

CHRISTOPH MÄRZ

Spectral asymptotics for Hill's equation near the potential maximum

Journées Équations aux dérivées partielles (1990), p. 1-10

http://www.numdam.org/item?id=JEDP_1990____A15_0

© Journées Équations aux dérivées partielles, 1990, tous droits réservés.

L'accès aux archives de la revue « Journées Équations aux dérivées partielles » (<http://www.math.sciences.univ-nantes.fr/edpa/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Spectral Asymptotics for Hill's Equation near the Potential Maximum

Christoph März

Fachbereich Mathematik

Johann Wolfgang Goethe-Universität, Frankfurt/Main

Postfach 11 19 32

D- 6000 Frankfurt am Main 11

1. Hypotheses and General Facts on Periodic Schrödinger Operators

In this note we are interested in the spectrum near the potential maximum of a one-dimensional semiclassical Schrödinger operator

$$(1.1) \quad P = P(h) = -\frac{h^2}{2} \frac{d^2}{dx^2} + V(x),$$

where the potential $V: \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following hypotheses:

(H1) V is real analytic,

(H2) V is 2π -periodic,

(H3) $V(x) \leq 0$ with equality exactly at the points $2\pi k, k \in \mathbb{Z}$,

(H4) $V''(0) < 0$; without loss of generality we may assume $V''(0) = -1$.

It is well known that P is selfadjoint with domain $H^2(\mathbb{R}) = \{u \in L_2(\mathbb{R}) \mid u', u'' \in L_2(\mathbb{R})\}$ and that P is unitarily equivalent to the direct integral

$$(1.2) \quad \int_{[0,1[}^{\oplus} P_{\vartheta} d\vartheta;$$

where $P_{\vartheta} u = Pu$ on $\mathcal{X}_{\vartheta}^2 = \{u \in H_2^{\text{loc}} \mid u^{(k)}(x-2\pi) = e^{2\pi i \vartheta} u^{(k)}(x) \text{ for } k = 0, 1, 2\}$. So each P_{ϑ} can be viewed as a selfadjoint, semibounded, elliptic operator on a compact manifold, that has therefore a pure point spectrum of the form

$$(1.3) \quad \sigma(P_{\vartheta}) = \{E_1(\vartheta) \leq E_2(\vartheta) \leq \dots\} \quad (\vartheta \in [0,1[).$$

The so-called bands

$$(1.4) \quad B_k = \{E_k(\vartheta) \mid \vartheta \in [0,1]\}$$

are closed intervals of non-vanishing length, and build up the spectrum of P

$$(1.5) \quad \sigma(P) = \bigcup_k B_k,$$

which in addition is absolutely continuous.

In one dimension two bands do not overlap except possibly at their endpoints, otherwise they are separated by open intervals, called gaps G_k . Let $\tau_{2\pi}(\mu)$ denote the operator of translation by -2π , acting on the two-dimensional space of solutions of $(P - \mu)u = 0$, defined by $(\tau_{2\pi}(\mu)u)(x) = u(x + 2\pi)$. Then we have the following simple criterion:

$$(1.6) \quad \mu \in \sigma(P) \iff \tau(\mu; h) = \frac{1}{2} \text{trace}(\tau_{2\pi}(\mu)) \in [-1, 1].$$

2. Former Results in Different Regions

The applicability of several methods, for instance the WKB-method, usually employed in the study of the spectrum of a semiclassical Schrödinger operator near some level μ , is strongly governed by *turning points*, i.e. zeros x_μ of $V - \mu$. We will restrict ourselves to turning points in a complex neighbourhood of the period $[0, 2\pi]$. When μ is negative and sufficiently small, we typically have real *simple* turning points, that is with $V'(x_\mu) \neq 0$. Here a classical particle has to change its direction, while the region beyond such a turning point is forbidden.

In view of different geometric situations, the study is roughly divided into the following regions for the energy level μ :

- (I) $C_0 > \mu > \varepsilon_0 > 0$,
- (II) $\varepsilon_0 \geq \mu \geq -\varepsilon_0$,
- (III) $0 > -\varepsilon_0 > \mu > -\varepsilon_1$,

where the constants $\varepsilon_0, \varepsilon_1$ and C_0 are determined by the potential.

In case (I) there do not exist any real turning points, and therefore there is no obstruction to employ the standard WKB-method, such that we obtain that the gaps are of size $O(h^\infty)$, while the band lengths are of order of magnitude $O(h)$.

