K$  is *fast* if there exists a positive integer  $n$  such that for each  $w \in \text{sub}(K^*)$  there exist  $x, y \in \Sigma^*$  such that  $|xy| \leq n$  and  $xwy \in K^*$ .

(3)  $K$  is *rich* if there exists a positive integer  $e$  such that  $\alpha_m(K^*) \geq \sigma^m/e$  for infinitely many positive integers  $m$ .

## RESULTS

In this section we investigate the problem how various properties of a code (such as: fast, dense, rich, regular and maximal) influence each other. Once this relationship is explored we can settle the problem of completing a regular code to a regular maximal code.

Our first result is known (see [SM]). However for the sake of completeness we provide its proof (which is different from the proof in [SM]).

**THEOREM 1:** *Each maximal code is dense.*

*Proof:* First we prove the following result.

**CLAIM 1:** *Let  $C$  be a code that is not dense. There exists an unbordered word  $w_c$  such that  $w_c \notin \text{sub}(C^*)$ .*

*Proof of Claim 1:* Since  $C$  is not dense, there exists a word  $z \notin \text{sub}(C^*)$ . Let  $b \in \Sigma$  be such that  $b \neq \text{first}(z)$  and let  $w_c = zb^{|z|}$ . Clearly  $w_c$  is unbordered. Moreover  $w_c \notin \text{sub}(C^*)$ , because  $z \notin \text{sub}(C^*)$ .

Thus claim 1 holds. ■

Now we prove theorem 1 as follows.

Let  $C$  be a maximal code.

Assume to the contrary that  $C$  is not dense. Then let  $w_c$  be an unbordered word satisfying the statement of claim 1.

Consider  $D = C \cup \{w_c\}$ . Let  $y$  be an arbitrary word in  $D^+$ . Since  $w_c$  is unbordered,  $y$  has a unique representation of the form  $y = x_0 w_c x_1 w_c \dots w_c x_n$ , where  $n \geq 0$  (that is if  $y = u_0 w_c u_1 w_c \dots w_c u_m$  where  $m \geq 0$  then  $m = n$  and  $u_i = x_i$  for  $1 \leq i \leq n$ ). Since  $C$  is a code and  $w_c \notin \text{sub}(C^*)$ ,  $y$  has a unique representation in  $D$ . Thus  $D$  is a code.

Since  $C \subseteq D$  and  $w_c \notin \text{sub}(C^*)$  we get a contradiction (to the fact that  $C$  is maximal).

Consequently  $C$  must be dense and theorem 1 holds. ■

**THEOREM 2:** *Each rich code is maximal.*

*Proof:* Let  $C$  be a rich code and let  $e$  be a positive integer constant satisfying the definition of richness for  $C$ .

Assume to the contrary that  $C$  is not maximal. Let  $z$  be a word such that  $B = C \cup \{z\}$  is a code; let  $|z| = t$ .

Let  $k$  be a positive integer. Let  $n_1, \dots, n_k$  be a sequence of positive integers such that:

$$n_1 < n_2 < \dots < n_k \quad \text{and} \quad \alpha_{n_i}(C^*) \geq \frac{\sigma^{n_i}}{e}. \quad (1)$$

(Since  $C$  is rich and  $e$  satisfies the definition of richness of  $C$ , such a sequence exists.)

Consider  $r = n_1 + n_2 + \dots + n_k + kt$ . Clearly:

$$\alpha_r(B^*) \leq \sigma^r. \quad (2)$$

On the other hand let us consider an arbitrary permutation  $i_1, \dots, i_k$  of the set  $\{1, \dots, k\}$ . Let  $y_{i_1} \in L_{n_{i_1}}(C^*), \dots, y_{i_k} \in L_{n_{i_k}}(C^*)$  and let  $\gamma(i_1, \dots, i_k) = y_{i_1} z y_{i_2} z \dots y_{i_k} z$ . Since  $B$  is a code, if  $(j_1, \dots, j_k)$  is a permutation of  $\{1, \dots, k\}$  different from  $(i_1, \dots, i_k)$ , then  $\gamma(i_1, \dots, i_k) \neq \gamma(j_1, \dots, j_k)$ . Consequently from (1) it follows that:

$$\frac{\sigma^{n_1}}{e} \frac{\sigma^{n_2}}{e} \dots \frac{\sigma^{n_k}}{e} k! \leq \alpha_r(B^*). \quad (3)$$

From (2) and (3) it follows that:

$$k! \leq e^k \sigma^{tk} = (e \sigma^t)^k. \quad (4)$$

Since  $e \sigma^t$  is a constant (independent of  $k$ ), there exists a positive integer  $k_0$  such that, for all  $s > k_0$ ,  $s! > (e \sigma^t)^s$ . Consequently (4) yields a contradiction ( $k$  was chosen to be an arbitrary positive integer).

