Michael Beals

Semilinear wave equations with angularly smooth data

Journées Équations aux dérivées partielles (1984), p. 1-5
<http://www.numdam.org/item?id=JEDP_1984____A10_0>

Semilinear wave equations with angularly smooth data

Michael Beals
Rutgers University

Consider the problem

(*) $$\Box u \equiv \left(\frac{\partial^2}{\partial t^2} - \sum_1^n \frac{\partial^2}{\partial x_i^2}\right) u = f(t,x,u,Du).$$

$$u|_{t=0} = u_0 \ , \ u_t|_{t=0} = u_1 \ , \ u_i \in H^{s-i}(\mathbb{R}^n) \ , \ i = a,1, \ s > \frac{n}{2} + 2.$$

Here f is a C^∞ function of its arguments. As is well known, a unique solution to (*) exists with $u \in C([0,T] ; H^s(\mathbb{R}^n)) \cap C'([0,T] ; H^{s-1}(\mathbb{R}^n))$ for some $T > 0$. Throughout this paper we restrict our attention to $0 \leqslant t \leqslant T$, and consider the question of the propagation of singularities starting from time $t = 0$.

In the case of one space dimension, the singular support of u is contained in the union of the characteristics over the singular support of the data (Rauch-Reed [8]). That is, the propagation of singularities for the nonlinear equation (for small time) is the same as in the linear case. Even in one space dimension, though, if the order of the strictly hyperbolic equation is greater than two, singularities not present in the linear case ("anomalous singularities") will appear, propagating along characteristics issuing from the points where singularities corresponding to the linear case cross. On the other hand, these crossings and later crossings of the anomalous singularities are the only sources of new singularities in the cas $n = 1$ (Rauch-Reed [9]).

If $n > 1$, the nonlinear picture is considerably different. In general data singular at only one point will give rise to anomalous singularities on the entire interior of the light cone over that point (Beals [1]). Thus for more than one space dimension, no condition on only the location of the singularities of the data will guarantee the absence of anomalous singularities on the union of the interiors of the light cones over the singularities of the data.

There are many natural types of initial conditions in which one knows much more than merely the location of the singularities ; for example :

(1) data which are smooth parallel to an initial (n-1) manifold.

(2) radial data.

For $n = 2$ the second case was considered in Berning-Reed [3] ; we will put their result in a more general context in Theorem 1 below. Bony [4] has a general theorem for handling the first situation : suppose the data are "conormal distributions" associated with the smooth hypersurface $S \subset \{t=0\}$. (For example, if $S = \{x_1=0\}$, this assumption means that $(x_1 \frac{\partial}{\partial x_1})^{\alpha_1} (\frac{\partial}{\partial x_2})^{\alpha_2} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n} u_i \in H^{s-i}(\mathbb{R}^n)$ for $i = 0, 1$, all α). Then, for small time, (sing supp $u) \subset \Sigma_1 \cup \Sigma_2$, where Σ_i are the two characteristic hypersurfaces obtained via the Hamiltonian flow-out from S. (There are analogous results for higher order equations, for the intersections of the flows from two such hypersurfaces (Bony [4]). and for three such hypersurfaces (Melrose-Ritter [7], Bony [5])). Bony's method also allows similar conclusions if the data are conormal with respect to a point, for example the origin : if

$$(x_j \frac{\partial}{\partial x_j})^{\alpha} u_i \in H^{s-i}(\mathbb{R}^n) \text{ for } i = 0, 1, \text{ all } \alpha,$$

then the singular support of u is contained in the light cone (surface) over the origin (Bony [6]).

The conormal hypotheses above are very restrictive ; unlike the case $n = 1$, they force the singularities of the data to be contained in submanifolds. Rauch-Reed [10] have considered the case of "striated" data, in which conditions are placed only on derivatives in directions parallel to a family of smooth hypersurfaces. We consider here the case of "angularly smooth" data, in which the family of hypersurfaces (spheres) flow out into surfaces (cones) which form caustics at $x = 0$. The basic idea is that by using arguments from the $n = 1$ case, we can control derivatives in one direction, allowing the relaxation of the conormal hypothesis.

Definition.

Data (u_0,u_1) are said to be angularly smooth if $(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j})^{\alpha} u_i \in H^{s-i}(\mathbb{R}^n)$ for $i = 0,1$, all α.

Definition.

