JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

ERIK SKIBSTED

Time-dependent approach to radiation conditions

Journées Équations aux dérivées partielles (1990), p. 1-13

<http://www.numdam.org/item?id=JEDP_1990____A10_0>

© Journées Équations aux dérivées partielles, 1990, tous droits réservés.

L'accès aux archives de la revue « Journées Équations aux dérivées partielles » (http://www.math.sciences.univ-nantes.fr/edpa/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

TIME-DEPENDENT APPROACH TO RADIATION CONDITIONS

Erik Skibsted

Matematisk Institut
Aarhus Universitet
8000 Aarhus C
Denmark

In the first part of this note I shall give an account of some recent results of Ira Herbst and myself [H-S]. These results deal with radiation conditions for two-body long-range Schrödinger operators from a time-dependent point of view.

The second part contains various N-body propagation estimates [S] which I believe (besides having interest of their own) may serve as a basis for an extension of the method and results of [H-S] to N-body long-range Hamiltonians (cf Th. 2.5).

1. Radiation conditions

At first we shall recall some of the history of radiation conditions, then we state some new results (Th. 1.4) fitting into this history. At last we shall give a brief account of the proof, which involves propagation estimates for certain "radiation operators" (Th. 1.6). These estimates might equally well be called radiation conditions. The traditional approach to radiation conditions is purely stationary and rather P.D.E. oriented.

We consider $H = -\Delta + V$ on the Hilbert space $L^2 = L^2(\mathbb{R}^n)$, n arbitrary, with the potential $V = V(x)$ smooth and satisfying for some $0 < \varepsilon_0 < 1$

$$|\partial_x^\alpha V(x)| \leqq C_\alpha \langle x\rangle^{-|\alpha|-\varepsilon_0}, \qquad \forall \text{ multiindices } \alpha\,;$$

$\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$, $\quad |\alpha| = \sum_i \alpha_i$.

Clearly H is selfadjoint on the standard Sobolev space of degree two. As for the results to be presented, one can add local singularities, not to be discussed here.

The set of bounded operators on L^2 is denoted by $\mathcal{B}(L^2)$ and the resolvent of H by $R(z)$, $z \notin \mathbb{R}$.

With the above condition on V, H is purely absolutely continuous on $\mathbb{R}^+$ (the positive reals). In fact, as is well-known, the limiting absorption principle (L.A.P.) holds:

For $\lambda \in \mathbb{R}^+$ and $\delta > \frac{1}{2}$

$$\lim_{\varepsilon \downarrow 0} \langle x\rangle^{-\delta} R(\lambda+i\varepsilon)\langle x\rangle^{-\delta} \quad \text{exists in } \mathcal{B}(L^2). \tag{1.1}$$

The proof of L.A.P. due to Ikebe and Saito [I-S] has as a consequence the following

Theorem 1.1. Suppose $0 < \varepsilon < \varepsilon_0/2$ and $\langle x\rangle^{(1+\varepsilon)/2} f \in L^2$. Then the function $u = R(\lambda+i0)f = \lim_{\varepsilon'\downarrow 0} R(\lambda+i\varepsilon')f$ (defined by (1.1)) is the unique solution to

$$(H-\lambda)u = f$$

such that

(1) $\langle x\rangle^{-(1+\varepsilon)/2} u \in L^2$,

(2) $\langle x\rangle^{(-1+\varepsilon)/2}(p - \sqrt{\lambda}\,\frac{x}{|x|})u \in L^2 \oplus \cdots \oplus L^2$, $\quad p = -i\nabla_x$.

The estimate (2) is a radiation condition. In order to state a stronger radiation condition due to Isozaki [I1], we need the following

Lemma 1.2. There exists $Y(x,\lambda) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$, real, such that

(1) Given $\Delta \subset \mathbb{R}^+$ compact $\exists R_\Delta > 0$: for $r = |x| > R_\Delta$ and $\lambda \in \Delta$

$$2\sqrt{\lambda}\,\frac{\partial}{\partial r} Y(x,\lambda) = V(x) + |\nabla_x Y(x,\lambda)|^2,$$

