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ON SEQUENCES DEFINED BY D0L POWER SERIES

JUHA HONKALA[1]

Abstract. We study D0L power series over commutative semirings. We show that a sequence $(c_n)_{n\geq 0}$ of nonzero elements of a field A is the coefficient sequence of a D0L power series if and only if there exist a positive integer k and integers β_i for $1 \leq i \leq k$ such that $c_{n+k} = c_{n+k-1}^{\beta_1} c_{n+k-2}^{\beta_2} \cdots c_n^{\beta_k}$ for all $n \geq 0$. As a consequence we solve the equivalence problem of D0L power series over computable fields.

AMS Subject Classification. 68Q45.

1. INTRODUCTION

D0L power series were defined in Honkala [3] and studied in detail in Honkala [4]. The study of these series gives an interesting counterpart to the customary theory of D0L languages.

The sequences of coefficients of D0L power series over the rationals were characterized in Honkala [4]. In this note we extend this characterization for arbitrary commutative semirings. As a consequence we obtain recursive formulas for these sequences over fields which are multiplicative versions of the recursive formulas satisfied by linear recurrence sequences studied, *e.g.*, in combinatorics. We show also that it is decidable whether or not two given D0L power series over a computable field are equal.

For further background and motivation we refer to Honkala [2,4] and the references given therein.

It is assumed that the reader is familiar with the basics of the theories of semirings, formal power series and L systems (see Kuich and Salomaa [5], Rozenberg and Salomaa [6,7]). Notions and notations that are not defined are taken from these references.

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2. Definitions

Suppose A is a commutative semiring and X is a finite alphabet. The set of *formal power series with noncommuting variables* in X and coefficients in A is denoted by $A \ll X^* \gg$. The subset of $A \ll X^* \gg$ consisting of all series with a finite support is denoted by $A < X^* >$. Series of $A < X^* >$ are referred to as *polynomials.*

Assume that X and Y are finite alphabets. A semialgebra morphism h: $A < X^* > \longrightarrow A < Y^* >$ is called a *monomial morphism* if for each $x \in X$ there exist a nonzero $a \in A$ and $w \in Y^*$ such that $h(x) = aw$. If $h : A < X^* > \longrightarrow A < Y^* >$ is a monomial morphism, the *underlying monoid morphism* $\overline{h} : X^* \longrightarrow Y^*$ is defined by $\overline{h}(x) = \text{supp}(h(x))$ for $x \in X$. A series $r \in A \ll X^* \gg$ is called a D0L *power series* over A if there exist a nonzero $a \in A$, a word $w \in X^*$ and a monomial morphism $h : A < X^* > \longrightarrow A < X^* >$ such that

$$r = \sum_{n=0}^{\infty} ah^n(w) \tag{1}$$

and, furthermore,

$$\text{supp}(ah^i(w)) \neq \text{supp}(ah^j(w)) \text{ whenever } 0 \leq i < j.$$

(Note that, if A is a field, this condition is equivalent with the local finiteness of the family $\{h^n(w)\}_{n \geq 0}$.)

Consider the series r given in (1) and denote

$$ah^n(w) = c_n w_n$$

where $c_n \in A$ and $w_n \in X^*$ for $n \geq 0$. Then we have

$$r = \sum_{n=0}^{\infty} c_n w_n. \tag{2}$$

In what follows the righthand side of (2) is called the *normal form* of r. A sequence $(c_n)_{n \geq 0}$ of elements of A is called a D0L *multiplicity sequence* over A if there exists a D0L power series r such that (2) is the normal form of r.

For examples of D0L power series see Section 4.

3. D0L multiplicity sequences over commutative semirings

A sequence $(a_n)_{n \geq 0}$ of nonnegative integers is called a *modified PD0L length sequence* if there exists a nonnegative integer t such that $a_0 = a_1 = \ldots = a_{t-1} = 0$ and $(a_{n+t})_{n \geq 0}$ is a PD0L length sequence. (For the definition of a PD0L length sequence see Rozenberg and Salomaa [6].) A sequence $(a_n)_{n \geq 0}$ of nonnegative integers is a modified PD0L length sequence if and only if the sequence $(a_{n+1} - a_n)_{n\geq 0}$ is $\mathbf{N}$-rational (see Rozenberg and Salomaa [6], p. 157).

The following theorem characterizes D0L multiplicity sequences over a commutative semiring.

