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PROPAGATION OF ANALYTIC SINGULARITIES

UP TO NON SMOOTH BOUNDARY

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1.- Propagation for sheaves

We shall follow the notations of [K-S 1]. In particular if  $X$  is a real manifold, we denote by  $D^b(X)$  the derived category of the category of complexes of sheaves with bounded cohomology, and if  $F \in D^b(X)$  we denote by  $SS(F)$  its microsupport. Recall that  $SS(F)$  is a closed conic involutive subset of  $T^*X$ . We shall also make use of the bifunctor  $\mu_{\text{hom}}$ , from  $D^b(X)^0 \times D^b(X)$  to  $D^b(T^*X)$ , a slight generalization of the functor of Sato's microlocalization.

Let  $h$  be a real  $C^2$ -function defined on an open subset  $U$  of  $T^*X$ ,  $H_h$  its hamiltonian vector field. If  $(x; \xi)$  is a system of homogeneous symplectic coordinates, with  $\omega_X = \sum_j \xi_j dx_j$ , then:

$$(1.1) \quad H_h = \sum_j \left( \frac{\partial h}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial h}{\partial x_j} \frac{\partial}{\partial \xi_j} \right) .$$

If  $p \in U$  we denote by  $b_p^+$  the positive half integral curve of  $H_h$  issued at  $p$ . We define similarly  $b_p^-$  and  $b_p = b_p^- \cup b_p^+$ . We also set for  $* = 0, +, -$ :

$$(1.2) \quad V_* = \{p \in U ; h(p) \geq 0 \quad (* = +) \quad \text{or} \\ h(p) \leq 0 \quad (* = -) \quad \text{or} \quad h(p) = 0 \quad (* = 0)\}.$$

The following result is easily deduced from [K-S 1, Th. 5.2.1].

Theorem 1.1. Let  $F$  and  $G$  belong to  $D^b(X)$  with  
 $SS(G) \cap U \subset V_-$ ,  $SS(F) \cap U \subset V_+$ . Let  $j \in \mathbb{Z}$  and let  
 $u$  be a section of  $H^j(\mu\text{hom}(G,F))$  on  $U$ . Then  
 $p \in \text{supp}(u)$  implies  $b_p^+ \subset \text{supp}(u)$ .

(Remark that  $\text{supp}(u)$  is contained in  $V_0$ ).

## 2.- Wave front sets at the boundary [S 1]

Let  $M$  be a real analytic manifold of dimension  $n$ ,  $X$  a complexification of  $M$ ,  $\Omega$  an open subset of  $M$ . We introduce :

$$(2.1) \quad C_{\Omega}|_X = \mu\text{hom}(\mathbb{Z}_{\Omega}, \mathcal{O}_X) \otimes \underline{\omega}_{M/X}[n]$$

where  $\underline{\omega}_{M/X}$  is the relative orientation sheaf.

Let  $\pi$  denote the projection  $T^*X \longrightarrow X$ , and let  $B_M = R\Gamma_M(\mathcal{O}_X) \otimes \underline{\omega}_{M/X}[n]$  denote the sheaf of Sato's hyperfunctions on  $M$ . There is a natural isomorphism :

$$(2.2) \quad \alpha : \Gamma_{\Omega}(B_M) \xrightarrow{\sim} \pi_* H^0(C_{\Omega}|_X).$$

Hence a hyperfunction  $u$  on  $\Omega$  defines a section  $\alpha(u)$  of  $H^0(C_{\Omega}|_X)$  all over  $T^*X$ . We set :

$$(2.3) \quad SS_{\Omega}(u) = \text{supp}(\alpha(u)).$$

Since  $H^0(C_\Omega|_X)$  is supported by the conormal bundle  $T_M^*X$ ,  $SS_\Omega(u)$  is a closed conic subset of  $T_M^*X$ . It coincides with the classical analytical wave front set above  $\Omega$ , but it may be strictly larger than its closure in  $T_M^*X$  (cf. [S 1]).