(2) With Δ given as above

$$|\partial_x^\alpha \partial_\lambda^k Y(x,\lambda)| \leqq C_{\alpha,k} \langle x\rangle^{1-|\alpha|-\varepsilon_0}, \quad \forall \lambda \in \Delta.$$

Definitions. For $\lambda \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$

(1) $S(x,\lambda) = \sqrt{\lambda}|x| - Y(x,\lambda)$

(notice that S satisfies the eikonal equation for $|x|$ large, i.e. $|\nabla_x S(x,\lambda)|^2 + V(x) = \lambda$),

(2) With $\chi(|x|^2 > 1)$ given as a smooth characteristic function for the region $|x|^2 > 1$ (avoiding the singularity at $x = 0$)

$$\tilde{\nabla} S(x,\lambda) = \chi(|x|^2 > 1)\nabla_x S(x,\lambda),$$

(3) $\gamma(\lambda) = p - \tilde{\nabla} S(x,\lambda)$.

Theorem 1.3 ([I1]). For any $\varepsilon, \lambda > 0$ and $\frac{1}{2} > s > -\frac{1}{2}$

$$\langle x\rangle^{s}\gamma(\lambda)R(\lambda+i0)\langle x\rangle^{-s-1-\varepsilon} \in \mathcal{B}(L^2, L^2 \oplus \cdots \oplus L^2).$$

Definitions.

(1) $\gamma_{\shortparallel}(\lambda) = \frac{1}{2}\{\tilde{\nabla} S(x,\lambda)\gamma(\lambda) + \gamma(\lambda)\tilde{\nabla} S(x,\lambda)\}$,

(2) $\tilde{\gamma}_{\shortparallel}(\lambda) = \frac{1}{2}\{\chi(|x|^2 > 1)\frac{x}{|x|}\gamma(\lambda) + \gamma(\lambda)\chi(|x|^2 > 1)\frac{x}{|x|}\}$.

Theorem 1.4 ([H-S]). For any $\varepsilon, \lambda > 0$

(1) $\langle x\rangle^{s}\gamma_{\shortparallel}(\lambda)R(\lambda+i0)\langle x\rangle^{-s-1-\varepsilon} \in \mathcal{B}(L^2)$, $\quad \frac{3}{2} > s > -\frac{1}{2}$,

(2) $\langle x\rangle^{s}\tilde{\gamma}_{\shortparallel}(\lambda)R(\lambda+i0)\langle x\rangle^{-s-1-\varepsilon} \in \mathcal{B}(L^2)$, $\quad \frac{1}{2} + \varepsilon_0 > s > -\frac{1}{2}$.

The time-dependent proof of Theorem 1.4 (1) (to be outlined at the end of this section) includes a new proof of Theorem 1.3. As for Theorem 1.4 (2) we mention that the statement is an easy consequence of (1.1), Lemma 1.2 and Theorems 1.3, 1.4 (1). As an application of Theorem 1.4 (2) we have the following result, which also is proved by Isozaki [I.1], but in a rather complicated way (since the proof is based on the weaker radiation condition of Theorem 1.3).

Corollary 1.5. Let $\varepsilon, \lambda > 0$ and $\frac{1}{2} + \varepsilon_0 > s > \frac{1}{2}$ be given. For $\psi \in L^2$, put

$$g = \langle x\rangle^{-s-1-\varepsilon}\psi$$

and

$$f_g(r) = r^{(n-1)/2}e^{-iS(r\cdot,\lambda)}(R(\lambda+i0)g)(r\cdot)$$

($r = |x|$ and the dot indicates a function on the unit-sphere S^{n-1} in $\mathbb{R}^n$).

Then for any input ψ

(1) $\quad \exists F_{g,\lambda} = L^2(S^{n-1}) - \lim\limits_{r\to\infty} f_g(r)$,

(2) $\quad \mathrm{Im}\langle g, R(\lambda+i0)g\rangle = \sqrt{\lambda}\,\|F_{g,\lambda}\|^2_{L^2(S^{n-1})}$

(Im = imaginary part).

