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ANDERS MELIN

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Intertwining methods in the problem of inverse scattering.

Anders Melin, University of Lund, Sweden

0. Introduction.

We shall consider the *Schrödinger operator* $H_v = -\Delta + v(x)$ in $\mathbb{R}^n$, $n=3,5,\ldots$. The potential v(x) will have a short range and for simplicity we assume that there are positive constants C_α and ε such that

$$(0.1) \qquad |D^\alpha v(x)| \le C_\alpha (1+|x|)^{-2-\varepsilon-|\alpha|}, \quad x \in \mathbb{R}^n .$$

By $\|v^{(n-2)}\|$ we denote the L^1 norm of the differential of order n-2 of v, and we shall say that v is small if this norm is small. Throughout operators in $\mathbb{R}^n$ will be identified with their distribution kernels, and the *intertwining* relation

$$(0.2) \qquad H_v A = A H_0$$

may then be expressed in the form of a differential equation

$$(0.2)' \qquad (\Delta_x - \Delta_y - v(x))A(x,y) = 0 .$$

We shall see that these equations always have solutions which are invertible operators in some weighted L^p-spaces. In the case of a small potential at least this gives us rather explicit expressions for the *wave operators*

$$W_\pm = \lim_{t \to \pm\infty} e^{itH_v} e^{-itH_0}$$

and also for the *scattering operator*

$$S = W_+^* W_- \quad .$$

From these expressions it will easily follow that (in contrast to the one-dimensional situation) the potential is always uniquely determined from the scattering operator. We shall also discuss some equations of Gelfand-Levitan-Marchenko type which give relations between A and S.

In the paper [4] the author studied the inverse scattering problem on the real line and also the characterization problem for scattering matrices. This was carried out by deriving some sharp estimates for the intertwining operators. That research, as well as its generalization to higher dimensions, was to a large extent inspired by Faddeev's papers about one- and multi-dimensional scattering [1],[2]. However, in contrast to Faddeev , we use explicit formulas for fundamental solutions of $\Delta_x - \Delta_y$ with support conditions. Using this we may construct intertwining operators in a rather straightforward and elementary way, where the main problem consists of estimating some big integrals. For other methods in inverse scattering we refer to Saito [7],[8] and Newton [5],[6]. We also remark here that the method of using intertwining relations is fairly old (cf. Marchenko [3]).

1. Fundamental solutions of the ultrahyperbolic operator.

We consider $\Delta_x - \Delta_y$ in $\mathbb{R}^n x \mathbb{R}^n$ where n= 3,5,... . For ω a unit vector in $\mathbb{R}^n$ we define $L_\omega \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ by

(1.1) $$L_\omega(u) = \int_{<x,\omega>\leq 0} (<\omega, \partial_x + \partial_y>^{n-2} u)(x, x - 2<x,\omega>\omega)\,dx.$$

We observe that $|y| = |x|$, $y - x \in \overline{\mathbb{R}_+ \omega}$ in $\mathrm{supp}(L_\omega)$ and that

(1.2) $$L_\omega((\Delta_x - \Delta_y)u) = \int_{<x,\omega>=0} (\Delta^{(n-1)/2} U)(x)\,dx ,$$

where $U(x) = u(x,x)$. When proving (1.2) we may assume $\omega = e_n$ and setting $x' = (x_1, \ldots, x_{n-1})$ we have

$$\{(\partial_{x_n} + \partial_{y_n})^{n-2}(\Delta_x - \Delta_y)u\}(x', x_n; x', -x_n) = \sum_{j<n} \partial w_j/\partial x_j + \partial w/\partial x_n ,$$

where $w_j(x) = \{(\partial_{x_n} + \partial_{y_n})^{n-2}(\partial_{x_j} - \partial_{y_j})u\}(x', x_n; x', -x_n)$ and

$$w(x) = ((\partial_{x_n} + \partial_{y_n})^{n-1} u)(x', x_n; x', -x_n).$$

The w_j do not contribute to the right-hand side of (1.1) with u replaced by $(\Delta_x - \Delta_y)u$ and the contribution from w equals

$$\int_{x_n=0} \partial_{x_n}^{n-1} U(x)\,dx = \int_{x_n=0} \Delta^{(n-1)/2} U(x)\,dx .$$

