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A GENERAL CLASS OF GEVREY-TYPE PSEUDO DIFFERENTIAL OPERATORS

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Recently much attention has been paid to the study of new classes of analytic and Gevrey-type pseudo differential operators; see for example Matsuzawa [8], Iftimie [5], Bolley -Camus-Métivier [2].

We shall consider here symbols $a(x,\xi)$ of general Gevrey type for which

$$(1) \qquad |D_x^\alpha D_\xi^\beta a(x,\xi)| \leqslant C^{|\alpha|+|\beta|+1} \alpha!\beta! \ \varphi(\xi)^{m-|\alpha|} \psi(\xi)^{m'+|\alpha|-|\beta|}$$

<u>if</u> $C'|\beta| \leqslant \varphi(\xi)$.

The weight functions φ, ψ are continuous in R^n and satisfy for suitable positive constants ε, ε' independent of $\xi, \eta \in R^n$:

$$(2) \quad \varepsilon(1+|\xi|)^{\varepsilon} \leqslant \varphi(\xi) \leqslant \varepsilon' \psi(\xi)$$

$$(3) \quad \varepsilon \leqslant \varphi(\xi)\ \varphi(\eta)^{-1} \leqslant \varepsilon' , \qquad \varepsilon \leqslant \psi(\xi)\ \psi(\eta)^{-1} \leqslant \varepsilon'$$

<u>if</u> $|\xi-\eta| \leqslant \varepsilon\ \psi(\xi)$.

To these conditions, which are quite common for general pseudo differential operators, we add the technical assumption:

(4) <u>for every</u> δ <u>there exists</u> δ' <u>such that</u> $\psi(\eta) \leqslant \delta|\xi-\eta|$ <u>implies</u> $\varphi(\eta)\ \psi(\xi) \leqslant \delta'\ |\xi-\eta|\ \varphi(\xi)$.

From the C^∞ point of view our symbols can be regarded as elements of a class of Beals [1] $S^{\lambda}_{Ⓗ,\vartheta}$, with Ⓗ $= \psi$, $\vartheta = \varphi/\psi$. The reason why we prefer here to refer to the function $\varphi = \vartheta\psi$ for the estimates in (1) is that a peculiar property of the pseudo differential operator

$$(5) \qquad a(x,D)f(x) = (2\pi)^{-n} \int e^{ix\xi}\ a(x,\xi)\ \hat{f}(\xi)\ d\xi$$

associated with $a(x,\xi)$ turns out to be the continuity from

G_ψ to G_φ, where G_ψ, G_φ are the inhomogeneous Gevrey classes related to the weight functions ψ, φ, respectively.

Let us begin by giving a general definition of such classes in terms of Fourier transform. Let φ (or ψ) be a weight function as in (2), (3). More generally, let $\lambda: R^n \to R_+$ be Lipschitzian, in the sense that $|\lambda(\xi)-\lambda(\eta)| \leqslant C|\xi-\eta|$ for some constant C independent of ξ,η, and assume also $\varepsilon(1+|\xi|)^\varepsilon \leqslant \lambda(\xi)$ for some $\varepsilon > 0$. Let X be open in R^n.

Definition 1. We say that $f \in \mathcal{D}'(X)$ is of class G_λ at $x_o \in X$ if there is a neighborhood U of x_o, $U \subset X$, and a bounded sequence $f_j \in \mathcal{E}'(X)$ such that $f = f_j$ in U and

$$(6) \qquad |\hat{f}_j(\xi)| < c(cj/\lambda(\xi))^j, \qquad j = 1,2,.. \quad .$$

We denote by $G_\lambda(X)$ the set of all $f \in \mathcal{D}'(X)$ which are of class G_λ at every $x_o \in X$.

When $\lambda(\xi) = (1+|\xi|)^\rho$, $0 < \rho \leqslant 1$, $G_\lambda(X)$ is the standard class $G^{1/\rho}(X)$ of all the functions $f \in C^\infty(X)$ which satisfy in every $K \subset\subset X$ the estimates

$$(7) \qquad |D^\alpha f(x)| < C^{|\alpha|+1} (\alpha!)^{1/\rho}$$

(cf. Hörmander [4], Proposition 2.4).

In particular for $\lambda(\xi) = 1 + |\xi|$ we have $G_\lambda(X) = \mathcal{A}(X)$, the set of all the real analytic functions in X.

Classes $G_\lambda(X)$ with inhomogeneous λ have been considered by several authors under different definitions; see for example Liess [6] and the references there. The advantage of the present definition is that it can be microlocalized in a natural way, adapting the procedure used by Rodino [10] in the C^∞ framework. Fix $\Gamma \subset R^n_\xi$ and set for $\varepsilon > 0$

(8) $$\Gamma_{\varepsilon\lambda} = \{\xi \in R^n,\ \text{dist}\ (\xi,\Gamma) < \varepsilon\ \lambda(\xi)\}.$$

Definition 2. We shall say that f is G_λ- smooth at $\{x_o\} \times \Gamma$ and we shall write formally $WF_\lambda\ f \cap (\{x_o\} \times \Gamma) = \phi$ if the estimates (6) are satisfied in $\Gamma_{\varepsilon\lambda}$, for a sufficiently small $\varepsilon > 0$.