Now let  $P$  be a differential operator defined on  $X$ , and assume for simplicity that the principal symbol  $\sigma(P)$  never vanishes identically. Let  $\mathcal{O}_X^P$  denote the sheaf of holomorphic solutions of the equation  $Pf = 0$ . Replacing  $\mathcal{O}_X$  by  $\mathcal{O}_X^P$  in the preceding discussion, we define :

$$(2.4) \quad C_\Omega^P|_X = \mu\text{hom}(\mathbb{Z}_\Omega, \mathcal{O}_X^P) \otimes \omega_{M/X} [n] .$$

Let  $B_M^P$  denote the sheaf of hyperfunction solutions of the equation  $Pu = 0$ . There is a natural isomorphism :

$$(2.5) \quad \alpha : \Gamma_\Omega(B_M^P) \xrightarrow{\sim} \pi_* H^0(C_\Omega^P|_X) .$$

If  $u$  is a hyperfunction on  $\Omega$  solution of the equation  $Pu = 0$ , we set :

$$(2.6) \quad SS_\Omega^P(u) = \text{supp}(\alpha(u)) .$$

Remark that

$$(2.7) \quad SS_\Omega^P(u) \subset SS(\mathbb{Z}_\Omega) \cap \text{char}(P)$$

(where  $\text{char}(P) = \sigma(P)^{-1}(0)$ ), but in general  $SS_\Omega^P(u)$  is no more contained in  $T_M^*X$ .

I don't know if  $SS_\Omega^P(u) \cap T_M^*X = SS_\Omega(u)$ , but this is true when  $M \setminus \Omega$  is convex (locally, up to analytic diffeomorphisms).

Of course the preceding discussion extends to solutions of general systems of differential equations (cf. [S 1]).

Now assume  $\partial\Omega = N$  is a real analytic hypersurface and let  $Y$  be a complexification of  $N$  in  $X$ . Assume  $P$  of order  $m$ ,  $Y$  is non characteristic for  $P$ , and a normal vector field to  $N$  in  $M$  is given, so that the induced system  $(D_X/D_X^P)_Y$  is isomorphic to  $D_Y^m$ ; (as usual,  $D_X$  denotes the ring of differential operators).

Let  $\rho$  and  $\bar{w}$  denote the natural maps associated to  $Y \longrightarrow X$ :

$$(2.8) \quad T^*Y \xleftarrow[\rho]{} Y \times_X T^*X \xrightarrow[\bar{w}]{} T^*X .$$

Let  $u \in \Gamma(\Omega; B_M^P)$  be a hyperfunction on  $\Omega$  solution of  $Pu = 0$ , and let  $b(u) \in \Gamma(N; B_N^m)$  be its traces. Recall (cf. [S 1], [S 2]) :

Theorem 2.1. In the preceding situation, one has :

$$SS_N(b(u)) = \rho \bar{w}^{-1} SS_\Omega^P(u) .$$

In other words, the analytic wave front set of  $b(u)$  is exactly the projection of  $SS_\Omega^P(u)$ . Remark that if  $\text{char}(P) \cap SS(\mathbb{Z}_\Omega)$  is contained in  $T_M^*X$ ,  $SS_\Omega^P(u)$  may be replaced by  $SS_\Omega(u)$  in Theorem 2.1.

Remark moreover that  $b(u)$  does not make sense when  $\partial\Omega$  is not smooth, but  $SS_\Omega(u)$  always does.

### 3.- Transversal propagation for non smooth boundaries

Let  $M$  be a real analytic manifold,  $X$  a complexification of  $M$ ,  $\Omega$  an open subset of  $M$ .

If  $x \in M$ , the cone  $N_x(\Omega)$  is defined in [K-S 1]. Recall that  $N_x(\Omega)$  is an open convex cone of  $T_x^*M$ , and  $\theta \in N_x(\Omega)$ ,  $\theta \neq 0$  implies that there exists a convex open cone  $\gamma$  (in a system of local coordinates around  $x$ ) such that  $\theta \in \gamma$  and  $\Omega + \gamma \subset \Omega$ .

We shall have to consider the real underlying structure of  $T^*X$ . Recall that if  $\omega_X$  is the complex canonical 1-form on  $T^*X$ , this real symplectic structure is defined by  $2\text{Re } \omega_X$ .

If  $h$  is a real  $C^2$ -function on  $T^*X$ , we denote by  $H_h^{\mathbb{R}}$  its real Hamiltonian vector field.

