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THE μ-CALCULUS ALTERNATION-DEPTH HIERARCHY IS STRICT ON BINARY TREES

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Abstract. In this paper we give a simple proof that the alternation-depth hierarchy of the μ-calculus for binary trees is strict. The witnesses for this strictness are the automata that determine whether there is a winning strategy for the parity game played on a tree.

Résumé. Nous donnons dans cet article une preuve simple que la hiérarchie d'alternance de points fixes pour le μ-calcul sur les arbres binaires est infinie. Les témoins en sont les automates qui déterminent l'existence d'une stratégie gagnante pour le jeu de parité joué sur un arbre binaire.

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Introduction

In the μ-calculus, the least and greatest fixed-point operators μ and ν are in some sense analogous to quantifiers in logics, and it is very natural to classify the formulas of the μ-calculus according to the alternation of these operators as it is done for logical formulas. This classification is called the alternation-depth hierarchy and its classes are denoted by Σ_n and Π_n.

Thus, a fundamental problem is the strictness of this hierarchy from the point of view of expressiveness: is it true or not that there exists an n such that any formula is equivalent to some formula in Σ_n. A necessary and sufficient condition for the strictness of the hierarchy is the existence for each n of a formula F_n that is in Σ_n but not in Π_n. Such a formula F_n is called Σ_n-hard.

When restricted to binary trees, the formulas of the μ-calculus are equivalent to alternating tree automata and the alternation-depth hierarchy corresponds to the Rabin index hierarchy of automata [15]. In 1986, Niwiński [14] showed that the

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alternation hierarchy of the μ-calculus on binary trees *without intersection* (*i.e.*, the Rabin index hierarchy of non deterministic automata) is strict. But the fixed-point terms he considered for proving this result are all equivalent to co-Büchi terms (*i.e.* in Π_2) with intersection. Since then, the question of the strictness of the hierarchy of the μ-calculus on binary trees *with intersection* (*i.e.*, alternating automata) was open [4, 15].

Recently, and simultaneously, Bradfield [6] and Lenzi [10] have proved that the alternation-depth hierarchy of the modal μ-calculus is strict. In [7], Bradfield shows that the formula

$$B_n = \mu x_1.\nu x_2.\cdots.\theta x_n.[c]x_n \vee \langle a_1\rangle x_1 \vee \cdots \vee \langle a_n\rangle x_n$$

is Σ_n-hard, as well as the Walukiewicz's formula

$$W_n = \mu x_1.\nu x_2.\cdots.\theta x_n.\left(E \Rightarrow \langle\rangle \bigwedge_{i=1}^{n}(R_i \Rightarrow x_i)\right) \wedge \left(O \Rightarrow [] \bigwedge_{i=1}^{n}(R_i \Rightarrow x_i)\right),$$

that states that the first player has a winning strategy in a parity game [17] and that is nothing but the extension of the Emerson-Jutla's formula [8] to games that are not bipartite.

The Lenzi's Π_n- and Σ_n-hard formulas are formulas on n-ary trees. They are defined inductively by

$$\begin{aligned} L_0(P) = L_0^d(P) &= P, \\ L_{n+1}(P) &= \nu x_{n+1}.(P \wedge L_n^d(a_{n+1}.x_{n+1})), \\ L_{n+1}^d(P) &= \mu x_{n+1}.(P \vee L_n(a_{n+1}.x_{n+1})), \end{aligned}$$

where a node has property $a_i.x$ if its ith successor has property x.

Since Lenzi's formulas are formulas on n-ary trees, and since one can encode n-ary trees into binary trees, one can deduce from Lenzi's result that the alternation hierarchy of the μ-calculus is also strict on binary trees. Such a transformation is not so easy for Bradfield's and Walukiewicz's formulas [5].

In this note we offer a direct proof that Walukiewicz's formulas are hard on binary trees. It combines two arguments already used by Bradfield in [7].