<u>Proof</u>. For r sufficiently large

$$\frac{d}{dr}f_g(r) = ir^{(n-1)/2}e^{-iS(r\cdot,\lambda)}(\tilde{\gamma}_{\shortparallel}(\lambda)R(\lambda+i0)g)(r\cdot).$$

Hence for r, r_1 large, $r > r_1$,

$$\|f_g(r) - f_g(r_1)\|_{L^2(S^{n-1})} \leqq \int_{r_1}^{r} \|\frac{d}{dl}f_g(l)\|\,dl$$

$$\leqq \left(\int_{r_1}^{r} l^{-2s}dl\right)^{\frac{1}{2}} \|\langle x\rangle^s\tilde{\gamma}_{\shortparallel}(\lambda)R(\lambda+i0)g\|$$

$$\leqq Cr_1^{-s+\frac{1}{2}}\|\psi\|, \text{ proving (1).}$$

Let $B_R(0)$ and $S_R(0)$ be the ball and sphere with radius R and centre 0, respectively.

Then by the Green identity, for R large and with $h = R(\lambda+io)g$,

$$\mathrm{Im}\langle g, R(\lambda+io)g\rangle_{B_R(0)}$$

$$= \mathrm{Im}\langle (H-\lambda)h, h\rangle_{B_R(0)}$$

$$= -\mathrm{Im}\langle (\frac{d}{dr}h)(\cdot), h(\cdot)\rangle_{S_R(0)}$$

$$= -\mathrm{Im}\langle (\{i\tilde{\gamma}_{\shortparallel}(\lambda) - \frac{n-1}{r} + i\frac{\partial}{\partial r}S(x,\lambda)\}h)(\cdot), h(\cdot)\rangle_{S_R(0)}.$$

Using this together with Lemma 1.2 and Theorem 1.4, we obtain on a sequence $R = R_m \to \infty$ that

$$\lim_{m\to\infty} \mathrm{Im}\langle g, R(\lambda+i0)g\rangle_{B_{R_m}(0)} = \sqrt{\lambda}\,\|F_{g,\lambda}\|^2,$$

proving (2).

Remark. As mentioned by Isozaki, Corollary 1.5 may serve as a basis for constructing a diagonalizing operator $\tilde{T}$ of the continuous part of H obtained as the extension of T given by

$$Tg = \int_{\mathbb{R}^+} \oplus d\lambda \left(\frac{\sqrt{\lambda}}{2\pi}\right)^{\frac{1}{2}} F_{g,\lambda} \in L^2(\mathbb{R}^+, L^2(S^{n-1})).$$

See [I1] for further information.

In order to explain the proof of Theorem 1.4 (1), we focus on a compact interval $\Delta \subset \mathbb{R}^+$. We choose $\chi_1(\xi) \in C_0^\infty(\mathbb{R}^n)$ such that

$$\chi_1(\xi) = 0 \quad \text{in a n.b.h. of } 0, \tag{1.2}$$

$$\chi_1(\xi) = 1 \quad \text{if } |\xi|^2 \in \Delta, \tag{1.3}$$

$\chi_2(x) \in C^\infty(\mathbb{R}^n)$ such that

$$\chi_2(x) = 1 \quad \text{outside a compact set,} \tag{1.4}$$

$$\chi_2(x) = 0 \quad \text{in } B_R(0) \text{ for some large } R. \tag{1.5}$$

We shall introduce a symmetrized pseudodifferential operator (Ps.D.Op.) $\overline{\gamma}$ corresponding to the symbol

$$\gamma(x,\xi) = \xi - \chi_1(\xi)\chi_2(x)(\nabla_x S)(x,\xi^2 + V(x)). \tag{1.6}$$

For that, let S_ℓ^m be the (symbol) class of $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$-functions $p(x,\xi)$ with

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leqq C_{\alpha,\beta} \langle x\rangle^{\ell-|\alpha|} \langle\xi\rangle^m. \tag{1.7}$$

The corresponding class of Ps.D.Op.'s is given by

$$(P(X,D)\psi)(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} p(x,\xi)\psi(y)\,dy\,d\xi \tag{1.8}$$

and is denoted by $\mathbf{S}_\ell^m$.