If  $(z; \zeta)$  is a system of homogeneous holomorphic symplectic coordinates on  $T^*X$ , such that  $\omega_X = \sum_j \zeta_j dz_j$ , and  $z = x + iy$ ,  $\zeta = \xi + i\eta$ , then

$$(3.1) \quad H_h^{\mathbb{R}} = \sum_j \left( \frac{\partial h}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial h}{\partial x_j} \frac{\partial}{\partial \xi_j} + \frac{\partial h}{\partial \eta_j} \frac{\partial}{\partial y_j} - \frac{\partial h}{\partial y_j} \frac{\partial}{\partial \eta_j} \right).$$

Now let  $P$  be a differential operator on  $X$ ,  $u$  a hyperfunction on  $\Omega$ , solution of the equation  $Pu = 0$ . Let  $p \in T_M^*X$ ,  $x_0 = \pi(p)$ .

Theorem 3.1. Assume :

- a)  $\text{Im } \sigma(P) \Big|_{T_M^*X} = 0$
- b)  $\pi(H_{\text{Im } \sigma(P)}^{\mathbb{R}}(p)) \in N_{x_0}(\Omega)$ .

Let  $b_p^+$  be the positive half integral curve of  $H_{\text{Im } \sigma(P)}^{\mathbb{R}}$  issued at  $p$ . Then  $p \in \text{SS}_\Omega(u)$  implies  $b_p^+ \subset \text{SS}_\Omega(u)$ .

Proof

We may assume  $X$  is open in  $\mathbb{C}^n$  and  $M = X \cap \mathbb{R}^n$ . Then there exists a convex open cone  $\gamma$  such that  $\Omega + \gamma \subset \Omega$  (in a neighborhood of  $x_0$ ) and  $\pi(H_{\text{Im } \sigma(P)}^{\mathbb{R}}(p)) \in \gamma$ . This last condition implies :

$$\langle d_{\xi} \text{Im } \sigma(P)(x, i\eta), \xi \rangle \geq c|\xi|$$

for some  $c > 0$ , and all  $\xi \in \gamma^{\circ}$  ( $\gamma^{\circ}$  is the polar set to  $\gamma$ ).

Hence :

$$(3.2) \quad \text{Im } \sigma(P)(x, \xi + i\eta) \leq 0$$

for  $(x, \xi + i\eta)$  in a neighborhood of  $p$ ,  $\xi \in \gamma^{\text{oa}}$ , where  $\gamma^{\text{oa}} = -\gamma^{\circ}$ .

Since  $\Omega + \gamma \subset \Omega$ , we have (cf. [K-S 1]) :

$$SS(\mathbb{Z}_{\Omega}) \subset T_M^*X + \gamma^{\text{oa}}.$$

Thus :

$$(3.3) \quad \text{Im } \sigma(P) \leq 0 \quad \text{on } SS(\mathbb{Z}_{\Omega})$$

in a neighborhood of  $p$ .

Now let  $u \in \Gamma(\Omega; B_M)$  be a solution of the equation  $Pu = 0$ .

Then  $u$  defines a section  $\alpha(u) \in \Gamma(T^*X; H^n(\mu\text{hom}(\mathbb{Z}_{\Omega}, \theta_X^P)))$  and  $p \in SS_{\Omega}(u)$  implies  $p \in SS_{\Omega}^P(u)$ , that is,  $p \in \text{supp}(\alpha(u))$ .

Since  $SS(\theta_X^P) = \text{char}(P) \subset \{\text{Im } \sigma(P) = 0\}$ , we may apply Theorem 1.1 and we obtain :

$$b_p^+ \subset SS_{\Omega}^P(u).$$

But  $b_p^+ \setminus \{p\}$  is contained in  $\pi^{-1}(\Omega)$  and

$SS_{\Omega}^P(u) = SS_{\Omega}(u) = SS_M(u)$  above  $\Omega$ . Thus  $b_p^+ \subset SS_{\Omega}(u)$ ,

which is the desired result.

4.- Diffraction

We keep the notations of §3, but we assume :

$$(4.1) \quad \Omega = \{x \in M ; x_1 > 0\}$$

$$(4.2) \quad \sigma(P) = \zeta_1^2 - g(z, \zeta')$$

where  $z = (z_1, z')$  ,  $\zeta = (\zeta_1, \zeta')$  .