The *reduction argument* is that every Σ_n-formula F reduces to the Walukiewicz's formula W_n via some mapping G_F, *i.e.*, $M \models F \Leftrightarrow G_F(M) \models W_n$. The *diagonal argument* is just the fact that any mapping G_F has a fixed point M_F. Thus, if the negation of the Walukiewicz's formula W_n is equivalent to a Σ_n-formula F, we get $M_F \models F \Leftrightarrow M_F \models \neg F$, a contradiction. The existence of this fixed point is a consequence of the celebrated Banach's fixed-point theorem, since the mappings G_F are contracting on the compact metric space of binary trees.

It turns out that the same diagonal argument can be applied to *weak alternating automata* introduced by Muller *et al.* to characterize the weakly definable sets of trees [12], providing a direct proof of the Mostowski's result on the hierarchy of weak alternating automata [11], instead of relying on a result of Thomas [16].

1. Alternating parity automata

An alternating parity automaton is an alternating automaton (see [13]) where the acceptance criterion is given by a parity condition. Namely, it is a tuple $\langle A, Q, \delta, n, r\rangle$ where

- the *alphabet* A is a finite set of binary symbols,
- Q is a finite set of states,
- n is a natural number ($n > 0$), called the *type* of $\mathcal{A}$ and r is a mapping from Q to $\{1, \dots, n\}$,
- δ associates with each $q \in Q$ and each $a \in A$ an element of the free distributive lattice generated by $Q \times \{1,2\}$.

Indeed, each $\delta(q,a)$ can be seen as a finite disjunction of finite conjunctions of elements in $Q \times \{1,2\}$. Without loss of generality, we may assume that $\delta(q,a)$ is a finite *non empty* disjunction of finite *non empty* conjunctions.

Example: Walukiewicz's automata.
Let A_n be the alphabet $\{c_i, d_i \mid 1 \leq i \leq n\}$, and let T_n be the set of binary trees over A_n (*i.e.*, the set of mappings $t : \{1,2\}^* \to A_n$). The Walukiewicz automaton $\mathcal{W}_n$ is $\langle A_n, Q_n, \delta_n, n, r_n\rangle$ where

- $Q_n = \{q_1, \dots, q_n\}$ and for any $q_i \in Q_n$, $r_n(q_i) = i$,
- for any $q \in Q$, $\delta(q, c_i) = (q_i, 1) \wedge (q_i, 2)$ and $\delta(q, d_i) = (q_i, 1) \vee (q_i, 2)$.

The set $R_{\mathcal{A},q}(t)$ of *runs* of $\mathcal{A}$ from the state q on a tree t built on the alphabet A is the set of (unordered) trees, labeled by states in Q, defined recursively as follows: $\rho \in R_{\mathcal{A},q}(a(t_1,t_2))$ if and only if

- the root of ρ is labeled by q,
- among the conjunctions that are added up to form $\delta(q,a)$, there exists a conjunction $(q_1, x_1) \wedge \cdots \wedge (q_k, x_k)$ such that the root of ρ has exactly k subtrees that are respectively in $R_{\mathcal{A},q_j}(t_{x_j})$ ($j = 1, \dots, k$).

A branch b of such a run ρ (that is always infinite because of our assumption) is *μ-accepting* (resp. *ν-accepting*) if the minimum value of $r(q)$ where q ranges over the set of states that occur infinitely often on b is even (resp. odd).

A run ρ is *μ-accepting* (resp. *ν-accepting*) if each of its branches is μ-accepting (resp. ν-accepting). Finally $L^{\mu}_{\mathcal{A}}(q)$ (resp. $L^{\nu}_{\mathcal{A}}(q)$) is the set of all trees t such that $R_{\mathcal{A},q}(t)$ contains at least one μ-accepting run (resp. ν-accepting).