Now we note that there are constants c_n' and c_n'' such that

(1.3) $$c_n' \int_{S^{n-1}} d\omega \Big(\int_{<x,\omega>=0} f(x)\,dx \Big) = \int_{\mathbb{R}^n} |x|^{-1} f(x)\,dx ,$$

and

(1.4) $$c_n'' \int |x|^{-1} \Delta^{(n-1)/2} g(x)\,dx = g(0) .$$

This together with (1.2) shows that

$$E_\theta = 2c_n'c_n'' \int_{\langle\omega,\theta\rangle \geq 0} L_\omega d\omega$$

is a fundamental solution of $\Delta_x - \Delta_y$ when $\theta \in S^{n-1}$, and

(1.5) $$|x|=|y| \ , \langle y-x,\theta\rangle \geq 0 \text{ in } \operatorname{supp}(E_\theta) \ .$$

In addition to these conditions E_θ satisfies a growth condition at infinity which makes E_θ unique.

Remark. Let $\hat{E}_\theta$ be the Fourier transform of E_θ and $Y_\pm$ denote the characteristic function for $\mathbb{R}_\pm$. Then

(1.6) $$\hat{E}_\theta(\xi,\eta) = \lim_{\varepsilon\to 0} \left(Y_+(\langle\xi+\eta,\theta\rangle)(|\eta|^2-|\xi|^2-i\varepsilon)^{-1} + Y_-(\langle\xi+\eta,\theta\rangle)(|\eta|^2-|\xi|^2+i\varepsilon)^{-1}\right) .$$

2. Construction of intertwining operators.

Let M denote the set of all measurable functions $R(x,y)$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$||R||_M = \max \left(\sup_x \int |R(x,y)|dy, \ \sup_y \int |R(x,y)|dx \right) < \infty .$$

Any such R defines a continuous operator on L^p when $1 \leq p \leq \infty$ and

$$\|R\|_{L^p \to L^p} \leq ||R||_M \quad .$$

Theorem 2.1 . Let $\theta \in S^{n-1}$. *Then* (0.2)' *has a unique solution* $A = A_\theta = \delta(x-y) + R(x,y)$ *such that*

(i) $\langle y-x,\theta\rangle \geq 0$ *in* $\operatorname{supp}(A)$.

(ii) *There is a constant* $\lambda = \lambda_v$, *with* $\lambda = 0$ *for* v *small, such that*

(2.1) $$||e^{-\lambda\langle y-x,\theta\rangle}R||_M < 1/2 \ .$$

(iii)' $$\int e^{-\lambda\langle y-x,\theta\rangle}|R(x,y)|dx \to 0 \quad \text{as } |y| \to \infty \ , \ y/|y| \to -\theta \ ,$$

(iii)'' $$\int e^{-\lambda\langle y-x,\theta\rangle}|R(x,y)|dy \to 0 \quad \text{as } |x| \to \infty \ , \ x/|x| \to \theta \ .$$

There are several consequences of this result. If v is small, then A is an isomorphism on any L^p and H_v is conjugated to H_0 under A and we have no bound states then. When v is big A will still be an isomorphism if we replace L^p by

$$L^p_{\lambda,\theta} = \{ f \ ; \ e^{\lambda\langle x,\theta\rangle} f \in L^p \} \ .$$

From this fact one may obtain information about the eigenfunctions and eigenvalues of H_v. In particuler, $-\lambda^2$ gives a lower bound for the spectrum.