<u>Definitions</u>:

(1) $\bar{\gamma} = \frac{1}{2}(\gamma(X,D) + \gamma(X,D)^*)$, $\gamma(x,\xi)$ given by (1.6),

(2) For any m-tuple $\alpha = \alpha(m) = (\alpha_1,\ldots,\alpha_m)$, α_j integers such that $1 \leqq \alpha_j \leqq n$, $\bar{\gamma}^{\alpha} = \bar{\gamma}_{\alpha_1} \ldots \bar{\gamma}_{\alpha_m}$; i.e. an ordered product of components of $\bar{\gamma}$.

We have the following propagation estimates for $\bar{\gamma}^{\alpha}$.

<u>Theorem 1.6</u>: Let $f \in C_0^\infty(\overset{\circ}{\Delta})$ ($\overset{\circ}{\Delta}$ = interior of Δ), $\varepsilon > 0$, $\alpha(m)$ an m-tuple as above and $\ell \in \mathbb{R}$ with $m > \ell \geq 0$ be given. Then

$$\langle x\rangle^{\ell}\, \bar{\gamma}^{\alpha}\, e^{-itH}\, f(H)\langle x\rangle^{-m} = O(t^{-m+\ell+\varepsilon})$$

in $\mathcal{B}(L^2)$ for $t \to +\infty$.

The proof of Theorem 1.6 involves various other propagation estimates (cf. Section 2) and as a key ingredient the following differential inequality for fixed m and any $\varepsilon > 0$:

$$\frac{d}{dt}\sum_{\alpha(m)} ||\bar{\gamma}^{\alpha(m)}\varphi(t)||^2 \leq \frac{-2(m-\varepsilon)}{t} \sum_{\alpha(m)} ||\bar{\gamma}^{\alpha(m)}\varphi(t)||^2$$
$$+ C\, t^{-2m-1+\varepsilon}\, ||\psi||^2;$$

$$\varphi(t) = e^{-itH} f(H)\langle x\rangle^{-m-2}\psi, \qquad \psi \in L^2.$$

The factor $-2(m-\varepsilon)/t$ comes about by computing the "leading term" of the commutator $i[H,\bar{\gamma}]$ to be

$$-2|p|\chi_1(p)\chi_2(x)|x|^{-1}\,\bar{\gamma},$$

and subsequently by replacing by $-(1-\varepsilon/m)t^{-1}\bar{\gamma}$.

The next step is to put $\ell = m - 2\varepsilon - 1$ in Theorem 1.6, multiply by $e^{it(\lambda+i\varepsilon_1)}$ and integrate from 0 to $+\infty$. We obtain the resolvent estimate

$$\| \langle x \rangle^{m-1-2\varepsilon-\alpha}_{\gamma} R(\lambda+i\varepsilon_1) f(H) \langle x \rangle^{-m} \| \leq C < \infty, \quad \forall \lambda \in \mathbb{R}, \ \varepsilon_1 > 0. \tag{1.9}$$

By interpolating between (1.9) (with m large) and (1.1) one obtains

<u>Theorem 1.7</u>: Let $f \in C_0^\infty(\mathring{\Delta})$, $\delta > \frac{1}{2}$, $m \geq s \geq 0$ and $\alpha(m)$ be given. Then

$$\| \langle x \rangle^{s-\delta-\alpha(m)}_{\gamma} R(z) f(H) \langle x \rangle^{-s-\delta} \| \leq C < \infty, \quad \forall z \text{ with } \operatorname{Im} z > 0.$$

Let $\lambda \in \mathring{\Delta}$. Choose then $f \in C_0^\infty(\mathring{\Delta})$ which is one in a n.b.h. of λ. Clearly Theorem 1.4 (1) follows if

$$\langle x \rangle^{s} \gamma_{''}(\lambda) R(\lambda+io) f(H) \langle x \rangle^{-s-1-\varepsilon} \in \mathcal{B}(L^2), \ \frac{3}{2} > s > -\frac{1}{2}.$$

This estimate follows from Theorem 1.7 by using the following identity (not to be discussed):

$$\gamma_{''}(\lambda) = -\frac{\bar{\gamma}^2}{2} + D_1(\lambda)\bar{\gamma} + D_2(\lambda)(H-\lambda) + D_3(\lambda) + D_4(\lambda),$$

where for some m

$$D_1(\lambda) \in S^m_{-1},$$

$$D_2(\lambda) \in S^m_{0},$$

$$D_3(\lambda) \in S^m_{-2},$$

$$D_4(\lambda) \in S^m_{0}$$

with vanishing symbol in $|\xi|^2 \in \Delta$.