Example: Walukiewicz's tree languages.
In an automaton $\mathcal{W}_n$ we have, by definition, $\delta_n(q_i, a) = \delta_n(q_j, a)$ for any $a \in A_n$ and any $i, j \in \{1, \dots, n\}$. It follows that for $\theta = \mu, \nu$ and for any i, j, $L^{\theta}_{\mathcal{W}_n}(q_i)$ $= L^{\theta}_{\mathcal{W}_n}(q_i)$. We denote this language by W^{θ}_n. It is interesting to notice that the intersections of these W^{θ}_n with the set of trees over the alphabet $\{c_i \mid i = 1, \dots, n\}$

are exactly the tree languages defined by Niwiński in [14] to show the strictness of the hierarchy of non deterministic automata.

The trees that belongs to W_n^θ can be characterized in terms of game. Let t be a tree in T_n. A node $u \in \{1,2\}^*$ is a C-node (resp. D-node) if $t(u) \in \{c_1, \dots, c_n\}$ (resp. $\{d_1, \dots, d_n\}$). Let two players C and D. Initially, a token is at the root ε of t. If the token is at C-node (resp. D-node) u then player C (resp. D) moves the token to node $u1$ or $u2$. A play is an infinite word of $\{1,2\}^\omega$, *i.e.*, a branch of t, and D wins the play if the least index that occurs infinitely often on this branch is even or odd, according to the value of θ. A strategy s for D is a mapping that associates $s(u) \in \{1,1\}$ with each D-node u. The strategy s is winning if D wins any play consistent with this strategy, *i.e.*, at the D-node u, D moves the token to $us(u)$. What the Walukiewicz automata does is just to guess a strategy (by the rules $\delta(q,d_i) = (q_i,1) \vee (q_i,2)$) and accepts t if the strategy it guessed is winning. Therefore $t \in W_n^\theta$ if and only if D has a winning strategy in the game on t.

Let $\widetilde{\mathcal{A}}$ be the automaton obtained from $\mathcal{A}$ by exchanging $\vee$ and $\wedge$. The complementation theorem of Muller and Schupp [13] now reads as follows, where T_A is the set of all binary trees built on the alphabet A.

Proposition 1.1. $L^\mu_{\widetilde{\mathcal{A}}}(q) = T_A - L^\nu_{\mathcal{A}}(q), \quad L^\nu_{\widetilde{\mathcal{A}}}(q) = T_A - L^\mu_{\mathcal{A}}(q)$.

Definition 1.2. We denote by Σ_n (resp. Π_n) the family of binary tree languages in the form $L^\mu_{\mathcal{A}}(q)$ (resp. $L^\nu_{\mathcal{A}}(q)$) for some automaton $\mathcal{A}$ of type n. In particular, $W_n^\mu \in \Sigma_n$ and $W_n^\nu \in \Pi_n$.

As a consequence of the previous proposition, we get

Proposition 1.3. *For any tree language L over the alphabet A,*

$$L \in \Sigma_n \Leftrightarrow T_A - L \in \Pi_n.$$

Definition 1.4. We say that a language L is Σ_n-*hard* if it is in Σ_n and not in Π_n (or, equivalently, its complement $T_A - L$ is not in Σ_n).

2. The reduction argument

It is well-known (see [9] and [8], for instance) that the acceptance of a tree t by an automaton $\mathcal{A}$ can be expressed as the existence of a winning strategy in a game G associated with $\mathcal{A}$ and t. When $\mathcal{A}$ is a parity automaton, the game G is a parity game and the existence of a (memoryless) winning strategy is expressed by the Walukiewicz's formulas. The same argument is used by Bradfield [7] to show hardness of Walukiewicz's formulas. We show that when $\mathcal{A}$ is of rank n, we can construct an associated game that is indeed a binary tree in T_n, and the existence of a winning strategy is asserted by the membership of this tree to a Walukiewicz's language.

Let $\mathcal{A}$ be an automaton of rank n over the alphabet A. For any state q of $\mathcal{A}$, we define recursively the mapping $G_{\mathcal{A},q} : T_A \to T_n$ as follows.

