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SUBELLIPTIC ESTIMATES

by J. J. KOHN

Consider the mapping $Q : C_o^\infty(\mathbb{R}^n)^m \times C_o^\infty(\mathbb{R}^n)^m$ given by

(1) $$Q(u,v) = \sum_{i,j=1}^{m} \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} (a_{\alpha\beta}^{ij} D^\alpha u_i, D^\beta v_j),$$

with $a_{\alpha\beta}^{ij} \in C^\infty(\mathbb{R}^n)$, here $(\ ,\ )$ denotes the L_2-inner product on $\mathbb{R}^n$. We will assume that

(2) $$Q(u,v) = \overline{Q(v,u)}$$

<u>Definition</u> : Q is <u>subelliptic</u> at $(x_o,\eta_o) \in \mathbb{R}^n \times (\mathbb{R}^n - \{0\})$ if there exist positive constants C, C' and ε and a classical symbol $p(x,\eta)$ of order zero (i.e. $p \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n - \{0\})$ and $p(x,t\eta) = p(x,\eta)$ for $t > 0$) such that $p(x,\eta) = 1$ in a conic neighborhood of (x_o,η_o) and

(3) $$\| Pu \|^2 \leq CQ(u,u) + C'\| u \|^2$$

for all $u \in C_o^\infty(\mathbb{R}^n)^m$, where P is pseudo-differential operator with symbol $p(x,\eta)$ and $\| f \|_\varepsilon^2 = \Sigma \| f_j \|_\varepsilon^2$, denotes the Sobolev ε-norms.

It is shown in [1] that subelliptic estimates imply regularity of solutions of the satisfying

(4) $$Q(u,v) = (f,v)$$

for all $v \in C_o^\infty(\mathbb{R}^n)^m$. Here we will outline a microlocal version of the method for obtaining sufficient conditions for subellipticity which is developped in [2]. The advantages of the present treatment is that it can be used to study C-R structures and that it gives results, in at least some cases, when pseudo-convexity fails.

The principal example of the Q comes from the C-R structure described as follows. Let $n = 2k + 1$ and let $L_1,\dots,L_k$ be complex valued vector fields on $\mathbb{R}^n$ such that $[L_i,L_j] = \sum_{s=1}^{k} b_{ij}^s L_s$ and such that $L_1,\dots,L_k$, $\bar{L}_1,\dots,\bar{L}_k$ are linearly independent. We define $Q : C_o^\infty(\mathbb{R}^n)^k \times C_o^\infty(\mathbb{R}^n)^k \to \mathbb{C}$ by

$$Q(u,v) = \sum_{i<j} (\bar{L}_i u_j - \bar{L}_j u_i, \bar{L}_i v_j - \bar{L}_j v_i) + (\sum_i L_i u_i, \sum_j L_j v_j). \tag{5}$$

This quadratic form controls the regularity of the system $L_i W = f_i$, $i = 1,\dots,k$.

Another example of a Q which can be treated by the methods which we describe below comes from the Hörmander operator $\sum_{j=1}^{k} X_j^2$, when the X_j are real first order pseudo-differential operators in $\mathbb{R}^n$ and $Q : C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n) \to \mathbb{C}$ is given by

$$Q(u,v) = \sum_{j=1}^{k} (X_j u, X_j v) . \tag{6}$$

Here subellipticity of Q implies hypoellipticity of the Hörmander operator.

<u>Definition</u> : If Q is given by (1) and if $p(x,\eta)$ is a C^∞ function defined in a conic neighborhood of $(x_o,\eta_o) \in \mathbb{R}^n - \{0\})$ which is homogeneous of zero order in η (i.e. $p(x,\eta) = p(x,t\eta)$ for $t > 0$), we say that p is a <u>subelliptic multiplier</u> for Q at (x_o,η_o) if there exists a pseudo-differential operator P such that the symbol of P equals p in a conic neighborhood of (x_o,η_o) and such that there exist constants C, C' and ε so that (3) is satisfied for all $u \in (C_o^\infty(\mathbb{R}^n))^m$. We say that two subelliptic multipliers are equivalent if they are equal on some conic neighborhood of (x_o,η_o). We denote the set of equivalence classes of subelliptic multipliers by $\mathcal{P}(Q;(x_o,\eta_o)) = \mathcal{P}$.

<u>Proposition</u> : $\mathcal{P} = \mathcal{P}(Q;(x_o,\eta_o))$ has the following properties

(a) $\mathcal{P}$ is an ideal in the ring $\mathcal{R}$. Where $\mathcal{R}$ denotes the ring of real-valued C^∞ functions defined in conic neighborhoods of (x_o,η_o) which are homogeneous of order zero.

(b) $\sqrt[\mathbb{R}]{\mathcal{P}} = \mathcal{P}$. Here $\sqrt[\mathbb{R}]{\mathcal{P}}$ denotes the real radical of $\mathcal{P}$, that is if $g \in \mathcal{R}$ then $g \in \sqrt[\mathbb{R}]{\mathcal{P}}$ if and only if there exists an integer m and $p \in \mathcal{P}$ such that $|g|^m \leq |p|$ in a conic neighborhood of (x_o,η_o).

