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ON LANGUAGES SATISFYING "INTERCHANGE LEMMA" (*)

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Abstract. – This paper deals with the closure properties of the family of languages satisfying the "interchange lemma" conditions and with a short comparison between these conditions and other iteration conditions on formal languages.

Résumé. – Dans cet article, on étudie les propriétés de clôture pour la famille des langages qui satisfont les conditions du « lemme de l'échange » et on compare ces conditions et d'autres conditions d'itération dans les langages formels.

1. INTRODUCTION

To substitute a subword of a word from a given language with an other string, such that the new word remains in language is a topic with many significances.

W. Ogden, R. Ross, K. Winklmann [8] have shown, in a rather wide sense, that if L is a context-free language and $Q \subseteq L_n = \{x \in L : \lg(x) = n\}$ is an enough large set then there are k strings in Q , z_i , $1 \leq i \leq k$, so that:

(i) $z_i = x_i w_i y_i$, $1 \leq i \leq k$;

(ii) $\lg(x_i) = \lg(x_j)$, $\lg(y_i) = \lg(y_j)$, $1 \leq i, j \leq k$;

(iii) $x_i w_j y_i \in L$ for any $i, j \in \{1, 2, \dots, k\}$.

Using this result, they prove that $C_2(V) = \{z \in V^+ : z = xy^2w, y \neq \varepsilon\}$ is not context-free for any alphabet V such that $|V| \geq 3$.

In analytical linguistics we find other applications [6].

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We used the following notations:

$\lg(x)$ is the length of the string x ,

ε is the empty word; $\lg(\varepsilon) = 0$,

CF is the family of all context-free languages,

$|A|$ is the cardinality of the finite set A ,

$N = \{0, 1, 2, \dots\}$.

2. PRELIMINARIES

Let $p \geq 2, q \geq 1$ be two integers. A language $L \subseteq V^*$ is in $IL(p, q)$ iff for any integers n, m so that $p \leq m \leq n$ and any subset $Q \subseteq L_n$, there are $k \geq |Q|/n^q$ strings $z_i \in Q, 1 \leq i \leq k$, such that the following conditions are satisfied:

- (i) $z_i = x_i w_i y_i, x_i, w_i, y_i \in V^*, 1 \leq i \leq k$;
- (ii) $\lg(x_i) = \lg(x_j), \lg(y_i) = \lg(y_j)$, for any $1 \leq i, j \leq k$;
- (iii) $m/p < \lg(w_i) \leq m, 1 \leq i \leq k$;
- (iv) $x_i w_j y_i \in L_n$, for any $i, j \in \{1, 2, \dots, k\}$.

It is clear that $IL(p, q) \subseteq IL(p', q')$ for any $p' \geq p, q' \geq q$ and $L \in IL(p, q)$ iff $L - \{\varepsilon\} \in IL(p, q)$. Therefore, we consider only the languages which have not the empty word. We denote $IL = \bigcup_{p \geq 2, q \geq 1} IL(p, q)$.

The inclusion $CF \subseteq IL$ is immediate as a consequence of [8]. If L is a context-free language then $L \in IL(\max(2, q), 3)$ where q is the constant from the "interchange lemma" with respect to L . The language L is *polynomially bounded* iff exist the integers $p, q \geq 1$ such that for any $n \geq p, |L_n| \leq n^q$.

It is obvious that if L is polynomially bounded then $L \in IL$.

A language L is *bounded* iff exist $w_1, w_2, \dots, w_k \in V^*, w_i \neq \varepsilon, 1 \leq i \leq k$, such that $L \subseteq \{w_1\}^* \{w_2\}^* \dots \{w_k\}^*$.

PROPOSITION 1: *Any bounded language is polynomially bounded but not vice versa.*

Proof: Let $L \subseteq \{w_1\}^* \dots \{w_s\}^*$ be a bounded language. Then $|L_n| \leq (n+1)^s \leq n^{s+1}$ for any $n \geq 2^s$, hence L is polynomially bounded.

