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Analyticity for certain solutions of nonhypoelliptic differential operators  
on the Heisenberg group

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We consider left invariant differential operators on the Heisenberg group  $G$  with Lie algebra  $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$   $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ ,  $[\mathfrak{g}_2, \mathfrak{g}] = 0$ , where  $X_1, X_2, \dots, X_{2n}$  is a basis of  $\mathfrak{g}_1$  and  $T$  a basis of  $\mathfrak{g}_2$ . Let  $p$  be an elliptic, homogeneous non-commuting polynomial in  $2n$  variables, i.e.  $p(\xi_1, \xi_2, \dots, \xi_{2n}) \geq C |\xi|^d$ ,  $C > 0$ . An operator of the form  $L = p(X_1, X_2, \dots, X_{2n})$  will be said to be homogeneous and elliptic in the generating directions. It is known that  $L$  is analytic-hypoelliptic and  $C^\infty$  hypoelliptic if and only if the  $L^2$  nullspace of  $L$  is nontrivial (see [10], [9], [7], [8], and [5]). The results announced here show that even if  $L$  is not hypoelliptic, it has a left inverse, modulo the projection onto its kernel, which preserves real analyticity, locally. More precisely, our main result is the following.

Theorem 1. Let  $L$  be a homogeneous, left invariant differential operator on the Heisenberg group  $G$  elliptic in the generating directions.  
Then there are distributions  $k_1$  and  $k_2$  such that

$$(1) \quad Lf * k_1 = f - \Pi_1 f$$

$$(2) \quad L(f * k_2) = f - \Pi_2 f$$

for  $f \in C_0^\infty(G)$ , where  $\Pi_1$  and  $\Pi_2$  are the orthogonal projections onto the  $L^2$  nullspaces of  $L$  and its adjoint  $L^*$ , respectively, and

(\*) denotes group convolution. Furthermore, the operators  $f \rightarrow f * k_i$  and  $f \rightarrow \Pi_i f$ ,  $i = 1, 2$ , all preserve analyticity, locally.

Corollary. If  $u$  and  $f$  are smooth functions of compact support on  $G$  and

$$(3) \quad Lu = f \quad \text{in} \quad U,$$

where  $U$  is an open set, then  $u_1 = (I - \Pi)u$  is analytic in every subset of  $U$  where  $f$  is, and  $u_1$  also satisfies (3).

In the special case where  $L = \square_b^0$ , the boundary Laplacian operator acting on 0-forms (see [2]) the analog of Theorem 1 was given by Greiner, Kohn, and Stein [4], who derived explicit formulas for  $k_i$  and  $\Pi_i$ . The analyticity of the projections  $\Pi_i$  was proved by Geller [3], who also proved the existence of distributions  $k_i$ , satisfying (1) and (2) and preserving local smoothness. The general result was conjectured by Stein [3]. See also Melin [6] for related results.

To prove Theorem 1, we use a standard reduction to the case where  $L$  is self adjoint and of high degree, in addition to satisfying the conditions of Theorem 1. The following is partly based on an idea of Beals and Greiner [1].

Theorem 2. Let  $L$  be a self-adjoint differential operator of high homogeneous degree  $d$  satisfying the conditions of Theorem 1. Then there is a closed contour  $\Gamma$  around 0 in  $\mathcal{C}$  such that  $L_\alpha = L - \alpha(-iT)^{d/2}$  is hypoelliptic for all  $\alpha \in \Gamma$ . There exist distributions  $k_\alpha$ ,  $\alpha \in \Gamma$ ,

such that  $L_\alpha k_\alpha = \delta$  and for any  $f \in C_0^\infty(G)$  and any multi-index  $\beta$  the function  $\alpha \rightarrow \|D^\beta(f * k_\alpha)\|_{L^\infty}$  is bounded for  $\alpha$  on  $\Gamma$ . Hence define  $K, S: C_0^\infty(G) \rightarrow C^\infty(G)$  by

$$Kf = \frac{1}{2\pi i} \int_\Gamma \alpha^{-1} f * k_\alpha d_\alpha$$

and

$$Sf = \frac{1}{2\pi i} \int_\Gamma T^{d/2} f * k_\alpha d_\alpha$$

Then

$$(4) \quad LKf = K * Lf = \mathcal{F} - Sf, \quad f \in C_0^\infty(G),$$

and  $S = \prod$ , the orthogonal projection onto the  $L^2$  kernel of  $L$ .

Furthermore,  $K$  and  $S$  preserve real analyticity, locally.

The proof of Theorem 2 first requires constructing the  $k_\alpha$ . For this we follow the method given by Métivier [7], checking that the  $k_\alpha$  so obtained vary well with  $\alpha$ . The first identity in (4) follows from the self adjointness of  $L$ , while the second is immediately obtained by writing  $L = L_\alpha + \alpha(-iT)^{d/2}$ . The proof that  $S = \prod$  is accomplished by applying the irreducible unitary representations to both operators. Then the equality reduces to a resolvent identity, and the original identity follows by the Plancherel formula for  $G$ .

Finally, to show that  $K$  and  $S$  preserve real analyticity, it suffices to show that the operators  $f \rightarrow f * k_\alpha$ , each of which preserves real analyticity, satisfy estimates uniform in  $\alpha$  for  $\alpha$  on  $\Gamma$ . For this, we use the methods of the second author [9] to estimate the  $L^2$  norms of derivatives of  $f * k_\alpha$ , checking again that the constants obtained may be chosen independent of  $\alpha$ .

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