In case (III) we have at least two real turning points b_μ^-, b_μ^+ near 0. Since further turning points between b_μ^+ and $b_\mu^- + 2\pi$ would hinder a systematic study, we will exclude the corresponding μ -regions. In other words, we assume that there is only *one well* I_μ over the period interval:

$$(2.1) \quad I_\mu = [b_\mu^-, b_\mu^+ + 2\pi] = \{x | V(x) \leq \mu; x \in [0, 2\pi]\}.$$

This situation has been investigated by Harrell [Ha], Simon [Si] and Outassourt [Ou]. We only mention here that [Ou] applies the method of the interaction matrix due to Helffer/Sjöstrand [He, Sj 1] in order to compute precise asymptotic formulas for the width $B_p(h)$ of the p -th band, concretely

$$(2.2) \quad B_p(h) = h^{\frac{1}{2p}} \frac{\pi^{-\frac{1}{2}}}{p!} 2^{p+3} e^{(2p+1)A} e^{-S(\mu)/h} (1 + O_p(h)),$$

where A is determined by the potential and

$$(2.3) \quad S(\mu) = \int_{b_\mu^-}^{b_\mu^+} (2[V(x) - \mu])^{\frac{1}{2}} dx$$

is the *Agmon distance* between the wells I_μ and $I_\mu + 2\pi$ with μ contained in the band.

Now the zone, given by case (II), is the region under consideration in this note. Then the situation concerning the turning points is as follows: When one is passing from $\mu < 0$ to $\mu > 0$, one has a change from two real turning points to two purely imaginary turning points near the origin, where for $\mu = 0$ there is exactly one double turning point at the origin.

(Clearly the same situation near the origin is given in the case of a double well potential V , recently studied by Gérard/Grigis [Gé,Gr] and Horn [Ho].)

One approach for the treatment of equations with turning points is given by *R.E. Langer's method of the comparison equation*. In our case (II) this comparison equation is given by *Weber's equation*

$$(2.4) \quad -\frac{d^2 u}{dx^2} - \left(\frac{x^2}{2} - \frac{S(\mu)}{\pi h}\right)u = 0.$$

Here (for $\mu > 0$) $S(\mu)$ is defined by

$$(2.5) \quad S(\mu) := i \int_{a_\mu^-}^{a_\mu^+} (2|V(y) - \mu|)^{\frac{1}{2}} dy.$$

Weinstein/Keller [We,Ke] use this method in order to compute asymptotically a fundamental system of solutions of the Schrödinger equation and, with respect to which they determine the translation matrix, such that they obtain the beautiful formula

$$(2.6) \quad \tau(\mu;h) \sim \left(1 + e^{2S(\mu)/h}\right)^{\frac{1}{2}} \cos\left\{\frac{1}{h}C(\mu)\right\},$$

where

$$(2.7) \quad C(\mu) := \int_0^{2\pi} (2[\mu - V(x)]_+)^{\frac{1}{2}} dx.$$

The role of the " \sim " is not quite clear, but it seems that their study is only valid up to the second order. Nevertheless following Lynn/Keller [Ly,Ke] it should be possible to carry out the study up to the order $\mathcal{O}(h^\infty)$. Finally they estimate very briefly the size of the bands $B_k(h)$ and the gaps $G_k(h)$ and get $|B_k(h)| \sim \frac{1}{2}|G_k(h)|$ in the region $\mu \leq 0$, which does not coincide with our results.

3. Formula for the Trace and Theorems

Our analysis will yield

$$(3.1) \quad \tau(\mu;h) = \left(1 + e^{-2\pi \frac{\mu'}{h}}\right)^{\frac{1}{2}} \cos\left\{\frac{1}{h}\left[C(\mu) + \mu'_0(\log|\mu'_0| - 1) - \mu' \log h\right] + \arg\left[\Gamma\left(\frac{1}{2} - i\frac{\mu'}{h}\right)\right] + \text{hr}(\mu;h)\right\} + \mathcal{O}\left(e^{-\frac{\varepsilon_0}{h}}\right).$$

Explanation of this formula:

* The error term $\mathcal{O}\left(e^{-\frac{\varepsilon_0}{h}}\right)$ is due to the method and uniform with respect to μ .

* $r(\mu;h)$ is an analytic symbol of order 0 (in the sense of Sjöstrand [Sj 1]).