Conversely, we consider a 2-True language over $\{a, b, c\}$ obtained by the iteration of a morphism [2]. This language is polynomially bounded but it is not bounded.

PROPOSITION 2: *In IL there are strict context sensitive, recursive recursively enumerable, non recursively enumerable languages.*

Proof: It is known that there are subsets $P \subseteq N$ so that $L_P = \{a^n : n \in P\}$ is context sensitive, recursive, recursively enumerable or non recursively enumerable language. Now, $L_P \in IL$ holds because L_P is polynomially bounded.

Examples: $L_1 = \{z \in V^+ : z = xy^2w, x, y, w \in V^*, y \neq \varepsilon\}$, $|V| \geq 6$ is not in *IL* as a consequence of [8].

$L = \{\alpha \# \alpha : \alpha \in \{a, b\}^*\}$ is not in *IL*.

Proof: We assume that $L \in IL(p, q)$. Then let n be an odd integer such that $2^{(n-1)/2}/n^q > 2^{n(p-1)-p}/(2p)$ and $Q = L_n$. Obviously, $|Q| = 2^{(n-1)/2}$.

CLAIM 1: *If $\{x_1 w_1 y_1, x_2 w_2 y_2\} \subseteq L_n$ with $\lg(x_1) = \lg(x_2)$, $w_1 \neq w_2$ and*

$$0 < \lg(w_1) = \lg(w_2) \leq n/2 \text{ then } x_1 w_2 y_1 \notin L \text{ and } x_2 w_1 y_2 \notin L.$$

We consider two cases:

Case 1: w_1 and w_2 are subwords of α ; in this case by the interchange of w_1 with w_2 the obtained words are not of the form $\beta \# \beta$.

Case 2: $w_1 = \delta_1 \# \delta_2$, $w_2 = \gamma_1 \# \gamma_2$ where $\lg(\delta_1) = \lg(\gamma_1)$, $\lg(\delta_2) = \lg(\gamma_2)$. If $x_1 w_2 y_1 \in L$ then, because $\lg(w_2) \leq n/2$, we have $\gamma_1 = \delta_1$ and $\gamma_2 = \delta_2$, hence, $w_1 = w_2$ which is contradictory.

CLAIM 2: *Let n_1, n_2 be two nonnegative integers such that $n/(2p) < n - n_1 - n_2$ and $w \in \{a, b, c\}^*$, $\lg(w) = n - n_1 - n_2$. If T_w is defined as follows:*

$$T_w = \{z \in Q : z = xwy, \lg(x) = n_1, \lg(y) = n_2\}$$

then

$$|T_w| \leq 2^{(n(p-1)-p)/(2p)}.$$

Again, we consider two cases:

Case 1: w is over $\{a, b\}$. If $z \in T_w$ then $z = \beta w \gamma \# \beta w \gamma$, $\beta, \gamma \in \{a, b\}^*$ so, $|T_w| \leq 2^{(n-1)/2 - \lg(w)}$ and because $\lg(w) > n/(2p)$ it follows that

$$|T_w| < 2^{(n-1)/2 - n/(2p)} = 2^{(n(p-1)-p)/(2p)}.$$

Case 2: w does contain a letter $\#$. In this case if $z \in T_w$ then $z = w_2 \beta w_1 \# w_2 \beta w_1$, where $w = w_1 \# w_2$ and $\beta \in \{a, b\}^*$. Hence

$$|T_w| \leq 2^{(n-1)/2 - \lg(w_1 w_2)} \leq 2^{(n-1)/2 - n/(2p)}.$$

Now, we choose $m=n/2$. It must exist $k > 2^{(n(p-1)-p)/(2p)}$ strings in Q satisfying the conditions $(*)$. But, among these strings there are two strings $z_1 \in T_{w_1}$ and $z_2 \in T_{w_2}$ such that $w_1 \neq w_2$. Thus the claim 2 and the point (iv) from $(*)$ are contradictory. Hence, $L \notin IL$.