Clearly subellipticity of Q at (x_o,η_o) is equivalent to $1 \in \mathcal{P}(Q;(x_o,\eta_o))$. The proposition given below shows how certain types of a priori estimates lead to conditions which imply that $1 \in \mathcal{P}$.

<u>Theorem</u> : Suppose that $A_1,\dots,A_N$ are pseudo-differential operators with symbols $a_1,\dots,a_N \in \mathcal{P}(Q;(x_o,\eta_o))$ such that there exist C and C' so that

$$\sum_1^N \|A_j Pu\|_1^2 \leqslant CQ(u,u) + C'\|u\|^2 \tag{7}$$

for all $u \in (C_o^\infty(\mathbb{R}^n))^m$. Suppose further that, for $i = 1,\dots,M$, $B_i : C_o^\infty(\mathbb{R}^n)^m \to C_o^\infty(\mathbb{R}^n)$ are first order differential operators such that

$$\|B_i Pu\|^2 \leqslant CQ(u,u) + C'\|u\|^2 \tag{8}$$

for all $u \in C_o^\infty(\mathbb{R}^n)^m$ and

$$\|B_i' Pv\|^2 \leqslant C \sum_1^N \|A_j v\|_1^2 + C'\|v\|^2 \tag{9}$$

for all $v \in C_o^\infty(\mathbb{R}^n)$. Here P denotes a zero order pseudo-differential operator whose symbol equals one in a conic neighborhood of (x_o,η_o). The operators B_i can be written as

$$B_i u = \sum_{k=1}^m B_i^k u_k \tag{10}$$

and B_i' is then given by

$$B_i' v = ((B_i^1)'v,\ (B_i^2)'v,\dots,(B_i^m)'v), \tag{11}$$

where $(B_i^k)'$ denotes the formal adjoint of B_i^k.

Suppose that $p_1,\dots,p_m \in \mathcal{P}(Q,(x_o,\eta_o))$ then for each i we have $\det\{p_j, \sigma(B_i^k)\} \in \mathcal{P}(Q,(x_o,\eta_o))$, here det denotes the determinant of the $m \times m$ matrix, $\{p,q\} = p_x q_\eta - p_\eta q_x$ denotes the Poisson bracket and $\sigma(B_i^k)$ denotes the symbol of B_i^k.

<u>Corollary</u> : Suppose that Q satisfies the hypothesis of the theorem at (x_o,η_o). Let $\mathcal{P}_o \subset \mathcal{P}_1 \subset \dots \subset \mathcal{P}_r \subset \mathcal{P}(Q;(x_o,\eta_o))$ be the ideals defined as follows

$$\mathcal{P}_o = \sqrt[\mathbb{R}]{(a_1,\dots,a_N)}\ , \tag{12}$$

where $(a_1,\dots,a_N)$ denotes the ideal in $\mathcal{R}$ generated by the a_j. For $r > 0$ we define

(13) $$\mathcal{P}_r = \sqrt[\mathbb{R}]{(\mathcal{P}_{r-1}, \{\det\{p_j, \sigma(B_i^k)\} \text{ for all } p_1, p_m \in \mathcal{P}_{r-1})}.$$

Then $1 \in \mathcal{P}_r$ implies that Q is subelliptic at (x_o, η_o).

Returning now to the C-R structures, with Q defined by (5), let γ be a differential form such that in a neighborhood of $x_o \in \mathbb{R}^n$ we have $\langle\gamma, L_i\rangle = \langle\gamma, \bar{L}_i\rangle = 0$ with $\gamma = -\bar{\gamma}$ and $|\gamma| = 1$. Then γ is determinated uniquely up to sign. Let $c_{ij} = \langle \gamma, [L_i, \bar{L}_j]\rangle$ this is the Levi-form and we say that the C-R structure is pseudo-convex at γ if $(C_{ij}) \geqslant 0$.

Let U be a neighborhood of x_o and V^+ be a conic neighborhood of $(x_o, [\gamma]_{x_o})$ such that V^+ is also a conic neighborhood of $(x, [\gamma]_x)$ for all $x \in U$. Let $V^- = \{(x,\eta) \mid (x,-\eta) \in V^+\}$ and let U' be a neighborhood of x_o with $\bar{U}' \subset U$. Consider zero order pseudo-differential operators P^o, P^+ whose symbols $p^o(x,\eta)$, $p^+(x,\eta)$ and $p^-(x,\eta)$ are zero for $x \notin U'$ and $p^o(x,\eta) = 0$ if $(x,\eta) \in V^+ \cup V^-$, $p^+(x,\eta) = 0$ if $(x,\eta) \in V^-$ and $p^-(x,\eta) = 0$ if $(x,\eta) \in V^+$. We always have