(3.2) $\mu' := F(\mu;h) = f_0(\mu) + hf_1(\mu) + h^2 f_2(\mu) + \dots$ is a classical analytic symbol (c.a.s); here $\mu'_0 := f_0(\mu)$, and it can be shown that $f_1 \equiv 0$. It can be shown that

$$(3.3) \quad S(\mu) = -\pi\mu' + \mathcal{O}(h^2) \quad \text{and}$$

* $\mu \mapsto C(\mu) + \mu'_0(\log|\mu'_0| - 1)$ is analytic for μ sufficiently small.

We still have to take into account the different asymptotic behaviour of the Γ -function in the following regions:

- (i) Let be $|\mu| \leq Ch$ for C arbitrarily large, but fixed. Viewing here the Γ -function as a holomorphic function, that is real on the real axis, we have $\arg\left[\Gamma\left(\frac{1}{2} - i\frac{\mu'}{h}\right)\right] = \mathcal{O}\left(\frac{\mu'}{h}\right)$.
- (ii) Let C be large enough. Then by the complex version of Stirling's formula we have in the region $Ch \leq |\mu'| \leq \frac{1}{C}$: $\arg\left[\Gamma\left(\frac{1}{2} - i\frac{\mu'}{h}\right)\right] = \frac{\mu'}{h}\left(1 - \log\left(\frac{|\mu'|}{h}\right)\right) + \frac{h}{\mu'}F\left(\frac{\mu'}{h}\right)$, where F is the real part of a function, that is holomorphic and bounded in $\left|\frac{\text{Im}z}{\text{Re}z}\right| \leq \frac{1}{C}$.

These observations allow it to simplify the phase of the cosine such that we get the following theorems:

Theorem 1: Let $C > 0$ be arbitrarily large, but fixed. Then the spectrum of P in $[-Ch, Ch]$ for h sufficiently small is the union of disjoint closed bands. Let μ' be defined as above for $\mu \in [-Ch, Ch]$. If μ' lies in a gap, then the length of this gap is given by

$$\frac{2h}{\left\{\log\left(\frac{1}{h}\right)\right\}} \left(\arccos \left[\left(1 + e^{-2\pi\frac{\mu'}{h}}\right)^{-\frac{1}{2}} \right] \right) + \mathcal{O}\left(\frac{h}{\left(\log\left(\frac{1}{h}\right)\right)^2}\right),$$

If μ' lies in a band, the length of this band is

$$\frac{2h}{\left\{\log\left(\frac{1}{h}\right)\right\}} \left(\arcsin \left[\left(1 + e^{-2\pi\frac{\mu'}{h}}\right)^{-\frac{1}{2}} \right] \right) + \mathcal{O}\left(\frac{h}{\left(\log\left(\frac{1}{h}\right)\right)^2}\right).$$

In particular: If $\mu' = o(h)$, then we see that the length of the gaps is tending to the length of the bands.

Theorem 2: If $C > 0$ is large enough and $h > 0$ is sufficiently small, then the spectrum of P in $\left[-\frac{1}{C}, -Ch\right]$ is the union of bands B_k separated by open gaps G_k with

$$|B_k| = \frac{2h}{C'(\mu)} \left(1 + \mathcal{O}\left(\frac{h^2}{\mu^2} \frac{1}{\log\left(\frac{1}{|\mu|}\right)}\right) \right) \arcsin \left[\left(1 + e^{-2\pi\frac{\mu'}{h}}\right)^{-\frac{1}{2}} \right]$$

for arbitrary $\mu \in B_k$, where μ' is given as above and $C(\mu)$ is defined by (2.7).

The distance between the centers of two consecutive bands is:

$$\left(1 + \mathcal{O}\left(\frac{h}{|\mu|} \frac{1}{\log\left(\frac{1}{|\mu|}\right)} \left(\frac{h}{|\mu|} + \frac{1}{\log\left(\frac{1}{|\mu|}\right)} \right) \right) \right) \frac{\pi h}{C'(\mu)}.$$

Remark: If C is very large, we conclude from our remarks above

$$\arcsin \left[\left(1 + e^{-2\pi\frac{\mu'}{h}}\right)^{-\frac{1}{2}} \right] \sim e^{-\frac{S(\mu)}{h}},$$

consequently

$$|B_k| \sim \frac{2h}{C'(\mu)} e^{-\frac{S(\mu)}{h}} \left(1 + \mathcal{O}\left(\frac{h^2}{\mu^2} \frac{1}{\log\left(\frac{1}{|\mu|}\right)}\right) \right).$$

This corresponds to the size of splitting in Theorem 3.1 of Gérard/Grigis, in view of the fact that $C'(\mu)$ is the half of the period of the Hamilton flow on the surface $\{p = \mu\}$.