2. Some N-body estimates

We consider the N-body Hamiltonian $H = -\Delta + V$ on the space $L^2 = L^2(X)$, where X is the C.M.-configuration space

$$\{x = (x^1,\dots,x^N) \mid x^i \in \mathbb{R}^n,\ \sum_{i=1}^N m_i x^i = 0\}$$

of N n-dimensional particles with masses m_i. The inner product in X is given by

$$x.y = \sum_{i=1}^N 2m_i x^i y^i .$$

For any cluster decomposition a we put

$$X_a = \{x \in X \mid x^i = x^j \text{ if } i,j \in C \text{ for some } C \in a\}$$

and

$$X^a = \text{the orthogonal complement in } X.$$

Corresponding to $X = X^a \oplus X_a$ we write for $x \in X$ $x = x^a + x_a$.

The cluster decomposition $(1)\dots(\hat{i})\dots(\hat{j})\dots(N),(ij)$, where $\hat{}$ indicates omission, is denoted by (ij).

The potential $V = \sum_{(ij)} V_{ij}(x^{(ij)})$ satisfies the following conditions:

On $L^2(X^{(ij)})$ (or as function on $X^{(ij)}$)

(1) $V_{ij}(-\Delta+1)^{-1}$ and $(x\nabla V_{ij}(-\Delta+1)^{-1}$ are compact,

(2) $((x\nabla)^n V_{ij})(-\Delta+1)^{-1}$ is bounded for any n,

(3) $\exists\, R_0 > 0,\ 1 > \varepsilon_0 > 0$:

$V_{ij}(x)$ is smooth in $|x| > R_0$

and

$$|\partial_x^\alpha V_{ij}(x)| \leqq C_\alpha \langle x\rangle^{-|\alpha|-\varepsilon_0}.$$

We shall present four propagation estimates. The first two are (more or less) due to Sigal and Soffer [S-S]. The last two follows by a certain extension of the method of [S-S] together with an application of a certain vector field on X constructed recently by Graf [G]. All results will appear in [S] with full proofs.

Definition. For $\varepsilon > 0$ and any selfadjoint operator A, $F(A < -\varepsilon)$ denotes the spectral projection $\chi_I(A)$, $I = (-\infty, -\varepsilon)$.

Theorem 2.1. Let $E, \varepsilon > 0$. Then for any $f \in C_0^\infty(\mathbb{R})$ supported in a small n.b.h. of E and any $s' > s > 0$,

$$F\left(\frac{x^2}{4t^2} - E < -\varepsilon\right) e^{-itH} f(H) \langle x \rangle^{-s'} = O(t^{-s})$$

for $t \to \infty$.

Theorem 2.2. Let $E, \varepsilon > 0$. Then $\exists E' \geqq E$: for any $f \in C_0^\infty(\mathbb{R})$ supported in a small n.b.h. of E and any $s \geqq \ell \geqq 0$,

$$\langle x \rangle^{\ell} F\left(E' - \frac{x^2}{4t^2} < -\varepsilon\right) e^{-itH} f(H) \langle x \rangle^{-s} = O(t^{-s+\ell})$$

for $t \to \infty$.

Remark. As noted in [S-S] one can obtain (with some more work) the conclusion of Theorem 2.2 with the explicit value $E' = E - \inf \sigma_c(H)$, $\sigma_c(H)$ = the continuous spectrum of H. This is the best we can hope of without an additional localization, cf. Theorem 2.3. For $N = 2$, Theorems 2.1 and 2.2 can also be proved by a method of Isozaki and Kitada [I-K], cf. [H-S].

Definition. Let $\chi_{fr}(=\chi_{free})$ be a $C^\infty(X)$-function, homogeneous of degree zero outside the unit-sphere in X and satisfying the support condition

$$\operatorname{supp}\chi_{fr} \cap \bigcup_{\substack{a\neq \\ (1)\dots(N)}} X_a = \emptyset$$

(i.e. χ_{fr} is supported in the "free" region, where the potential goes to zero).