We can see $\delta(q,a)$ as a finite binary tree whose internal nodes are labelled by $\vee$ and $\wedge$ and leaves by elements of $Q \times \{1,2\}$. Then $G_{\mathcal{A},q}(a(t_1,t_2))$ is the tree obtained by substituting in $\delta(q,a)$

- c_i for $\wedge$, d_i for $\vee$, where $i = r(q)$,
- the tree $G_{\mathcal{A},q'}(t_x)$ for (q',x).

The following characterization is nothing but another way of explaining when a tree is accepted by a parity automaton, like in [8], or, in other words, the winning strategy for D in the game on $G_{\mathcal{A},q}(t)$ allows us to construct an accepting run in $R_{\mathcal{A},q}(t)$, and conversely, from an accepting run, we can construct a winning strategy.

Proposition 2.1. *For $\theta = \mu,\nu$ and $t \in T_A$, $t \in L^\theta_{\mathcal{A}}(q) \Leftrightarrow G_{\mathcal{A},q}(t) \in W^\theta_n$.*

3. The diagonal argument

Since, in an automaton $\mathcal{A}$, substituting $\delta(q,a) \vee \delta(q,a)$ for $\delta(q,a)$ does not modify the set of runs, we may assume that each tree $\delta(q,a)$ has its root labelled by $\vee$. Therefore, if we consider that T_A and T_n are equipped with the usual ultrametric distance Δ defined by $\Delta(t,t') \leq 2^{-k} \Leftrightarrow \forall u \in \{1,2\}^*$, $|u| < k \Rightarrow t(u) = t'(u)$, that makes T_A and T_n complete, and even compact [2], it is easy to see that, for any automaton $\mathcal{A}$ and any state q, the mapping $G_{\mathcal{A},q} : T_A \to T_n$ is contracting, provided the above assumption on each $\delta(a,q)$.

Proposition 3.1. $\Delta(G_{\mathcal{A},q}(t), G_{\mathcal{A},q}(t')) \leq \Delta(t,t')/2$.

Proof. It is easy to prove by induction on k that $\Delta(t,t') \leq 2^{-k} \Rightarrow \Delta(G_{\mathcal{A},q}(t), G_{\mathcal{A},q}(t')) \leq 2^{-(k+1)}$.

In particular, if $\mathcal{A}$ is an automaton of rank n over the alphabet T_n, the contracting mapping $G_{\mathcal{A},q} : T_n \to T_n$ has a unique fixed point $t_{\mathcal{A},q}$ and we get, from Proposition 2.1,

Proposition 3.2. $t_{\mathcal{A},q} \in L^\theta_q(\mathcal{A}) \Leftrightarrow t_{\mathcal{A},q} \in W^\theta_n$.

Theorem 3.3. *The Walukiewicz's language W^μ_n is Σ_n-hard.*

Proof. Clearly $W^\mu_n \in \Sigma_n$. Now, assume that $T_n - W^\mu_n \in \Sigma_n$. Then there exists an automaton $\mathcal{A}$ of rank n and a state q such that $T_n - W^\mu_n = L^\mu_q(\mathcal{A})$ and we get $t_{\mathcal{A},q} \in T_n - W^\mu_n \Leftrightarrow t_{\mathcal{A},q} \in W^\mu_n$, an obvious contradiction. Hence $W^\mu_n \in \Sigma_n - \Pi_n$. By a similar reasoning, we get $W^\nu_n \in \Pi_n - \Sigma_n$. □

4. Universal languages

The previous diagonal argument can be easily extended to prove the hardness of some tree languages.

We say that a language L over an alphabet A is Σ_n*-universal* if there is a non expansive mapping $F : T_n \to T_A$ [1] such that

$$\forall t \in T_n, \quad t \in W_n^\mu \Leftrightarrow F(t) \in L.$$

Note that we do not require L to be in Σ_n.

Similarly, we say that L is Π_n*-universal* if

$$\forall t \in T_n, \quad t \in W_n^\nu \Leftrightarrow F(t) \in L.$$

In particular, taking F as the identity mapping over T_n, we get that W_n^μ is Σ_n-universal and that W_n^ν is Π_n-universal.