3. CLOSURE PROPERTIES

A faithful rational transduction [1] is said to be bi-faithful iff both of the morphisms h and g are strictly alphabetical. A class of languages closed under bi-faithful rational transductions is a bi-faithful rational cone.

We will prove that IL is a bi-faithful cone closed under restricted morphisms and as a consequence we will show that the programming language ALGOL 60 is not in IL . An other result it will be that the problem "Is L in IL " for a given context sensitive language L is an undecidable problem.

THEOREM 1: *IL is closed under intersection with regular languages.*

Proof: Let L be in $IL(p, q)$, $L \subseteq V^*$, and L'' be a regular language accepted by a deterministic finite automata $M=(V, K, s_0, f, F)$. Then, $L' = L \cap L'' \in IL(p', q')$ where $p' = \max(p, |K^2|)$, $q' = q + 1$. For any $p' \leq m \leq n$ and any $Q \subseteq L'_n \subseteq L_n$ there exist k strings in Q satisfying the conditions $(*)$ with respect to L . Let R be the set of all these strings $z_i = x_i w_i y_i$, $1 \leq i \leq k$. We set

$$T(s_i, s_j) = \{z \in R : f(s_0, x) = s_i \text{ and } f(s_i, w) = s_j\},$$

for all $(s_i, s_j) \in K^2$. Then $|R| = \sum_{(s_i, s_j) \in K^2} |T(s_i, s_j)|$ therefore exists a pair $(s_i, s_j) \in K^2$ such that $k/n \leq |R|/|K^2| \leq |T(s_i, s_j)|$. All elements of $T(s_i, s_j)$ satisfy $(*)$ and the proof is complete.

LEMMA 1: *IL is closed under substitution with finite, ε -free languages.*

Proof: Let $L' \subseteq \{a_1, a_2, \dots, a_s\}^*$ be a language in $IL(p', q')$ and $f(a_i) = L^{(i)}$ $1 \leq i \leq s$ be a substitution. Let us denote

$$t_1 = \max(|L^{(i)}|), 1 \leq i \leq s,$$

$$t_2 = \max\{\lg(x) : x \in L^{(i)}, 1 \leq i \leq s\} \text{ and } L = f(L').$$

Furthermore, we take $p = \max(p', t_2, t_1)$ and $q = q' + 5$. If $p \leq m \leq n$ and $Q \subseteq L_n$ there exists $Q_j \subseteq L_j$, $p' \leq j \leq n$ such that $Q = \bigcup_{j=p}^n (f(Q_j) \cap Q)$ and

$f(x) \cap Q \neq \emptyset$ for any $x \in Q_j$, $p' \leq j \leq n$. Hence, exists $j \in \{p', p'+1, \dots, n\}$ such that $|f(Q_j) \cap Q| \geq |Q|/n$. Moreover, $|Q_j| \geq |f(Q_j) \cap Q|/t_1 \cdot j$ and if $[a]_*$ means an integer so that $[a]_* \leq a < [a]_* + 1$ then there are k' strings $z_i = x_i w_i y_i$ in Q_j with respect to $(*)$ for $m = [m/t_2]_*$. Let A' be the set of all these strings and $A = \{f(z') \cap Q : z' \in A'\}$.

For any integers n_1, n_2 such that $m/p < n - n_1 - n_2 \leq m$ we construct the sets $T(n_1, n_2) = \{z \in A : z = xwy, \lg(x) = n_1, \lg(y) = n_2 \text{ and exists } z' = x'w'y' \in A' \text{ so that } z \in f(z') \text{ and } x \in f(x'), y \in f(y')\}$. Since $m/p' \cdot t_2 < \lg(w) \leq m/t_2$ it follows that $m/p < \lg(w) \leq m$ hence $A = \bigcup_{n_1, n_2} T(n_1, n_2)$. Therefore, exist n_1, n_2 such that

$|T(n_1, n_2)| \geq |A|/n^2$. Because all words of $T(n_1, n_2)$ do satisfy the conditions $(*)$ with respect to $L, L \in IL(p, q)$ follows.