It is natural then to introduce the space of the "microfunctions" at $\{x_o\}\times \Gamma$.

Definition 3. We denote by $C^\infty_{x_o,\Gamma,\lambda}$ the factor space $C^\infty_{x_o}/_\sim$, where $C^\infty_{x_o}$ is the set of the germs of C^∞ functions defined near x_o and $f \sim g$ in $C^\infty_{x_o}$ iff $WF_\lambda (f-g) \cap (\{x_o\}\times \Gamma) = \phi$.

It is convenient in certain applications to use also a different kind of microlocalization. Precisely, set for $\varepsilon > 0$

(8)' $$\Gamma_{[\varepsilon\lambda]} = \{\xi \in R^n,\ \lambda(\xi-\eta) < \varepsilon\ \lambda(\xi)\ \text{for some}\ \eta \in \Gamma\}.$$

Definition 2'. We shall say that f in strongly G_λ- smooth at $\{x_o\} \times \Gamma$ and we whall write formally $WF^*_\lambda\ f \cap (\{x_o\} \times \Gamma) = \phi$ if the estimates (6) are satisfied in $\Gamma_{[\varepsilon\lambda]}$, for a sufficiently small $\varepsilon > 0$.

For example, if Γ is the halfray generated by $\xi_o \neq 0$ and $\lambda(\xi) = (1 + |\xi|)^\rho$, $0 < \rho \leq 1$, then $WF^*_\lambda\ f \cap (\{x_o\} \times \Gamma) = \phi$ means that (x_o,ξ_o) is not in the Gevrey wave front set $WF_{1/\rho} f$ of Hörmander [4].

Note that strong G_λ-smoothness at $\{x_o\} \times \Gamma$ implies G_λ-smoothness there, but the converse is not true in general.

Let us now return to pseudo differential operators and give a precise definition of our classes from the microlocal point of view.

Assume φ and ψ satisfy the conditions (2), (3), (4). Let X be open in R^n_x and fix $\Gamma \subset R^n_\xi$.

Definition 4. We define $S^{m,m'}_{\varphi,\psi}(X,\Gamma)$ to be the set of all $a(x,\xi) \in C^\infty(X\times\Gamma)$ which can be extended for some $\varepsilon > 0$ to functions in $C^\infty(X\times\Gamma_{\varepsilon\psi})$ such that (1) is satisfied with suitable positive constants C, C' independent of $x \in X, \xi \in \Gamma_{\varepsilon\psi}$.

A symbol $a(x,\xi) \in S^{m,m'}_{\varphi,\psi}(X,\Gamma)$ can be further extended to a function $\tilde{a}(x,\xi) \in C^\infty(X\times R^n)$, by cutting off in the ξ variables, and $\tilde{a}(x,D)$ from (5) is then defined as a map from $C^\infty_o(X)$ to $C^\infty(X)$. The continuity property can now be expressed in the following microlocal form.

Theorem 5. Let $a(x,\xi)$ be in $S^{m,m'}_{\varphi,\psi}(X,\Gamma)$, and take $x_o \in X$, $\Lambda \subset \Gamma$. Then $\tilde{a}(x,D)$ defines by factorization an operator

(9) $$a(x,D) : C^\infty_{x_o,\Lambda,\psi} \to C^\infty_{x_o,\Lambda,\varphi}$$

which depends only on a and not also on the extensions $\tilde{a}$ of a.

The symbolic calculus for the operators $a(x,D)$ in (9) follows the lines of the calculus of the C^∞-general pseudo differential operators (cf. Beals [1]), with some evident complications in the estimates due to the factor $C^{|\alpha|+|\beta|+1}\alpha!\beta!$ which we expect in (1). From Theorem 5 and from symbolic calculus one deduces by means of a standard argument the following result on existence of parametrices.

Theorem 6. Consider $a(x,\xi) \in S^m_\psi(X,\Gamma) = S^{o,m}_{\psi,\psi}(X,\Gamma) \subset S^{o,m}_{\varphi,\psi}(X,\Gamma)$ and fix $x_o \in X$, $\Lambda \subset \Gamma$. Assume there exist a neighborhood U of x_o, $U \subset X$, real numbers m_1, m'_1 and positive costants c, c', ε, C

<u>such that</u>

(10) $\quad |a(x,\xi)| \geq c\ \varphi(\xi)^{m_1}\ \psi(\xi)^{m_1'}$ <u>for</u> $x \in U, \xi \in \Lambda_{\varepsilon\psi}$ <u>and</u>

$$|\xi| \geq C$$

(11) $\quad |D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C^{|\alpha|+|\beta|}\alpha!\beta!\,|a(x,\xi)|\ \varphi(\xi)^{-|\alpha|}\psi(\xi)^{|\alpha|-|\beta|}$

<u>for all</u> α <u>and all</u> x, ξ, β with $x \in U$, $\xi \in \Lambda_{\varepsilon\psi}$, $c'|\beta| \leq \varphi(\xi)$ <u>and</u> $|\xi| \geq C$.