To construct A_θ one makes use of the operator $Tu = E_\theta * (vu)$,where $(vu)(x,u) = v(x)u(x,y)$. With $u_o(x,y) = \delta(x-y)$ one finds that the series

$$R(x,y) = \sum_1^\infty T^N u_o$$

converges in $L^1_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$ and that $A = I + R$ solves (0.2). In fact, setting $R_N = T^N u_o$ one has the estimates

$$||e^{\lambda\langle x-y,\theta\rangle} R_N||_M \leq 3^{-N} \tag{2.2}$$

with λ as above. Moreover, one can prove inequalities of the form

$$||R_N o(I-\chi(\varepsilon D))||_M \leq 3^{-N} \quad , \quad N \geq N_o \ , \tag{2.3}$$

where $\chi \in C_o^\infty(\mathbb{R}^n)$ equals 1 near the origin and ε is small, and also that $I + R(I-\chi(\varepsilon D))$ is invertible in any L^p then.

To derive estimates of the form (2.2) for example one has to express $R_N(x,y)$ as the integral over $(S^{n-1})^N$ of expression of the following form:

$$J(\omega_1,\ldots,\omega_N) \int\cdots\int_{\langle X_j,\omega_j\rangle \geq 0} v_1(x+X_1,\omega)\ldots v_N(x+X_1+\ldots+X_N,\omega)\delta(y-x-2\textstyle\sum\langle X_j,\omega_j\rangle\omega_j)dX_1\ldots dX_N.$$

Here $\omega_j \in S^{n-1}$ and the norm of $J(\omega_1,\ldots,\omega_N)$ in $L^1((S^{n-1})^N)$ growths at most as C_n^N as N tends to infinity , while each $|v_j|$ may be majorized by $|v^{(n-2)}|$.

Remark. Using (2.3) and the fact that $I + R(I-\chi(\varepsilon D))$ is invertible one can find an operator B which is bounded and invertible in any L^p such that

$$H_v B = BH_0 + v(x)\chi(\varepsilon D) \tag{2.4}$$

with χ as above. Thus when v is sufficiently small at infinity the operator H_v is then conjugated to $-\Delta + K$, where K is of trace class for example.

3. Formulas for the wave operators and the scattering matrix.

We set

$$u_{\mp}(x,\eta) = \int A_{\pm\,\eta/|\eta|}(x,y)e^{i\langle y,\eta\rangle}dy. \tag{3.1}$$

These functions always exist for small v and are then bounded functions in both variables. The same is true for v large too as soon as we require η to be far away from the origin: $|\eta| \geq C_v$ where C_v depends on v. The following result shows that the $u_{\mp}$ are the generalized eigenfunctions (solutions of $H_v u = |\eta|^2 u$) that enter in the distorted Fourier transforms.

Theorem 3.1. The functions $u_{\pm}(x,\eta)$ are the kernels of the generalized (adjoint) Fourier transforms $W_{\mp} F^*$, where F denotes the standard Fourier transform.

We shall give a heuristic motivation for this result by considering wave packets. For the case of simplicity we also assume that v is small. Let $\hat{u} \in C_0^\infty(\mathbb{R}^n)$ be supported near $\mathbb{R}_+\theta$. Then $e^{-itH_0}u$ is concentrated near ∞ in the direction of $-\theta$ as t tends to $-\infty$. By the condition (iii)' of Theorem 2.1 we may expect that $Ae^{-itH_0}u$ is close to $e^{-itH_0}u$ for such t. Hence we have

$$W_- u \sim e^{itH_v} e^{-itH_0} u \sim e^{itH_v} A e^{-itH_0} u = Au$$

if we apply (0.2). The complete proof of the theorem requires a partition of unity on the Fourier transform side together with more precise estimates.

Since $(\Delta_x - \Delta_y)A_\theta = vA_\theta$ we may expect that A_θ is essentially determined from the behaviour of $(vA_\theta)^\wedge(\xi,\eta)$ restricted to the set $|\xi| = |\eta|$. This motivates us (apart from the normalization factor) to introduce the integral operators $K_\theta(r)$ on $L^2(S^{n-1})$ depending on $r \in \mathbb{R}_+$ and with kernels

$$K_\theta(r;\phi,\psi) = (-i\pi)(2\pi)^{-n} r^{n-2} (vA_\theta)^\wedge(r\phi,-r\psi) .$$

These are defined for all r when v is small and for large r in the general case.