THEOREM 2: *IL is a bi-faithfull rational cone. However, IL is not closed under arbitrary morphisms.*

Proof: IL is a bi-faithfull cone follows from Theorem 1 and Lemma 1. If $L = \{xy\#xz : x \in \{a, b\}^*, y, z \in \{c, d\}^*, \lg(x) = \lg(y) = \lg(z)\}$ is in $IL(6, 1)$ and h is a morphism from $\{a, b, c, d, \#\}^*$ into $\{a, b, \#\}^*$, $h(a) = a, h(b) = b, h(\#) = \#, h(c) = h(d) = \varepsilon$ then we have $h(L) = \{x\#x : x \in \{a, b\}^*\}$ which is not in IL . However, IL is closed under an other kind of morphism, the restricted morphism.

If $L \subseteq (V\{\varepsilon, \#, \#^2, \dots, \#^{k-1}\})^*$ for some $\# \notin V$ and some constant k and if h is a morphism defined on $(V \cup \{\#\})^*$ by $h(a) = a$ for $a \in V$ and $h(\#) = \varepsilon$ then h is said k -restricted on L . This implies that $\lg(x) \leq k \cdot \lg(h(x))$ for all $x \in L$; moreover, given any subword x of a word in L , such that $\lg(x) \geq k$, we know that $\lg(h(x)) \geq 1$.

Now, we prove:

THEOREM 3: *IL is closed under restricted morphisms.*

Proof: Let h be a t -restricted morphism on

$$L \subseteq (V\{\varepsilon, \#, \#^2, \dots, \#^{t-1}\})^*, \quad L \in IL(p, q).$$

Then $L' = h(L) \in IL(p \cdot t, q + 3)$. Let $p \cdot t \leq m \leq n$ and $Q \subseteq L_n, Q = \bigcup_{i=n}^{t \cdot n} h(Q_i)$, $Q_i \subseteq L_i, n \leq i \leq t \cdot n$. Furthermore, if $x_1, x_2 \in Q_i$ then $h(x_1) \neq h(x_2)$ for any $n \leq i \leq t \cdot n$ (i.e. $|h(Q_i)| = |Q_i|$). Since $|Q| \leq \sum_{i=n}^{t \cdot n} |Q_i|$ there is i between n and $t \cdot n$ such that $|Q_i| \geq |Q|/t \cdot n$. There are k strings $z_j = x_j w_j y_j, 1 \leq j \leq k$, in Q_i in keeping with the requirements of $(*)$, $|Q|/n^{q+2} \leq |Q_i|/n^q \leq k$. Now, exists

a set $F \subseteq \{z_1, z_2, \dots, z_k\}$ such that:

(i) if $z' = x' w' y'$, $z = x w y$ are in F then $N_{\#}(w') = N_{\#}(w)$ where $N_{\#}(v)$ is the number of occurrences of letter $\#$ in v ;

(ii) $|F| \geq k/n$.

All strings of $h(F)$ respect the conditions $(*)$ involving that the claim holds.

Now, following [4, pp. 20] we consider the ALGOL programs of the form:

```
begin integer x;
      y := 1
end
```

These programs are correct if and only if $x = y$. Let us denote R the set of all programs as above with arbitrary x, y over $\{a, b\}$, and let ALGOL be the set of all correct ALGOL programs. If h is a restricted morphism which erases all symbols different from $\#$; and those in x and y , $h(a) = a$, $h(b) = b$, $h(\cdot) = \#$, then we obtain $h(\text{ALGOL} \cap R) = \{x \# x : x \in \{a, b\}^*\}$ which is not in IL . Since IL is closed under intersection with regular sets and under restricted morphisms it follows that ALGOL is not in IL .