<u>Then there is</u> $b \in S^{-m_1,-m'_1}_{\varphi,\psi}(U,\Lambda)$ <u>such that</u> $b(x,D) \circ a(x,D)$: $C^\infty_{x_o,\Lambda,\psi} \to C^\infty_{x_o,\Lambda,\varphi}$ <u>is the natural inclusion</u>. <u>In particular</u>, <u>for any fixed extension</u> $\tilde{a}$ <u>of</u> a, <u>we have that</u> $WF_\psi \tilde{a}(x,D)f \cap (\{x_o\} \times \Lambda) = \phi$ <u>implies</u> $WF_\varphi f \cap (\{x_o\} \times \Lambda) = \phi$.

When $A = a(x,D)$ is a linear partial differential operator with analytic coefficients in X there are some obvious simplifications in the statement; namely, if for every $K \subset\subset X$ we have for large $|\xi|$ and suitable costants $|a(x,\xi)| \geq c|\xi|^r$ and

(11)' $\quad |D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C^{|\alpha|+1}\alpha!\ |a(x,\xi)|\varphi(\xi)^{-|\alpha|}\psi(\xi)^{|\alpha|-|\beta|}$

then $Af \in G_\psi(X)$ implies $f \in G_\varphi(X)$ for every $f \in \mathcal{D}'(X)$; in particular all solutions of $Af = 0$ are in $G_\varphi(X)$.

A simple example is given by the hypoelliptic operators with constant coefficients $P = p(D)$. Let $\delta(\xi)$ be the distance from $\xi \in R^n$ to the surface $\{\zeta \in C^n, p(\zeta) = 0\}$, and set $\psi(\xi) = 1 + \delta(\xi)$. It is well known that

(12) $$|D_\xi^\beta p(\xi)| \leq C\ |p(\xi)|\ \psi(\xi)^{-|\beta|}$$

and (11)'is then satisfied with $\psi = \varphi$. We conclude that $P f \in G_\psi(X)$ implies $f \in G_\psi(X)$ for any $X \subset R^n$ and all $f \in \mathcal{D}'(R^n)$.

An example of operator for which $\varphi \neq \psi$ (that means a loss of Gevrey regularity for the solutions) is given by

(13) $$A = 1 + |x|^{2k} p(D) ,$$

where $p(D)$ is hypoelliptic and $p(\xi) \geq 0$; the estimates (11)' are satisfied for $\psi(\xi)$ as in preceding example and any $\varphi(\xi)$ for which $p(\xi) < (\psi(\xi)/\varphi(\xi))^{2k}$.

Theorem 6, as well as Theorem 5, can be restated in terms of strong G_λ-smoothness, according to Definition 2'. A relevant application is given by the choice $\psi(\xi) = (1+|\xi|)^\rho$, $\varphi(\xi) = (1+|\xi|)^{\rho-\delta}$, $0 \leq \delta < \rho \leq 1$, which corresponds to the operators in [2], [5], [8]. Since the related Gevrey wave front sets are invariant under canonical transformations, geometric invariant statements are possible in this case; for example, let us consider a classical analytic symbol $a(x,\xi) \sim \sum_{j=0}^{\infty} a_{m-j}(x,\xi)$ and assume the principal part $a_m(x,\xi)$ vanishes exactly of order k, $k \geq 2$, on an involutive manifold $\Sigma \subset T^*X \setminus 0$. Noting a'_{m-1} the subprincipal symbol, set for any $\gamma \in \Sigma$ and for any C^∞ vector field Y defined in a neighborhood of γ

(14) $$I_a(\gamma,Y) = (k!)^{-1} (Y^k a_m)(\gamma) + a'_{m-1}(\gamma).$$

<u>Theorem 7.</u> <u>Assume</u> $I_a(\gamma,Y) \neq 0$ <u>for every</u> γ <u>and</u> Y. <u>Then,</u> <u>writing</u> $s = k/(k-1)$, <u>we have</u> $WF_s\, a(x,D)f = WF_s f$ <u>for all</u> $f \in \mathcal{E}'(X)$. <u>In particular</u> $a(x,D)f \in G^s(X)$ <u>implies</u> $f \in G^s(X)$.
In fact, after conjugation by a Fourier integral operator, $a(x,D)$ becomes an operator to which Theorem 6 applies with $\psi(\xi) = \varphi(\xi) = (1+|\xi|)^{1/s}$ (cf. Parenti-Rodino [9], where C^∞-

hypoellipticity was proved under the same assumptions). Similarly we can prove a G^2-hypoellipticity result for the operators in the classes of Boutet de Monvel-Grigis-Helffer [2].

Another application of Theorem 6 refers the choice $\phi(\xi) = \sum_{j=1}^{n} |\xi_j|^{1/M_j}$, where $M = (M_1, \ldots, M_n)$ is a n-tuple of rational numbers $\geqslant 1$; the related hypoellipticity results can be expressed in terms of the anisotropic Gevrey wave front set WF_M of Zanghirati [12], Rodino [11]. Details and proofs of the results announced here will be found in Liess-Rodino [7].

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