Thus  $C$  must be maximal and theorem 2 holds. ■

**THEOREM 3:** *Each regular code is fast.*

*Proof:* Obvious. ■

**THEOREM 4:** *Each dense and fast code is rich.*

*Proof:* Let  $C$  be a code that is dense and fast. Then there exists a finite set  $F$  of ordered pairs of words from  $\Sigma^*$  such that for each  $w \in \Sigma^*$  there exists  $(x, y) \in F$  such that  $xwy \in C^*$ . Let  $q = \max\{|xy| : (x, y) \in F\}$ ,  $f = \#F$  and  $d = f \sigma^q$ .

**CLAIM 2:** *For each positive integer  $n$  there exists a positive integer  $m \leq n + q$  such that  $\alpha_m(C^*) \geq \sigma^m/d$ .*

*Proof of claim 2:* Let for each  $w \in \Sigma^*$ ,  $pair(w)$  be a fixed element  $(x, y)$  of  $F$  such that  $xwy \in C^*$ .

Let  $n$  be a positive integer. Let:

$$E(n, x, y) = \{w \in L_n(\Sigma^*) : pair(w) = (x, y)\}.$$

Clearly for some  $(x_0, y_0) \in F$ ,  $\# E(n, x_0, y_0) \geq \sigma^n/f$ . Let  $p = |x_0 y_0|$ . Then  $\alpha_{n+p}(C^*) \geq \# E(n, x_0, y_0) \geq \sigma^n/f$ .

Hence:

$$\alpha_{n+p}(C^*) \geq \frac{\sigma^n}{f} = \frac{\sigma^{n+p}}{f\sigma^p} \geq \frac{\sigma^{n+p}}{f\sigma^q} \geq \frac{\sigma^{n+p}}{d}.$$

Thus if we choose  $m = n + p$  we get  $m \leq n + q$  and claim 2 holds. ■

Now theorem 4 follows directly from claim 2. ■

REMARK: Theorems 2 and 4 together are more general than theorem 7.4 (due to Schutzenberger) from [E]. However, it is pointed out by D. Perrin in [P3] that a proof of the general case can be retrieved from the proof of theorem 9.3 in [E]. ■

THEOREM 5: *Let  $C$  be a regular code. There exists a code  $D$  which is dense, fast, regular and such that  $C \subseteq D$ .*

*Proof:* Let  $C$  be a regular code.

We consider separately two cases.

(i)  $C$  is dense.

Then the theorem follows from theorem 3 (take  $D = C$ ).

(ii)  $C$  is not dense.

Then, by claim 1, there exists an unbordered word  $w_c$  such that  $w_c \notin \text{sub}(C^*)$ .

Let:

$$A = \{w_c x_1 w_c x_2 \dots w_c x_n w_c : n \geq 1, x_i \notin C^* \text{ and } w_c \notin \text{sub}(x_i)\}$$

and let  $D = C \cup \{w_c\} \cup A$ .

CLAIM 3:  $D$  is a code.

*Proof of Claim 3:* Let  $y \in D^+$ . Since  $w_c$  is unbordered,  $y$  has a unique representation of the form  $y = x_1 w_c x_2 w_c \dots w_c x_n$  (that is we can uniquely distinguish all occurrences of  $w_c$  in  $y$ ).

This representation provides the basis for the division of  $y$  into  $D$ -blocks which is obtained as follows:

(1) A subword  $w_c x_j w_c x_{j+1} \dots w_c x_{j+l} w_c$  constitutes a  $D$ -block (corresponding to  $A$ ) if  $2 \leq j \leq n-1$ ,  $j+l \leq n-1$ ,  $x_j, \dots, x_{j+l} \notin C^*$  and  $x_{j-1}, x_{j+l+1} \in C^*$ ; such a  $D$ -block is referred to as an  $A$ -block.

(2) All occurrences of  $w_c$  not involved in  $A$ -blocks are also  $D$ -blocks.

(3) All  $x_i$ 's which are not involved in  $A$ -blocks must be in  $C^*$  and so they are uniquely divisible in  $D$ -blocks (really  $C$ -blocks).

The definition of  $A$  and the fact that  $w_c \notin \text{sub}(C^*)$  and  $w_c$  is unbordered guarantee that such a division is unique.

Hence  $D$  is a code and claim 3 holds. ■

CLAIM 4:  $D$  is dense.

*Proof of claim 4:* Let  $u \in \Sigma^*$ .

Consider  $y = w_c u w_c$ . Reasoning as in the proof of claim 3 we get a (unique) representation of  $y$  in  $D^+$ .

Thus  $D$  is dense and claim 4 holds. ■

CLAIM 5:  $D$  is regular.