Theorem 1. *Suppose A is a commutative semiring. A sequence $(c_n)_{n\geq 0}$ of nonzero elements of A is a D0L multiplicity sequence over A if and only if there exist a positive integer k, nonzero $a_1, \ldots, a_k \in A$ and modified PD0L length sequences $(s_{in})_{n\geq 0}$ for $1 \leq i \leq k$ such that*

$$c_n = \prod_{i=1}^{k} a_i^{s_{in}} \tag{3}$$

for all $n \geq 0$.

Proof. Suppose first that $r = \sum_{n=0}^{\infty} ah^n(w)$ is a D0L power series over A with the normal form

$$r = \sum_{n=0}^{\infty} c_n w_n.$$

Without loss of generality we assume that $a = c_0 = 1$. Let $g : X^* \longrightarrow X^*$ be the underlying monoid morphism of the monomial morphism $h : A < X^* > \longrightarrow A < X^* >$. Then we have $g^n(w_0) = w_n$ for all $n \geq 0$. Let $\overline{X} = \{\overline{x} \mid x \in X\}$ be a new alphabet with the same cardinality as X. Define the monoid morphism $g_1 : (X \cup \overline{X})^* \longrightarrow (X \cup \overline{X})^*$ by

$$g_1(x) = \overline{x}g(x), \quad g_1(\overline{x}) = \lambda, \qquad x \in X.$$

For each $x \in X$ let $a_x \in A$ be such that $h(x) = a_x g(x)$. Define the semialgebra morphism $\alpha : A < (X \cup \overline{X})^* > \longrightarrow A$ by

$$\alpha(x) = 1, \quad \alpha(\overline{x}) = a_x, \qquad x \in X.$$

Then we have

$$h(u) = \alpha(g_1(u))g(u) \tag{4}$$

and

$$g_1(g(u)) = g_1^2(u) \tag{5}$$

for any word $u \in X^*$. Equation (5) implies inductively that

$$g_1(g^n(u)) = g_1^{n+1}(u) \tag{6}$$

for any $n \geq 1$ and $u \in X^*$. We claim that

$$c_n = \alpha(w_0 g_1(w_0) g_1^2(w_0) \ldots g_1^n(w_0)) \tag{7}$$

for all $n \geq 0$.

The claim is trivially true for $n = 0$. If the claim holds for $n = k$, we have by (4, 6) and (7)

$$\begin{aligned} h^{k+1}(w_0) &= h(c_k w_k) = c_k h(w_k) = c_k \alpha(g_1(w_k)) g(w_k) = c_k \alpha(g_1(g^k(w_0))) w_{k+1} \\ &= c_k \alpha(g_1^{k+1}(w_0)) w_{k+1} = \alpha(w_0 g_1(w_0) g_1^2(w_0) \dots g_1^{k+1}(w_0)) w_{k+1}, \end{aligned}$$

which implies the claim for $n = k + 1$ and, hence, for all $n \geq 0$.

Next, for each $x \in X$ define the sequence $(s(x)_n)_{n \geq 0}$ by

$$s(x)_n = \#_{\overline{x}}(w_0 g_1(w_0) g_1^2(w_0) \dots g_1^n(w_0))$$

where $\#_{\overline{x}} v$ stands for the number of the occurrences of the letter $\overline{x}$ in the word v. Because

$$s(x)_{n+1} - s(x)_n = \#_{\overline{x}}(g_1^{n+1}(w_0)),$$

the sequence $(s(x)_{n+1} - s(x)_n)$ is $\mathbf{N}$-rational for all $x \in X$. (Here $\mathbf{N}$-rationality follows because the sequence is an HD0L length sequence.) Hence the sequences $(s(x)_n)_{n \geq 0}$ are modified PD0L length sequences. By (7) we have

$$c_n = \prod_{x \in X} a_x^{s(x)_n}$$

for all $n \geq 0$. This concludes the proof in one direction.

Suppose then that there exist a positive integer k, nonzero $a_1, \dots, a_k \in A$ and modified PD0L length sequences $(s_{in})_{n \geq 0}$ for $1 \leq i \leq k$ such that (3) holds for all $n \geq 0$. We have to show that $(c_n)_{n \geq 0}$ is a D0L multiplicity sequence over A. Because D0L multiplicity sequences over A are closed under finite product provided that no term of the product sequence is zero, it suffices to consider the case $k = 1$. Denote $a = a_1$ and $s_n = s_{1n}$ for $n \geq 0$. Without restriction we suppose that (s_n) is a PD0L length sequence. If the set $\{s_n \mid n \geq 0\}$ is finite, $(c_n)_{n \geq 0}$ is clearly a D0L multiplicity sequence over A. Suppose therefore that $\{s_n \mid n \geq 0\}$ is an infinite set and let $G = (\Sigma, f, w_0)$ be a PD0L system defining the sequence $S(G) = (w_n)_{n \geq 0}$ with $|w_n| = s_n$ for $n \geq 0$. Define the monomial morphism $h : A < \Sigma^* > \longrightarrow A < \Sigma^* >$ by