For $p = (t_0,x_0) \in (0,\infty) \times \mathbb{R}^n \setminus 0$, let

$$L_p = \{(t_0-t,\ x_0 - \frac{tx_0}{|x_0|}) : t > 0\} \cup \{(t_0-t,\ x_0 + \frac{tx_0}{|x_0|}) : t > 0\}.$$

(These are the two backward characteristics through p which project on to the line from x_0 to 0). For p inside the light cone over 0, (so that L_p intersects $\{x = 0\}$), let N_p be the backward light cone from $L_p \cap \{x = 0\}$. Otherwise, let $N_p = \emptyset$. Finally, set

$$C_p = L_p \cup N_p.$$

Notice that if $\Box u = 0$ and u has angularly smooth data, then smoothness near $L_p \cap \{t = 0\}$ implies smoothness at p.

Theorem 1.

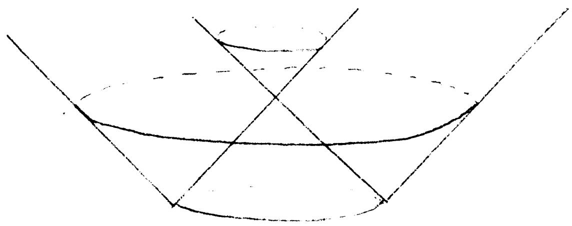
If u satisfies (*) with angularly smooth data, and if the data are C^∞ near $C_p \cap \{t=0\}$, then u is C^∞ near p.

Corollary 1.

If the data are angularly smooth and singular only at $x = 0$, then the solution is singular only on the surface of the light cone over $x = 0$.

Corollary 2.

If the data are angularly smooth and singular only at $|x| = 1$, then the solution is singular only on the surface of two cones, one of which forms a caustic at $t = 1$.



The proof of Theorem 1 involves three steps ; for details see [2].

A. If the data are angularly smooth, then for all α, the solution u satisfies

$$(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j})^\alpha u \in C([0,T] ; H^s(\mathbb{R}^n)) \cap C'([0,T] ; H^{s-1}(\mathbb{R}^n)).$$

The proof of this property involves imitating the usual contraction mapping argument for proving the existence of u. At this point is used the property that the commutators $[x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}, \Box]$ are microlocally in the span of $\{x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}\}_{j,k}$ and $\Box$, as in Bony [4], Melrose-Ritter [7]. (In fact, in our case the commutators are zero, which is more than is necessary).

B. If $u \in C^2(\mathbb{R}^{n+1})$ and $(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j})^\alpha u \in C^2(\mathbb{R}^{n+1})$ for all α, for $r > 0$ and $e \in S^{n-1}$ set $P_1 = \{(t,(r+t)e) : t \geq 0\}$, $P_2 = \{(t,(r-t)e) : 0 \leq t \leq r\}$. For u satisfying (*), with $u \in C^\infty$ near $(0,re)$, then $(\frac{\partial}{\partial t} + \frac{\partial}{\partial r})^j u \in C^2$ near P, for all j, and $(\frac{\partial}{\partial t} - \frac{\partial}{\partial r})^j u \in C^2$ near P_2 for all j. The proof in this case involves using the arguments of Rauch-Reed [8] with the angular variables $\theta_1,\ldots,\theta_{n-1}$ as parameters.

C. Notice that in polar coordinates locally away from $r = 0$, we have control over all derivatives $(\frac{\partial}{\partial\theta_1})^{\alpha_1} \ldots (\frac{\partial}{\partial\theta_{n-1}})^{\alpha_{n-1}} u$, so we only need control over $(\frac{\partial}{\partial t} + \frac{\partial}{\partial r})^j$ and $(\frac{\partial}{\partial t} - \frac{\partial}{\partial r})^j$ for all j. There are three possible locations for $p_0 = (t_0,x_0)$:

1. $t_0 < |x_0|$. Then L_p does not intersect $\{x = 0\}$, and from B it follows that $(\frac{\partial}{\partial t} + \frac{\partial}{\partial r})^j$ u and $(\frac{\partial}{\partial t} - \frac{\partial}{\partial r})^j$ u are in C^2 locally near p for all j. Hence u is smooth at p if the data are smooth at $L_p \cap \{t = 0\}$.