Theorem 3: If $C > 0$ is large enough and $h > 0$ is sufficiently small, then the spectrum of P in $[Ch, \frac{1}{C}]$ is the union of bands B_k separated by open gaps G_k with

$$|G_k| = \frac{2h}{C'(\mu)} \left(1 + O\left(\frac{h^2}{\mu^2} \frac{1}{\log(\frac{1}{\mu})}\right) \right) \arcsin \left[\left(1 + e^{2\pi \frac{\mu'}{h}} \right)^{-\frac{1}{2}} \right].$$

for arbitrary $\mu \in G_k$ and with μ' and $C(\mu)$ as above.

The distance between the centers of two consecutive gaps is:

$$\left(1 + O\left(\frac{h}{\mu} \frac{1}{\log(\frac{1}{\mu})} \left(\frac{h}{\mu} + \frac{1}{\log(\frac{1}{\mu})} \right) \right) \right) \frac{\pi h}{C'(\mu)}.$$

Here we notice that $C'(\mu)$ is the time that needs a classical particle of energy μ for passing over the period. So in view of the behaviour of the amplitude of $\tau(\mu; h)$ we conclude that up to this modification the bands and the gaps exchange their roles.

Description of the method

4. Reduction to a Normal Form - The Branching Model

From now on we will make an extensive use of the microlocal theory due to Sjöstrand (see [Sj 1]). The essential ideas and the terminology can be found in the appendices of [He, Sj 2] and [Mz].

The operator P given by (1.1) is now viewed as an h -pseudodifferential operator, whose (principal) symbol is

$$(4.1) \quad p(x, \xi) = \frac{1}{2} \xi^2 + V(x).$$

p has a non-degenerate saddle point at $(0,0)$. So we can apply the results of appendix b of [He, Sj 2]: There exists a real analytic canonical transformation κ from a neighbourhood of $(0,0)$ onto a neighbourhood of $(0,0)$ and a realvalued function $f_0(t)$, defined in a neighbourhood of 0 such that

$$(4.2) \quad f_0(0) = 0, \quad f_0'(0) = 1$$

and

$$(4.3) \quad f_0 \circ p \circ \kappa = p_0,$$

where

$$(4.4) \quad p_0(x, \xi) = x\xi$$

is the (principal) symbol of the dilation generator $P_0 = \frac{1}{2}(xhD + hDx)$. We also have

$$(4.5) \quad d\kappa|_{(0,0)} = \kappa_{\frac{\pi}{4}} = \text{the rotation by the angle } \frac{\pi}{4} \text{ around } (0,0).$$

Furthermore, there exist a realvalued (formal) classical analytic symbol

$$(4.6) \quad F(t; h) = \sum f_j(t) h^j,$$

defined for t in a neighbourhood of 0 , and a formal unitary Fourier integral operator U associated to the canonical transformation κ , mapping functions defined microlocally in some fixed neighbourhood of $(0,0)$ (in a sense that can be made precise by means of FBI-transformations) to functions defined microlocally in some other fixed neighbourhood of $(0,0)$ such that

$$(4.7) \quad U^{-1} F(P;h)U = P_0.$$

In this equation $F(P;h)$ is defined by a functional calculus based on Cauchy's integral formula such that both members may be considered as analytic pseudodifferential operators with symbols defined in a neighbourhood of $(0,0)$ and hence are acting on functions defined microlocally in some fixed neighbourhood of $(0,0)$.

Let $\Gamma: u \mapsto \bar{u}$ be the complex conjugation, \mathcal{F}_h the h -Fourier-transformation and put $A_0 = \mathcal{F}_h \Gamma$. Then, since $[P_0, A_0] = [P, \Gamma] = 0$, we are able to modify the proof of (4.7) such that we find U (as above) satisfying

$$(4.8) \quad \Gamma U = U A_0.$$

U may be represented more explicitly by the (formal) expression

$$(4.9) \quad Uu(x) = 2^{\frac{1}{4}} e^{\frac{i\pi}{8}} (2\pi h)^{-\frac{1}{2}} \int e^{\frac{i}{h} \psi(x,y)} \sigma(x,y;h) u(y) dy,$$

where the phase function ψ is analytic near $(0,0)$ and is generating κ :

$$(4.10) \quad \kappa: (y, -\psi'_y(x,y)) \mapsto (x, \psi'_x(x,y)),$$

where by (4.5)

$$(4.11) \quad \psi(x,y) = -\frac{x^2}{2} + \sqrt{2} xy - \frac{y^2}{2} + \mathcal{O}((x,y)^3).$$