Theorem 4.1. *A Σ_n-universal language is never in Π_n. A Π_n-universal language is never in Σ_n.*

Proof. Let L be a Σ_n-universal language. If it is in Π_n, $\overline{L} = T_A - L$ is in Σ_n, and by the reduction argument (Prop. 2.1) there exists a contracting mapping $G_{\overline{L}} : T_A \to T_n$ such that $t \notin L \Leftrightarrow G_{\overline{L}}(t) \in W_n^\mu$. Since L is Σ_n-universal, $G_{\overline{L}}(t) \in W_n^\mu \Leftrightarrow F(G_{\overline{L}}(t)) \in L$. Since $G_{\overline{L}}$ is contracting and F is non expansive, $F \circ G_{\overline{L}} : T_A \to T_A$ is contracting and has a fixed point $t_{\overline{L}}$ that satisfies $t_{\overline{L}} \notin L \Leftrightarrow t_{\overline{L}} \in L$, a contradiction.

The second part of the theorem is proved quite similarly.

Corollary 4.2. *If a Σ_n-universal language is in Σ_n then it is Σ_n-hard.*

Example: Bradfield's tree languages.
Let $\mathcal{B}_n$ be the automaton over the alphabet $A'_n = \{c_n\} \cup \{d_i \mid 1 \leq i \leq n\}$ defined by

- $Q = \{q_1, \dots, q_n\}$ and for any $q_i \in Q_n$, $r(q_i) = i$,
- for any $q \in Q$, $\delta(q, c_n) = (q_n, 1) \wedge (q_n, 2)$ and $\delta(q, d_i) = (q_i, 1) \vee (q_i, 2)$.

Because of the analogy between these automata and the Bradfield's formulas, we call them *Bradfield's automata.* The *Bradfield's languages* are the languages $B_n^\theta = L_{q_1}^\theta(\mathcal{B}_n)$ that are in Σ_n or Π_n according to whether θ is μ or ν. Indeed it is easy to see that $B_n^\theta = W_n^\theta \cap T'_n$ where T'_n is the set of all binary trees over the alphabet $A'_n \subseteq A_n$.

Proposition 4.3. *B_n^μ is Σ_n-universal. B_n^ν is Π_n-universal.*

Proof. Let $F : T_n \to T'_n$ be defined by

- F is the identity on A'_n,
- for $i < n$, $F(c_i(x_1, x_2)) = c_n(d_i(x_1, x_1), d_i(x_2, x_2))$.

We establish a correspondence between the runs of $\mathcal{W}_n$ on t and the runs of $\mathcal{B}_n$ on $F(t)$ as follows.

- We apply the rule $(q, d_i(t_1, t_2)) \to (q_i, t_j)$ in $\mathcal{W}_n$ if and only if we apply the rule $(q, d_i(F(t_1), F(t_2))) \to (q_i, F(t_j))$ in $\mathcal{B}_n$ (with $j = 1$ or 2).

[1] F is non expansive if $\forall t, t', \Delta(F(t), F(t')) \leq \Delta(t, t')$.

- We apply the rule $(q, c_n(t_1, t_2)) \to (q_n, t_1) \wedge (q_n, t_2)$ in $\mathcal{W}_n$ if and only if we apply the rule $(q, c_n(F(t_1), F(t_2))) \to (q_n, F(t_1)) \wedge (q_n, F(t_2))$ in $\mathcal{B}_n$.
- We apply the rule $(q, c_i(t_1, t_2)) \to (q_i, t_1) \wedge (q_i, t_2)$ in $\mathcal{W}_n$ (with $i < n$) if and only if we apply in $\mathcal{B}_n$ the derivation

$$\begin{aligned} & (q, c_n(d_i(F(t_1), F(t_1)), d_i(F(t_2), F(t_2)))) \\ \to \quad & (q_n, d_i(F(t_1), F(t_1))) \wedge (q_n, d_i(F(t_2), F(t_2))) \\ \to \quad & (q_i, F(t_1)) \wedge (q_i, F(t_2)). \end{aligned}$$