Moreover we assume :

$$(4.3) \text{ a) } \frac{\partial}{\partial x_1} g < 0 \quad \text{at } p \quad \text{or} \quad \text{b) } \frac{\partial}{\partial x_1} g \equiv 0 .$$

Theorem 4.1. Under these hypotheses, if  $p \in SS_\Omega(u)$  then  $b_p^+$   
 or  $b_p^-$  is contained in  $SS_\Omega(u)$  , in a neighborhood of  $p$ .

The idea of the proof is the following.

If  $\zeta_1 \neq 0$  at  $p$ , the result is a particular case of Theorem 3.1 . Otherwise define for  $* = 0, 1, -$  :

$$\Omega_* = \{z \in X ; x_1 > 0, y' = 0, y_1 \in \mathbb{R} (* = 0) \\ \text{or } y_1 \geq 0 (* = +) \text{ or } y_1 \leq 0 (+ = -)\}$$

Thus  $\text{Im } \sigma(P)$  is negative (resp. positive) on  $SS(\mathbb{Z}_{\Omega+})$   
 (resp.  $SS(\mathbb{Z}_{\Omega-})$ ) in a neighborhood of  $p$ . Then one can apply Theorem 1.1 to  $\mu_{\text{hom}}(\mathbb{Z}_{\Omega*}, O_X^P)$  ,  $* = +$  or  $-$  , and one obtain that if  $u|_{b_p}$  has compact support, then  $u \in H^{n-1}(\mu_{\text{hom}}(\mathbb{Z}_{\Omega_0}, O_X^P))$  , and it is not difficult to conclude using the holomorphic parameter  $z_1$  (cf. [S 2]) .

Remark that Theorem 4.1 has been first obtained by Kataoka [Ka] (under hypothesis (4.3) a)) then refined by G. Lebeau [Le] .



An application : Let  $(x_1, \dots, x_n)$  be the coordinates on  $\mathbb{R}^n$ , and let  $\Omega_1$  and  $\Omega_2$  be two open half spaces. Set  $\Omega = \Omega_1 \cup \Omega_2$  and let  $u$  be a hyperfunction on  $\Omega$ . One can easily prove :

$$(4.4) \quad SS_{\Omega}(u) = SS_{\Omega_1}(u) \cup SS_{\Omega_2}(u) .$$

Now assume  $\Omega_i = \mathbb{R} \times \Omega'_i$ , ( $i = 1, 2$ ) and  $u$  satisfies the wave equation  $Pu = 0$ , where  $P = D_1^2 - \sum_{j=2}^n D_j^2$ .

Applying Theorem 4.1 we get that  $p \in SS_{\Omega}(u) \implies b_p^+$  or  $b_p^-$  is contained in  $SS_{\Omega}(u)$ , where  $b_p^+$  and  $b_p^-$  are the half bicharacteristic curves of  $\text{Im } \sigma(P)$ .

Problem : to extend this result to the case where

$\mathbb{R}^n \setminus \Omega = \mathbb{R} \times A$ , and  $A$  is any convex closed subset of  $\mathbb{R}^{n-1}$ .

Remark that if  $\mathbb{R}^n \setminus \Omega = \mathbb{R} \times A$ , where  $A$  is polyedral, and if  $p \in SS_{\Omega}(u)$ ,  $b_p^+ \setminus \{p\} \subset \pi^{-1}(\Omega)$  then  $b_p^+ \subset SS_{\Omega}(u)$ , in view of Theorem 3.1.

BIBLIOGRAPHY

- [K-S 1] KASHIWARA, M. and SCHAPIRA, P. : Microlocal study of sheaves. Astérisque 128 (1985).
- [Ka] KATAOKA, K : Microlocal theory of boundary value problems. J. Fac. Sc. Univ. Tokyo Sect. A. I : 27, 355-399 (1980), II : 28, 31-56 (1981).
- [Le] LEBEAU, G. : Deuxième microlocalisation sur les sous-variétés isotropes. Ann. Inst. Fourier, Grenoble 35,2, 145-216 (1985).
- [S 1] SCHAPIRA, P. : Front d'onde analytique au bord II. Sem. E.D.P. Ecole Polytechnique, Exp. 13 (1986).
- [S 2] SCHAPIRA, P. : Microfunctions for boundary value problems. Volume dedicated to Pr. M. Sato. To appear.