The scattering operator S commutes with H_0. In order to describe it w.r.t. polar coordinates on the Fourier transform side we define the unitary map $\gamma: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}_+; L^2(S^{n-1}))$ by the formula

$$(\gamma f)(r,\omega) = r^{(n-1)/2} \hat{f}(r\omega)/(2\pi)^{n/2} .$$

For any operator Q on $L^2(\mathbb{R}^n)$ we set $Q^\gamma = \gamma Q \gamma^*$.

Theorem 3.2. The kernel of S^γ (considered as an operator on $L^2(\mathbb{R}_+)$ acting on vector valued functions) has the following form

$$S^\gamma(r,\rho) = \delta(r-\rho)\tilde{S}(r) = \delta(r-\rho)(I + T(r)) , \tag{3.2}$$

where $T(r;\phi,\psi) = K_\psi(r;\phi,\psi)$. (We assume r to be large if v is not small.)

Remark. $S(r)$ is the *scattering matrix* and $T(r)$ the *scattering amplitude*.

Proof. We assume v is small for simplicity and then we use the fact that

$$((S-I)f,g) = \lim_{\varepsilon\to 0} -i \int_{-\infty}^{\infty} e^{-\varepsilon|t|}(vW_- e^{-itH_0}f, e^{-itH_0}g)dt$$

for a dense set of f and g in L^2. Hence

$$(F(S-I)F^{*}f,g) = \lim_{\varepsilon\to 0} -i \int_{-\infty}^{\infty} e^{-\varepsilon|t|}(G_t f,g)dt ,$$

where $G_t = e^{itM}F(vW_-)F^{*}e^{-itM}$ and M = multiplication by $|\xi|^2$. The kernel of G_t equals

$$e^{it(|\xi|^2-|\eta|^2)}(vA_{\eta/|\eta|})^{\wedge}(\xi,-\eta)$$

by Theorem 3.1. The theorem follows therefore from the definition of K_θ and the fact that

$$\int_{-\infty}^{\infty} e^{-\varepsilon|t|}e^{it(|\xi|^2-|\eta|^2)}dt \to \pi|\xi|^{-1}\delta(|\xi|-|\eta|) \quad \text{as } \varepsilon\to 0.$$

Theorem 3.3. The potential is determined from the scattering matrix if $v\in L^1$ *(in addition to* (0,1)*).*

Proof. Set $R_\psi^\varepsilon = R_\psi(1-\chi(\varepsilon D))$ with ε and χ as above. For large r we have

$$(vA_\psi)^{\wedge}(r\phi,-r\psi) = (vu_o)^{\wedge}(r\phi,-r\psi) + (vR_\psi^\varepsilon)^{\wedge}(r\phi,-r\psi) .$$

Since $v(x)R_\psi^\varepsilon(x,y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ it follows from the Riemann-Lebesgue lemma that $(vR_\psi^\varepsilon)^{\wedge}(r\phi,-r\psi) \to 0$ as $r\to\infty$ (uniformly **w.r.t.** ϕ and ψ) . On the other hand

$$(vu_o)^{\wedge}(r\phi,-r\psi) = \hat{v}(r(\phi-\psi)),$$

and any $\zeta\in\mathbb{R}^n$ can be obtained as a limit as $r_j\to\infty$ of $r_j(\phi_j-\psi_j)$ where ϕ_j and ψ_j are sequences of unit vectors. Hence $\hat{v}$ is determined from $\mathcal{S}$.

4. The fundamental identity and a Gelfand-Levitan equation (in the case of small v).

In this section we shall restrict to the case of a small potential . What happens in the general case is not yet clear for us.

Theorem 4.1. (The fundamental identity) Let K_θ *be as in Section* 3. *Then*

$$(4.1)\qquad K_\theta(r;\phi,\psi) = K_\psi(r;\phi,\psi) - \int_{\langle\omega-\psi,\theta\rangle \geq 0} K_\omega(r;\phi,\omega)K_\theta(r;\omega,\psi)\,d\omega\ .$$

When proving this formula one may consider both sides of (4.1) as analytic functions of z after v has been replaced by zv. It suffices then to show that (4.1) holds for the Taylor expansion at the origin , and it turns out then that (4.1) follows from (1.6) and the equation $A_\theta = E_\theta * (vA_\theta) + \delta(x-y)$. The fundamental identity is also true for large v as long as we take r large.