$$h(\sigma) = a^{|f(\sigma)|-1} f(\sigma)$$

for $\sigma \in \Sigma$. It follows inductively that

$$a^{s_0} h^n(w_0) = a^{s_n} w_n$$

for $n \geq 0$. Hence the series r defined by

$$r = \sum_{n=0}^{\infty} a^{s_0} h^n(w_0)$$

is a D0L power series over A and the sequence $(c_n)_{n\geq 0} = (a^{s_n})_{n\geq 0}$ is a D0L multiplicity sequence over A. □

4. The equivalence problem of D0L power series

If the basic semiring A is a field, we obtain the following characterization of D0L multiplicity sequences.

Theorem 2. *Suppose A is a field. A sequence $(c_n)_{n\geq 0}$ of elements of A is a D0L multiplicity sequence over A if and only if there exist a positive integer k, nonzero $a_1, \ldots, a_k \in A$ and $\mathbf{Z}$-rational sequences $(s_{in})_{n\geq 0}$ for $1 \leq i \leq k$ such that*

$$c_n = \prod_{i=1}^{k} a_i^{s_{in}}$$

for all $n \geq 0$.

Proof. The left to right direction is immediate by Theorem 1. The other direction follows by Theorem 1 because every $\mathbf{Z}$-rational sequence can be expressed as the difference of two PD0L length sequences (see Rozenberg and Salomaa [6], p. 160). □

In case $A = \mathbf{Q}$, where $\mathbf{Q}$ stands for the field of rational numbers, the left to right direction of Theorem 2 was used in Honkala [3] to solve the equivalence problem of D0L power series over $\mathbf{Q}$. The solution is based on the fundamental theorem of arithmetic and does not generalize as it stands for an arbitrary computable field.

Example 1. Let A be the ring of numbers $n + m\sqrt{-3}$ where $n, m \in \mathbf{Z}$. Let $X = \{a, b\}$ and define the monomial morphisms $h_i : A < X^* > \longrightarrow A < X^* >$, $i = 1, 2$, by

$$h_1(a) = aba, \qquad h_1(b) = 16\lambda,$$
$$h_2(a) = (1 + \sqrt{-3})a, \qquad h_2(b) = -2(1 + \sqrt{-3})ba^2b.$$

Define the D0L power series r_1 and r_2 by

$$r_1 = \sum_{n=0}^{\infty} h_1^n(aba)$$

and

$$r_2 = \sum_{n=0}^{\infty} h_2^n(aba).$$

It follows inductively that

$$h_1^n(aba) = h_2^n(aba) = 16^{2^n-1}(aba)^{2^n}$$

for $n \geq 0$. Hence r_1 and r_2 are equivalent. Note that A is not a unique factorization domain. Therefore, to solve the equivalence problem of D0L power series over A it does not suffice to consider the factorizations of the coefficients into products of irreducible elements.

To solve the equivalence problem of D0L power series over an arbitrary computable field we first deduce a new characterization of D0L multiplicity sequences.

Lemma 3. *Suppose G is an abelian group and $(c_n)_{n\geq 0}$ is a sequence of elements of G. Then the following conditions are equivalent:*
(i) There exist a positive integer k, elements $a_1, \dots, a_k \in G$ and $\mathbf{Z}$-rational sequences $(s_{in})_{n\geq 0}$ for $1 \leq i \leq k$ such that

$$c_n = \prod_{i=1}^{k} a_i^{s_{in}}$$

for all $n \geq 0$.
(ii) There exist a positive integer t and integers $\beta_1, \dots, \beta_t$ such that

$$c_{n+t} = c_{n+t-1}^{\beta_1} c_{n+t-2}^{\beta_2} \cdots c_n^{\beta_t}$$

for $n \geq 0$.

Proof. Suppose first that (i) holds. Then there exist a positive integer t and integers $\beta_1, \dots, \beta_t$ such that

$$s_{i,n+t} = \beta_1 s_{i,n+t-1} + \dots + \beta_t s_{in}$$

for all $n \geq 0$, $1 \leq i \leq k$ (see Berstel and Reutenauer [1], Salomaa and Soittola [8]). Therefore

$$c_{n+t} = \prod_{i=1}^{k} a_i^{s_{i,n+t}} = \prod_{i=1}^{k} a_i^{\beta_1 s_{i,n+t-1} + \dots + \beta_t s_{in}}$$
$$= \prod_{i=1}^{k} (a_i^{s_{i,n+t-1}})^{\beta_1} \dots (a_i^{s_{in}})^{\beta_t} = c_{n+t-1}^{\beta_1} \cdots c_n^{\beta_t}$$

for $n \geq 0$. Hence (ii) holds true.