Theorem 2.3. Let $E,\varepsilon > 0$. Then for any $f \in C_0^\infty(\mathbb{R})$ supported in a small n.b.h. of E, any function χ_{fr} as above and any $s' > s > 0$

$$F(E - \frac{x^2}{4t^2} < -\varepsilon)\chi_{fr}(x)e^{-itH}f(H)\langle x\rangle^{-s'} = O(t^{-s})$$

for $t \to \infty$.

Introducing Ps.D.Op.'s on X by replacing $\mathbb{R}^n$ by X and utilizing (1.7) and (1.8), we have

Theorem 2.4. Suppose $P_-(X,D) \in S_0^0$ and that

$$\operatorname{supp} p_-(x,\xi) \subset \{(x,\xi) \subset X\times X \mid x\cdot\xi < (1-\varepsilon_1)|x||\xi|\}$$

for some $\varepsilon_1 > 0$.

Then for any $f \in C_0^\infty(\mathbb{R}^+)$, any χ_{fr} as in Theorem 2.3 and any $0 < s < s'$,

$$P_-(X,D)\chi_{fr}(x)e^{-itH}f(H)\langle x\rangle^{-s'} = O(t^{-s})$$

for $t \to +\infty$.

Remark. For $N = 2$ there already exist two proofs of Theorem 2.4 in the literature [J], [I2]. For $N \geqq 3$ there exist results with some similarity [M], however with very restrictive assumptions on either subsystems or the potential (still including the pure Coulomb case).

With a few modifications the estimates presented in Theorems 2.1 - 2.4 should suffice in applying the method in [H-S] to obtain radiation conditions for the free channel. In particular Theorem 1.6 with $\chi_{fr}(x)$ inserted in front of e^{-itH}, and with S (the solution to the eikonal equation) and $\overline{\gamma}^{\alpha}$ suitably modified, and the following statement, should hold (to be worked on).

<u>Theorem 2.5</u>. Let $\varepsilon, \lambda > 0$ $\frac{1}{2} + \varepsilon_0 > s > \frac{1}{2}$ and χ_{fr} as in Theorem 2.3 be given. For any $\psi \in L^2(X)$, put

$$g = \langle x\rangle^{-s-1-\varepsilon}\psi$$

and (in polar coordinates)

$$f_g(r) = r^{(\dim X-1)/2} e^{-iS(r\cdot,\lambda)} \chi_{fr}(r\cdot)(R(\lambda+i0)g)(r\cdot).$$

Then $\exists$

$$L^2(S^{\dim X-1}) - \lim_{r\to\infty} f_g(r).$$

REFERENCES:

[G] Graf, G.M.: Asymptotic completeness for N-body short-range quantum systems: A new proof. Preprint (1989).

[H-S] Herbst, I., Skibsted, E.: Time-dependent approach to radiation conditions. Preprint (1990).

[I-S] Ikebe, T., Saitō, Y.: Limiting absorbtion method and absolute continuity for the Schrödinger operator, J.Math. Kyoto Univ., 12(1972), 513-542.

[I1] Isozaki, H.: Eikonal equations and spectral representations for long-range Schrödinger Hamiltonians, J.Math. Kyoto Univ., 20-2(1980), 243-261.

[I2] Isozaki, H.: Differentiability of generalized Fourier transforms associated with Schrödinger operators, J.Math. Kyoto Univ., 25-4(1985), 781-806.

[I-K] Isozaki, H., Kitada, H.: Modified wave operators with time-dependent modifiers, J.Fac.Sci. Univ. Tokyo, Sect. 1A, Math.32(1985), 77-104.

[J] Jensen, A.: Propagation estimates for Schrödinger-type operators, Transactions of the American Mathematical Society 291-1(1985).

[M] Mourre, E.: Operateurs conjugues et proprietes de propagation, Commun.Math.Phys.91(1983), 279-300.

[S-S] Sigal, I.M., Soffer., A.: Local decay and propagation estimates for time-dependent and time-independent Hamiltonians. Preprint(1988).

[S] Skibsted, E.: Propagation estimates for N-body Schrödinger operators. In preparation.