We are going on with other closure properties of IL .

THEOREM 4: *IL is closed under union and concatenation but it is not closed under intersection and complementation.*

Proof: Let $E \in IL(p_1, q_1)$, $F \in IL(p_2, q_2)$ be two languages, E over V^* , F over U^* .

It is immediate that $E \cup F \in IL(\max(p_1, p_2), \max(q_1, q_2) + 1)$.

If $L = E \circ F$ and $p = \max(2 \cdot \max(p_1, p_2), |V|, |U|)$, $q = \max(q_1, q_2) + 3$ then $L \in IL(p, q)$. Let $p \leq m' \leq n$ and $Q \subseteq L_n$, then $Q = \bigcup_{i+j=n} (A_i B_j \cap Q)$, $A_i \subseteq E_i$,

$B_j \subseteq F_j$.

Hence, there are i and j so that $i+j=n$ and $|A_i B_j \cap Q| \geq |Q|/n$. If $i+j=n$ then we have $i \geq n/2$ or $j \geq n/2$. Let us assume that $i \geq n/2$. If $\alpha \in A_i$ then $|M(\alpha) = \{\beta \in B_j : \alpha\beta \in Q\}| \leq |U|^j < n^n$.

For each $1 \leq h \leq n$ we set M_h the union of all sets $\alpha M(\alpha)$ with $\alpha \in A_i$ and $n^{h-1} \leq |M(\alpha)| < n^n$. So, $A_i B_j \cap Q = \bigcup_{h=1}^n M_h$ and exists h between 1 and n such

that $|M_h| \geq |A_i B_j \cap Q|/n$. If $M = \{\alpha \in A_i : \alpha\beta \in M_h\}$ then there are $k_1 \geq |M|/n^{q_1}$ strings in M satisfying the conditions $(*)$ with respect to E for $m = m'$ if $i \geq m'$ or $m = i$ if $m' > i$. Since $k_1 n^{h-1} \geq |M_h|/n^{q_1+1}$ so

$$\left| \bigcup_{i=1}^{k_1} z_i M(z_i) \right| \geq |M_h|/n^{q_1+1}.$$

Now, it is easy to observe that all strings from $\bigcup_{i=1}^{k_1} z_i M(z_i)$ verify (*), so,

$E \circ F \in IL$.

For the last part of this theorem let us consider

$$L_1 = \{xy\#xz : x, y, z \in \{a, b\}^*, \lg(y) = \lg(z)\},$$

$$L_2 = \{xy\#zy : x, y, z \in \{a, b\}^*, \lg(x) = \lg(z)\}$$

which are in $IL(5, 1)$ (the details are left to the reader) and $L_1 \cap L_2 = \{x\#x : x \in \{a, b\}^*\}$ is not in IL . Using De Morgan relations the proof is complete.

For a language L and a word x , let us denote $\partial_x(L) = \{y : xy \in L\}$ for the *left derivative* with respect to x .

LEMMA 2: *IL is closed under the left derivative with respect to a given word.*

Proof is immediate.

At the end, we prove that the problem "Is L in IL ?" for an arbitrary context sensitive language L , is undecidable, using the following:

THEOREM 5 [5]: *Let \mathcal{L} be a family of languages effectively closed under union, concatenation with regular sets and the problems "Is the complementary of L empty or no?" is undecidable for L .*

If P is a non trivial property on \mathcal{L} such that:

- (a) P is true on the regular languages,
- (b) if L has the property P , R is a regular language and x is an arbitrary word, then $L \cap R$ and $\partial_x(L)$ have the property P , then P is undecidable on \mathcal{L} .

If \mathcal{L} is the family of the context sensitive languages and P is the property of a language that to be in IL then it satisfies all conditions of Theorem 5.

4. A SHORT COMPARISON WITH OTHER ITERATION CONDITION ON FORMAL LANGUAGES

We denote

OG – the family of languages satisfying Ogden's lemma, [7]

BH – the family of languages satisfying Bar Hillel's lemma,

SO – the family of languages satisfying Sokolowski's lemma [9].