Suppose then that (ii) holds. Assume first that $G = H$ where H is the multiplicative subgroup of the rationals generated by the first t primes $p_0, \dots, p_{t-1}$. If $x \in \mathbf{Q}$ is nonzero and p is a prime denote by $\nu_p(x)$ the p-adic value of x. Then

$$\nu_{p_i}(c_{n+t}) = \beta_1 \nu_{p_i}(c_{n+t-1}) + \dots + \beta_t \nu_{p_i}(c_n)$$

for all $n \geq 0$, $0 \leq i \leq t-1$. Hence the sequences $(\nu_{p_i}(c_n))_{n\geq 0}$ are $\mathbf{Z}$-rational. Because

$$c_n = \prod_{i=0}^{t-1} p_i^{\nu_{p_i}(c_n)}$$

for all $n \geq 0$, condition (i) holds.

Let then G be an arbitrary abelian group. Then there exists a group morphism $\psi : H \longrightarrow G$ such that $\psi(p_i) = c_i$ for $0 \leq i \leq t-1$. Define the sequence $(d_n)_{n\geq 0}$ in H recursively by

$$d_i = p_i, \qquad 0 \leq i \leq t-1$$

and

$$d_{n+t} = d_{n+t-1}^{\beta_1} d_{n+t-2}^{\beta_2} \cdots d_n^{\beta_t}$$

for $n \geq 0$. Then $\psi(d_n) = c_n$ for all $n \geq 0$. By the argument above there exist a positive integer k, elements $a_1, \ldots, a_k \in H$ and $\mathbf{Z}$-rational sequences $(s_{in})_{n\geq 0}$ for $1 \leq i \leq k$ such that

$$d_n = \prod_{i=1}^{k} a_i^{s_{in}}$$

for $n \geq 0$. Consequently,

$$c_n = \psi(d_n) = \prod_{i=1}^{k} \psi(a_i)^{s_{in}}$$

for $n \geq 0$. This shows that (i) holds. □

Theorem 4. *Suppose A is a field. A sequence $(c_n)_{n\geq 0}$ of nonzero elements of A is a D0L multiplicity sequence over A if and only if there exist a positive integer t and integers $\beta_1, \ldots, \beta_t$ such that*

$$c_{n+t} = c_{n+t-1}^{\beta_1} c_{n+t-2}^{\beta_2} \cdots c_n^{\beta_t} \tag{8}$$

for $n \geq 0$. Furthermore, if $(c_n)_{n\geq 0}$ and $(d_n)_{n\geq 0}$ are D0L multiplicity sequences over A there exist a positive integer t and integers $\beta_1, \ldots, \beta_t$ such that both (8) and

$$d_{n+t} = d_{n+t-1}^{\beta_1} d_{n+t-2}^{\beta_2} \cdots d_n^{\beta_t} \tag{9}$$

hold true for all $n \geq 0$.

Proof. The first claim follows by Theorem 2 and Lemma 3, the second claim by the proof of Lemma 3. □

Now we have the tools to solve the equivalence problem of D0L power series over computable fields.

Theorem 5. *Suppose A is a computable field. It is decidable whether or not two given D0L power series r_1 and r_2 over A are equal.*

Proof. Let

$$r_1 = \sum_{n=0}^{\infty} c_n u_n$$

and

$$r_2 = \sum_{n=0}^{\infty} d_n v_n$$

be the normal forms of r_1 and r_2, respectively. The proof of Theorem 3.2 in Honkala [3] implies that it sufffices to give a method to decide whether or not the D0L multiplicity sequences (c_n) and (d_n) are equal. (The main idea in Honkala [3] is first to decide whether or not $\{u_n \mid n \geq 0\} = \{v_n \mid n \geq 0\}$. This is an instance of D0L language equivalence problem. If the answer is positive, we can effectively decompose the sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ into finitely many pairwise identical sequences.) This decision can be made by Theorem 4. Indeed, there exist a positive integer t and integers $\beta_1, \ldots, \beta_t$ such that (8) and (9) hold. Then $c_n = d_n$ for all $n \geq 0$ if and only if $c_n = d_n$ for $0 \leq n < t$. □

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