This correspondence preserves the set of states that appears infinitely often on any branch, except that in $\mathcal{B}_n$ we may add infinitely often q_n. But since $r(q_n) = n$, the minimal rank of these sets remains unchanged. Therefore, one of two corresponding runs is θ-accepting if and only if so is the other one. Since B_n^μ is in Σ_n, we get, by Corollary 4.2:

Corollary 4.4. *B_n^μ is Σ_n-hard.*

Example: Lenzi's tree languages.

The Lenzi's formulas are formulas over n-ary trees. Translating these formulas into alternating parity automata over binary trees that encode n-ary trees, we get the following definition of Lenzi's automata $\mathcal{L}_n$, for $n \geq 2$.

The alphabet of $\mathcal{L}_n$ is $A_n'' = \{a_i \mid 0 \leq i \leq n\}$ and we denote by T_n'' the set of all trees over this alphabet. Its set of states is $\{q_i \mid 1 \leq i \leq n\}$ with $r(q_i) = i$, and three additional states $\{q_0, q_a, q_r\}$ such that

- q_0 accepts only trees in the form $a_0(t_1, t_2)$, its rank does not matter since it will occur at most once on any branch of a run,
- q_a, of rank 2, accepts any tree,
- q_r, of rank 1, accepts no tree.

It is not difficult to write down the rules implementing these conditions.

The other rules are, for $i = 1, \ldots, n$,

$$\delta(q_i, a_i) = \begin{cases} (q_{i+1}, 1) \vee (q_{i-1}, 2) & \text{if } i \text{ is odd,} \\ (q_{i+1}, 1) \wedge (q_{i-1}, 2) & \text{if } i \text{ is even,} \end{cases}$$

where we assume that $q_{i+1} = q_n$ whenever $i \geq n$.

When $i \neq j$ the rule $\delta(q_j, a_i)$ is not defined. We assume that in this case the automaton rejects the tree, *i.e.* $\delta(q_j, a_j) = (q_r, 1) \wedge (q_r, 2)$.

Let $L_n^\theta = L_{q_1}^\theta(\mathcal{L}_n)$.

The following proposition shows that $L_{n+2}^\mu \in \Sigma_{n+2} - \Pi_n$ and that $L_{n+2}^\mu \in \Pi_{n+2} - \Sigma_n$. They are not exactly hard in the sense above (*i.e.*, in $\Sigma_{n+2} - \Pi_{n+2}$ or in $\Pi_{n+2} - \Sigma_{n+2}$). However they also provide evidence for the strictness of the alternation hierarchy: if the hierarchy is not strict, there exists n such that $\Pi_n = \Sigma_n = \Sigma_{n+2}$.

Proposition 4.5. *L^{μ}_{n+2} is Σ_n-universal. L^{ν}_{n+2} is Π_n-universal.*

Proof. We recursively define a family of non expansive mappings $F_i : T_n \to T''_{n+2}$, for $i = 1, \ldots, n+2$, such that $t \in W^{\theta}_n \Leftrightarrow F_i(t) \in L^{\theta}_i(\mathcal{L}_{n+2})$, and we take $F = F_1$.

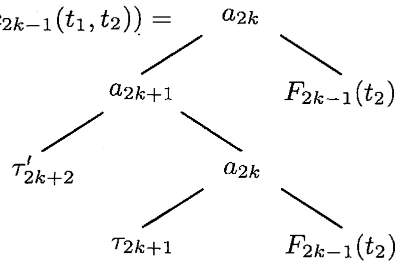
First we show that for each $i = 0, \ldots, n+2$ there exist a tree $\tau_i \in L^{\theta}_i(\mathcal{L}_{n+2})$ and a tree $\tau'_i \notin L^{\theta}_i(\mathcal{L}_{n+2})$. Obviously, τ'_i is an arbitrary tree whose the root is not a_i. τ_0 is an arbitrary tree whose the root is a_0, $\tau_{2i+1} = a_{2i+1}(\tau_{2i}, \tau_{2i})$, and τ_{2i+2} is the unique tree t such that