2. $p_0 = (t_0,0)$. If the data are smooth near $C_{p_0} = N_{p_0}$, from B it follows that $u \in H^\infty$ microlocally on all backward bicharacteristics from p_0 for $t < t_0$ and the same holds for $f(t,x,u,D_u)$. Now Hörmander's theorem, induction, and elliptic regularity allow the conclusion that $u \in C^\infty$ at p_0.

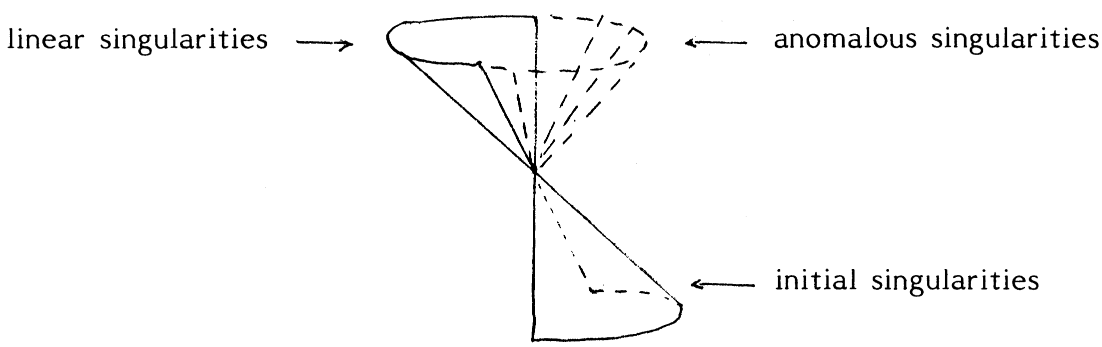
3. $p_0 = (t_0,x_0)$ is inside the forward light cone over 0, that is, $|x_0| < t_0$. For microlocal smoothness along the bicharacteristic which intersects $\{x = 0\}$, the problem can be restarted at time $t_0 - |x_0|$. If the data are smooth near C_{p_0}, then by case 2 u is smooth near $(t_0 - |x_0|, 0)$, and arguing as in B we can show that u is smooth near p_0.

Finally, in [2] it is shown that the nonlinear domain of dependence N_p does indeed affect the smoothness of u at p.

Theorem 2.

There is a solution u of (*), with angularly smooth data which is smooth at $L_p \cap \{t = 0\}$ (but not at $N_p \cap \{t = 0\}$) such that u is not smooth at p.

Thus nonlinear singularities can occur even with angularly smooth data, though they are restricted to a relatively small set.



The proof of Theoreme 2 involves construction of appropriate data ; call v the solution of $\Box v = 0$ with that data. If E is the forward fundamental solution for $\Box$ starting at $t = 0$, we write $u = v + E\, f(u) = v + E\, f(v) + E(f(u) - f(v))$. The nonlinear function f is chosen so that f(v) explicitly exhibits singularities not present in the linear case. The proof is completed by showing that the remainder term $E(f(u) - f(v))$ is strictly smoother than $E\, f(v)$ on the set of anomalous singularities.

References

[1] M. Beals, Self-spreading and strength of singularities for solutions to semilinear wave equations, Ann. of Math. 118 (1983), 187-214.

[2] M. Beals, Nonlinear wave equations with data singular at one point, Contemp. Math. 27 (1984), 83-95.

[3] J. Berning and M. Reed, Reflection of singularities of one-dimensional wave equations at boundaries, Journ. Math. Anal. Appl. 72 (1979), 635-653.

[4] J.M. Bony, Interaction des singularities pour les équations aux derivees partielles non lineaires, Sem. Joulaouic-Schwartz (1981-82), exp. n° 2.

[5] J.M. Bony, Interaction des singularities pour les equations de Klein-Jordon non lineaires, Sem. Joulaouic-Schwartz (1983-84), exp. n° 10.

[6] J.M. Bony, Propagation et interaction des singularities pour les solutions des equations aux derivees partielles non lineaires, (preprint).

[7] R. Melrose and N. Ritter, Interaction of nonlinear progressing waves, (preprint).

[8] J. Rauch and M. Reed, Propagation of singularities for semilinear hyperbolic equations in one space variable, Ann. of Math. 111 (1980), 531-552.

[9] J. Rauch and M. Reed, Nonlinear microlocal analysis of semilinear hyperbolic systems in one space dimension, Duke Math. J. 49 (1982), 397-475.

[10] J. Rauch and M. Reed, Striated solutions of semilinear two speed wave equations, (preprint).