Let $Y_\pm(s)$ be as in (1.6) and set

$$K_\theta^\pm(r;\phi,\psi) = Y_\pm(\langle\psi-\phi,\theta\rangle)K_\theta(r;\phi,\psi)\ .$$

Then $K_\theta = K_\theta^+ + K_\theta^-$ is the decomposition of $K_\theta(r)$ considered as a matrix into upper and lower triangular ones w.r.t. the ordering of S^{n-1} defined by θ . The equation (4.1) now takes the form

$$(4.1)'\qquad I + K_\theta^+(r) = \tilde{S}(r)(I-K_\theta^-(r))\ ,\ r>0\ .$$

This gives a factorization of $\tilde{S}(r)$ into a lower and upper triangular matrix. Note also that

$$(4.2)\qquad (I+K_\theta^+)^*(I+K_\theta^+) = (I-K_\theta^-)^*(I-K_\theta^-)$$

in view of the unitarity of the scattering matrix.

<u>Remark 4.2</u>. In the one-dimensional case the operator in (4.2) takes the form $\begin{pmatrix} 0 & \bar\rho \\ \rho & 0 \end{pmatrix}$, where ρ is the reflection coefficient.

We shall next show that it is possible to describe the operator in (4.2) without working on the Fourier transform side. When doing this we use the identity

$$(4.3)\qquad (A_\theta^*)^{-1} = A_{-\theta}\ .$$

This is a consequence of the uniqueness assertion of Theorem 2.1 since both sides of (4.3) are intertwining operators.The Gelfand-Levitan-Marchenko equation (w.r.t the direction $-\theta$) may be written as

$$(4.4)\qquad A_{-\theta}Q_\theta A_{-\theta}^* = I.$$

Here Q_θ is a positively definit operator and $A_{-\theta}Q_\theta^{1/2}$ is unitary . By (4.3)

$$(4.5)\qquad Q_\theta = A_\theta^* A_\theta$$

and Q_θ commutes with H_θ. It is natural therefore to consider Q_θ^γ .

Theorem 4.3. Assume that v *is small. Then*

$$(4.6)\qquad Q_\theta^\gamma(r,\rho) = \delta(r-\rho)\tilde{Q}_\theta(r) \ , \ r>0 \ ,$$

where

$$(4.7)\qquad \tilde{Q}_\theta(r) = (I+K_\theta^+(r))^*(I+K_\theta^+(r)) = (I-K_\theta^-(r))^*(I-K_\theta^-(r)) \ .$$

In principle this gives a method to compute v from the scattering matrix. When doing so Q_θ is first computed from (4.7) and the fundamental identity. After that (4.4) is converted into a linear equation for $A_{-\theta}$ as in the one-dimensional case by using the support condition on $A_{-\theta}$. Finally v is computed from $A_{-\theta}$.

REFERENCES

1. L.D. Faddeev, The inverse problem in quantum theory of scattering. J.Math.Phys., 4(1),1963,72-104.

2. L.D. Faddeev, Inverse problem of quantum scattering theory,II. J.Sov.Math., 5(1),1976, 334-396.

3. V.A. Marchenko, Some questions in the theory of linear differential operators of second order ,I (in Russian). Trudy Moskov.Mat.Obshch.,1, 1952,327-420.

4. A.Melin, Operator methods for inverse scattering on the real line. Comm.Partial Differential Equations,10(7),1985,677-766.

5. R.G.Newton, Inverse scattering.II. Three dimensions.J.Math.Phys.21(7),1980, 1698-1715.

6. R.G.Newton, An inverse spectral problem in three dimensions. SIAM-AMS Proceedings,vol 14,1984, 81-90.

7. Y.Saito, Some properties of the scattering amplitude and the inverse scattering problem. Osaka J.Math.,19,1982,527-547.

8. Y.Saito, An inverse problem in potential theory and the inverse scattering problem.J.Math.Kyoto Univ. 22,1982,307-321.