$$t = \begin{cases} a_{2i+2}(t, \tau_{2i+1}) & \text{if } 2i+2 = n+2, \\ a_{2i+2}(a_{2i+3}(\tau'_{2i+4}, t), \tau_{2i+1}) & \text{otherwise,} \end{cases}$$

with the convention, that will also be used in the sequel, that if an index becomes strictly greater than $n+2$ it has to be understood as having the value $n+2$, or in other words, each index i has to be read as $\min(i, n+2)$.

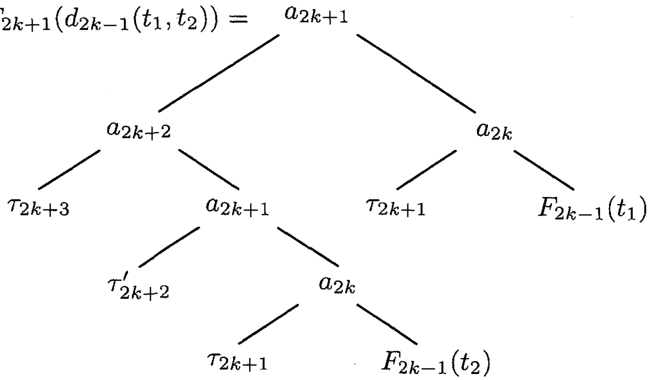
Now, for $2k-1 \leq n$, we define

$F_{2k}(c_{2k-1}(t_1, t_2)) =$



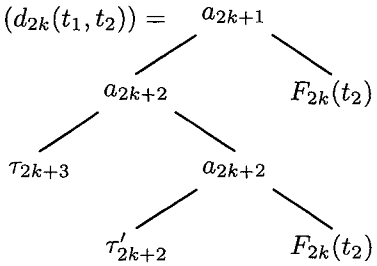
and

$F_{2k+1}(d_{2k-1}(t_1, t_2)) =$



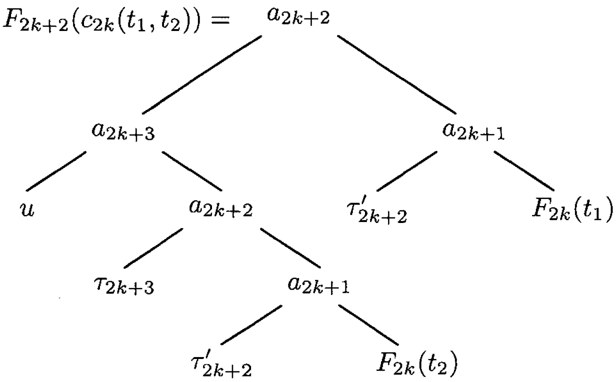
For $2k \leq n$, we define

$F_{2k+1}(d_{2k}(t_1, t_2)) =$



and

$F_{2k+2}(c_{2k}(t_1, t_2)) =$



where $u = \begin{cases} \tau_{n+2} & \text{if } 2k+2 = n+2, \\ \tau'_{2k+4} & \text{otherwise.} \end{cases}$

Now, for $T \in T_n$, let

$$\gamma(t) = \begin{cases} 2k+2 & \text{if the root of } t \text{ is } c_{2k} \text{ or } d_{2k+1}, \\ 2k+1 & \text{if the root of } t \text{ is } c_{2k-1} \text{ or } d_{2k}. \end{cases}$$