THEOREM 6:

1. OG and IL are incomparable.

2. *BH* and *IL* are incomparable.
3. $IL \subset SO$ and the inclusion is proper.
4. $CF \subset OG \cap BH \cap IL$ is proper.

Proof: We consider the languages:

$$L_1 = \{ a^n b^n c^n : n \geq 1 \}$$

$$L_2 = \{ xy^2z : x, y, z \in V^*, y \neq \epsilon, |V| = 6 \}.$$

Now, $L_1 \in IL$ but $L_1 \notin BH \cup OG$, $L_2 \notin IL$ but $L_2 \in BH \cap OG$.

Let L be in $IL(p, q)$, $L \subseteq V^*$ and $|V| \geq 3$. Let U be a subset of V , $|U| \geq 2$, $u_1, u_3 \in V^*$, $u_2 \in V^* - U^*$ (if $u_2 \in U^*$ then the Sokolowski's condition is trivial), so that $A = \{ u_1 x u_2 x u_3 : x \in U^+ \} \subseteq L$. Let n be an integer such that

$$p \cdot \max(\lg(u_1), \lg(u_2), \lg(u_3)) \leq n/2 < n - (\lg(u_1) + \lg(u_2) + \lg(u_3))$$

and verifying a similar condition as in Examples for the language L . We take $Q = A_n$, since $L \in IL$ there are k strings in Q satisfying the requirements (*), for $m = n/2$. Therefore (see Examples) there are $z_1 = x_1 w_1 y_1$, $z_2 = x_2 w_2 y_2$ in Q so that $w_1 \neq w_2$ and $x_1 w_2 y_1 \in L$. Hence, we get the desired x' , x'' from U^+ , $x' \neq x''$, such that $u_1 x' u_2 x'' u_3 \in L$. Because $L_3 \in SO$ the inclusion is proper.

Finally, $B_p = \{ (ab)^n : n \in P \} \cup (\{ a, b \}^* \{ aa, bb \} \{ a, b \}^*)$, where $P \subseteq N$ is the set of all prime numbers, is not context-free; from [3] B_p is in $OG \cap BH$ and the relation $B_p \in IL(2, 2)$ is immediate.

We mention here some *open problems*: Is *IL* closed under substitutions, inverse morphisms or under the operations \star ?

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REFERENCES

1. J. M. AUTEBERT and L. BOASSON, Generators of Cones and Cylinders, Formal Languages Theory: Perspectives and Open Problems, R. V. Book Ed., *Acad. Press*, 1980, pp. 49-88.
2. J. BERSTEL, Sur les mots sans carré définis par un morphisme, *Lecture Notes in Comput. Sci.*, 1979, 71, pp. 16-25.

3. L. BOASSON and S. HORVATH, On language satisfying Ogden's lemma, *R.A.I.R.O. Inform. Theor. Appl.*, 1978, 12, pp. 193-199.
4. J. DASSOW and Gh. PAUN, Regulated rewriting in formal language theory, *Akademie-Verlag*, Berlin, 1989.
5. S. A. GREIBACH, A note on undecidable properties of formal languages, *Math. Syst. Theory*, 1968, 2, 1, pp. 1-6.
6. S. MARCUS, Algebraic linguistics. Analytical models, New York, London, *Academic Press*, 1967.
7. W. OGDEN, A helpful result for proving inherent ambiguity, *Math. Syst. Theory*, 1968, 2, pp. 191-197.
8. W. OGDEN, R. ROSS and K. WINKLEMAN, An "interchange lemma" for context-free languages, *S.I.A.M. J. Comput.*, 1985, 14, pp. 410-415.
9. S. SOKOŁOWSKI, A method for proving programming language non context-free, *Inf. Proc. Lett.*, 1978, 7, pp. 151-153.