*Proof:* Obvious. ■

CLAIM 6:  $D$  is fast.

*Proof:* This follows from claim 5 and theorem 3. ■

Now theorem 5 follows from claims 3 through 5. ■

Our results yield two interesting corollaries. The first one solves an open problem from the theory of codes (see, e. g., [R] and [P2]). As a matter of fact it provides a more general result: Restivo has asked ([R]) whether an arbitrary *finite* code can be completed to a maximal regular code—we show that even an arbitrary *regular* code can be completed to a maximal regular code.

**COROLLARY 1:** *Let  $C$  be a code. If  $C$  is regular, then there exists a code  $D$  such that  $C \subseteq D$ ,  $D$  is maximal and  $D$  is regular.*

*Proof:* Let  $C$  be a regular code.

By theorem 5 there exists a regular code  $D$  such that  $C \subseteq D$ ,  $D$  is fast and dense.

Thus, by theorem 4,  $D$  is rich and so, by theorem 2,  $D$  is maximal.

Hence corollary 1 holds. ■

Secondly, we notice that theorems 1 through 4 provide an alternative proof of the theorem by Schutzenberger (see [E], p. 94).

**COROLLARY 2:** *Let  $C$  be a regular code. Then  $C$  is maximal if and only if  $C$  is dense.*

*Proof:* It follows directly from theorems 1 through 4. ■

DISCUSSION

We have established a number of relationships between dense, fast, rich, maximal and regular codes. Using these relationships we were able to demonstrate that each regular code is included in a maximal regular code.

In particular we have demonstrated that each rich code is maximal and each maximal code is dense. Hence each rich code is dense. We provide now a “direct” proof of this result—we believe it sheds a different light on this relationship.

COROLLARY 3: *Each rich code is dense.*

*Proof:* Let  $C$  be a rich code.

Assume that  $C$  is not dense. Hence there exists a word  $z \notin \text{sub}(C^*)$ ; let  $|z|=t$ . Let  $n$  be an arbitrary positive integer;  $n$  can be represented in the form  $n=k_1 t+k_2$  for some  $k \geq 0$  and  $k_2 < t$ . An arbitrary word from  $L_n(C^+)$  can be (starting from the left end) divided into  $k_1$  consecutive subwords of length  $t$  leaving a suffix of length  $k_2$ . Thus:

$$\alpha_n(C^+) < (\sigma^t - 1)^{k_1} \sigma^{k_2}.$$

Consequently:

$$\frac{\alpha_n(C^+)}{\sigma^n} < \frac{(\sigma^t - 1)^{k_1} \sigma^{k_2}}{\sigma^n} = \frac{(\sigma^t - 1)^{k_1} \sigma^{k_2}}{\sigma^{tk_1} \sigma^{k_2}} = \left(1 - \frac{1}{\sigma^t}\right)^{k_1}.$$

Hence:

$$\lim_{n \rightarrow \infty} \frac{\alpha_n(C^+)}{\sigma^n} = 0,$$

which contradicts the fact that  $C$  is rich.

Consequently  $C$  must be dense and the result holds. ■

To put some of the dependencies we have demonstrated in a better perspective we provide now the following result.

THEOREM 6: *There exists a maximal code which is not rich.*

*Proof:* Consider the family of all full binary trees in which leafs are labelled by  $a$  and all inner nodes are labelled by  $b$ . Consider now all postfix notations for these trees—in this way we get the language  $P \subseteq \{a, b\}^+$ . It is well known that  $P$  is a code (every forest of full binary trees has a unique representation in the postfix notation).

Consider an arbitrary word  $z \in \{a, b\}^+ - P$ . Clearly  $a^{|z|+1} z \in P^+$  (we parse  $a^{|z|+1} z$  from right to left assigning  $+1$  to  $a$  and  $-1$  to  $b$ ; then each subword yielding by summation weight  $+1$  is a tree corresponding to an element of



$P$ ). Hence  $P \cup \{z\}$  is not a code, because  $a^{|z|+1}z$  would have two different representations in  $P^+$ . Thus  $P$  is a maximal code.

On the other hand it is known (see, e. g., [F], ch. III, sect. 3) that:

$$\lim_{n \rightarrow \infty} \frac{\alpha_n(P^+)}{2^n} = 0.$$

(Here one considers random walks on the line of positive integers where  $a$  represents a “step up” and  $b$  represents a “step down”. It turns out that the probability of starting in 0 and not returning to 1 in up to  $n$  steps equals 1 in the limit.)

Hence  $P$  is not rich and the theorem holds. ■

Perhaps the most significant open question in the area of “extending codes to their maximal counterparts” is (see [P2]): can every biprefix regular code be extended to a maximal biprefix regular code? An answer to this question will certainly make the picture of the whole area clearer.

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