We have just defined $F_i(t)$ for $i = \gamma(t)$. For $i \neq \gamma(t)$, we set

$$F_{2j}(t) = \begin{cases} a_{2j}(\tau_{2j+1}, F_{2j-1}(t)) & \text{if } 2j > \gamma(t), \\ a_{2j}(F_{2j+1}(t), \tau_{2j-1}) & \text{if } 2j < \gamma(t), \end{cases}$$

$$F_{2j+1}(t) = \begin{cases} a_{2j+1}(\tau'_{2j+2}, F_{2j}(t)) & \text{if } 2j+1 > \gamma(t), \\ a_{2j+1}(F_{2j+2}(t), \tau'_{2j}) & \text{if } 2j+1 < \gamma(t). \end{cases}$$

We prove that $t \in L^\theta_{q_i}(\mathcal{W}_n) = W^\theta_n \Leftrightarrow F_i(t) \in L^\theta_{q_i}(\mathcal{L}_n)$. We remark that $F_i(b_k(t_1, t_2))$, for $b_k = c_k, d_k$, has the form $t(F_k(t_1), F_k(t_2))$. It is easy to see that an accepting run from state q_i on $F_i(t)$ is made up of accepting runs from q_j on the subtrees τ_j, of an accepting run from q_k on $F_k(t_1)$ and (or) on $F_k(t_2)$, and that on the path from the root to $F_k(t_1)$ and (or) $F_k(t_2)$, the only states that appears are q_i, q_k and some q_j with $j \geq \min(i, k)$.

5. Weak alternating automata

An alternating automaton $\mathcal{A} = \langle A, Q, \delta, n, r\rangle$ is said to be *weak* if δ has the following additionnal property:

for all $q \in Q$ and for all $a \in A$, if q' occurs in $\delta(q, a)$, then $r(q') \leq r(q)$.

It is obvious that if $\mathcal{A}$ is weak, then its dual $\widetilde{\mathcal{A}}$ is weak too.

The *weak alternation-depth hierarchy* is defined in the same way as the alternation depth hierarchy: $w\Sigma_n$ (resp. $w\Pi_n$) is the family of binary tree languages in the form $L^\mu_{\mathcal{A}}(q)$ (resp. $L^\nu_{\mathcal{A}}(q)$) where $\mathcal{A}$ is a weak automaton of type n. Since the family of languages accepted by weak alternating automata is exactly the family of languages L accepted by non deterministic Büchi automata whose the complement is also accepted by a non deterministic Büchi automaton, and since the family of languages accepted by a non deterministic Büchi automaton is exactly Σ_2 [1, 3], we have

$$\bigcup_{n>0} (w\Sigma_n \cup w\Pi_n) = \Sigma_2 \cap \Pi_2.$$

A direct consequence of the definition of a weak automaton $\mathcal{A}$ is that on any branch of $G_{\mathcal{A}}(q, t)$ the sequence of indices of the letters c_i, d_i $(i = 1, \dots, n)$ occurring on this branch is decreasing. Therefore, in such a case, Proposition 2.1 can be restated as:

$$\text{for } \theta = \mu, \nu \text{ and } t \in T_A,\ t \in L^\theta_{\mathcal{A}}(q) \Leftrightarrow G_{\mathcal{A},q}(t) \in L^\mu_{w\mathcal{W}_n}(q_n),$$

where $w\mathcal{W}_n$ is a variant of the Walukiewicz's automaton $\mathcal{W}_n$ that takes into account this property of decreasing indices, *i.e.*,

- $\delta(q_j, c_i) = (q_i, 1) \wedge (q_i, 2)$ and $\delta(q_j, d_i) = (q_i, 1) \vee (q_i, 2)$ if $j \geq i$.
- $\delta(q_j, c_i) = \delta(q_j, d_i) = (q_1, 1) \vee (q_1, 2)$ if $j < i$.

Obviously, $w\mathcal{W}_n$ is weak, and the above diagonal construction allows us to prove that $L^\mu_{w\mathcal{W}_n}(q_n)$ is not in $w\Pi_n$ (resp. $L^\nu_{w\mathcal{W}_n}(q_n)$ is not in $w\Sigma_n$). This is a new proof of a Mostowski's result [11]

Theorem 5.1 (Mostowski). *The alternation-depth hierarchy of weak automata is strict.*

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