(14) $$\|P^o u\|_1^2 \leqslant CQ(u,u), \quad \text{for all } u \in C_o^\infty(\mathbb{R}^n)^k$$

Furthermore if $(c_{ij}) \geqslant 0$ on U then

(15) $$\sum_{i,j=1}^{k} \|\bar{L}_j P^+ u_i\|^2 \leqslant CQ(u,u)$$

and

(16) $$\sum_{i,j=1}^{k} \|L_j P^- u_i\|^2 \leqslant CQ(u,u).$$

To apply our theorem at $(x_o, [\gamma]_{x_o})$ we let

(17) $$A_j = \Lambda^{-1}\bar{L}_j \quad \text{for } j = 1,\dots,k,$$

where Λ denotes the square root of the Laplacian. We define

$$B : C_o^\infty(\mathbb{R}^n)^k \to C_o^\infty(\mathbb{R}^n)$$

by

(18) $$Bu = \sum_{i=1}^{k} L_i u_i \; .$$

The theorem then implies that $\det(C_{ij}(x)) \in \mathcal{A}(Q,(x, [\gamma]_x)$ for $x \in U'$. Applying the corollary we define ideals of germs of C^∞ functions at x_o by

(19) $$I_1^+ = \sqrt[\mathbb{R}]{(\det(C_{ij}))}$$

and inductively

(20) $$I_r^+ = \sqrt[\mathbb{R}]{(I_{r-1}^+, \det(M_{r-1}^+))},$$

when M_{r-1} runs through all $k \times k$ submatrices of the infinite matrix

(21) $$\begin{pmatrix} C_{11} \cdots\cdots\cdots C_{1k} \\ C_{k1} \cdots\cdots\cdots C_{kk} \\ L_1(f) \cdots\cdots L_k(f) \\ L_1(g) \cdots\cdots L_k(g) \\ \vdots \qquad\qquad \vdots \end{pmatrix}$$

when $f,g,\ldots$ run through all the elements of I_{r-1}^+.

Hence $1 \in I_r^+$ implies subellipticity at $(x_o, [\gamma]_{x_o})$. Similarly to apply the theorem at $(x_o, -[\gamma]_{x_o})$ we set

(22) $$A_j = \Lambda^{-1} L_j \qquad \text{for } j = 1,\ldots,k$$

and $B_{ij} : C_o^\infty(\mathbb{R}^n)^k \to C_o^\infty(\mathbb{R}^n)$ is defined by

(23) $$B_{ij} u = \bar{L}_i u_j - \bar{L}_j u_i \quad \text{for } 1 \leqslant i < j \leqslant k.$$

The theorem then applies only when $k \geqslant 2$ (otherwise there are no B_{ij} and subellipticity does not hold). We then define ideals of germs of C^∞ functions at x_o by

(24) $$I_1^- = \sqrt[\mathbb{R}]{\left(\det\begin{pmatrix} C_{i_1 i_1} & C_{i_1 i_2} \\ C_{i_2 i_1} & C_{i_2 i_2} \end{pmatrix}\right.}$$

and

$$I_r^- = \sqrt[\mathbb{R}]{(I_{r-1}^-, \det(M_{r-1}^-))}, \tag{25}$$

where the M_r^- run through the 2×2 submatrices of (21) with $f,g,\ldots \in I_r^-$. Hence we see that $1 \in I_r^-$ implies subellipticity at $(x_o, -[\gamma]_{x_o})$.

I would conjecture that the conditions $1 \in I_r^+$ and $1 \in I_r^-$, for some r, are also necessary for subellipticity, this is true in the case of real analytic C-R structures.

The method outlined above will also give sufficient conditions in case the Levi form (C_{ij}) is a direct sum in all of U of a non negative semi definite and a non position semi definite form.

In the case of the Hörmander equation, where Q is given by (6). We set $A_j = \Lambda^{-1}X_j$ and $B_j = X_j$ and we obtain the Hörmander condition for subellipticity by applying the theorem.

An example which is related both to the Hörmander equation and to C-R structures is given by a first order pseudo differential operator L on $\mathbb{R}^n$. Here we consider $Q : C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n)$ given by

$$Q(u,u) = \|Lu\|^2 . \tag{26}$$

The subellipticity of this Q was initiated by Nirenberg and Treves and then taken up by Egorov and Hörmander (see [3]). It is known that a necessary condition for subellipticity is that on the characteristic of L we have

$$\{\sigma(L), \quad \sigma(L^*)\} \geqslant 0. \tag{27}$$

Furthermore, Egorov has shown that if subellipticity holds at (x_o, η_o) than

$$\|\overleftarrow{L}Pu\| \leqslant C(\|Lu\| + \|u\|). \tag{28}$$

Hence, if (28) holds problem is reduced to the case of (6) with $Q(u,u) = \|X_1 u\|^2 + \|X_2 u\|^2$ where $L = X_1 u + \sqrt{-1}X_2 u$.

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