(4.8) implies

$$(4.12) \quad (i) \quad y = \psi'_y(x, \psi'_y(x,y)), \quad (ii) \quad \psi'_x(x,y) = -\psi'_x(x, \psi'_y(x,y)).$$

$\sigma(x,y;h) \sim \sigma_0(x,y) + h\sigma_1(x,y) + h^2\sigma_2(x,y) + \dots$ is a c.a.s. with $\sigma_0(0,0) = 1$, and from (4.8) we get

$$(4.13) \quad \frac{\overline{\sigma_0(x,y)}}{|\psi''_{yy}(x,y)|^{\frac{1}{4}}} = \frac{\sigma_0(x, \psi'_y(x,y))}{|\psi''_{yy}(x, \psi'_y(x,y))|^{\frac{1}{4}}}.$$

We will sketch now, why $f_1 \equiv 0$. Let be $P = p(x,\xi) + p_1(x,\xi)h + \dots$ a real-valued classical analytic symbol, defined near $(0,0)$. Assume that $(0,0)$ is a saddle point for p with critical value 0 . In view of the definition of $f(P)$, when f is a holomorphic function near 0 , we get for the Weyl-symbol of $f(P)$

$$(4.14) \quad \sigma(f(P)) = f(P(x,\xi;h)) + \mathcal{O}(h^2).$$

Furthermore we may replace κ by an h -dependent canonical transformation \mathfrak{K}_U , such that

$$(4.14) \quad F(P(x,\xi;h);h) \cdot \mathfrak{K}_U = p_0 + \mathcal{O}(h^2).$$

We apply this to action integrals, i.e. integrals of the form $I_q(\mu) = \int \xi dx$, where the integration is taken over some closed not necessarily real curve in $q^{-1}(\mu)$. Let μ and μ' be related by (3.2), and let \tilde{p}_0 be the left hand side of (4.14). We then have

$$(4.15) \quad 2\pi i \mu'_0 = I_{P_0}(\mu'_0) = I_P(\mu) = I_{\tilde{p}_0}(\mu') = I_{P_0}(\mu') + O(h^2) = 2\pi i \mu' + O(h^2),$$

hence $f_1 = 0$. Transforming $I_P(\mu)$ into an integral between turning points we verify (3.3).

5. Treatment of the Equation $(P - \mu)u = 0$

We will now use the following special solutions of the equation $(P_0 - \mu')v = 0$:

$$(5.1) \quad u_{\pm}^0(x) = H(\pm x) |x|^{-\frac{1}{2} + i\frac{\mu'}{h}}, \quad w_{\pm}^0 = B_0 u_{\pm}^0,$$

where $B_0 = \Gamma \mathcal{F}_h$. Any solution $v \in \mathcal{S}'$ of $(P_0 - \mu')v = 0$ is of the form $v = \alpha_+ u_+^0 + \alpha_- u_-^0 = \beta_+ w_+^0 + \beta_- w_-^0$, where the coefficients are related by

$$(5.2) \quad \begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix} = B_{\mu'/h} \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}.$$

B is a unitary, symmetric matrix; so it is only necessary to know the matrix element b_{11} :

$$(5.3) \quad b_{11} = e^{i(\frac{\mu'}{h} \log h - \frac{\pi}{4})} e^{\frac{\pi \mu'}{2h}} (2\pi)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} + i\frac{\mu'}{h}\right).$$

With respect to our microlocal framework we see that u_+^0 , w_+^0 , u_-^0 and w_-^0 are defined microlocally in some μ' -independent neighbourhood Ω' of $(0,0)$ and that they are uniformly (with respect to μ') microlocally concentrated to small neighbourhoods of $\{(y,0) | y \geq 0\} \cup \{(0,\eta) | \eta \in \mathbb{R}\}$, $\{(0,\eta) | \eta \geq 0\} \cup \{(y,0) | y \in \mathbb{R}\}$, $\{(y,0) | y \leq 0\} \cup \{(0,\eta) | \eta \in \mathbb{R}\}$, $\{(0,\eta) | \eta \leq 0\} \cup \{(y,0) | y \in \mathbb{R}\}$ respectively, where these neighbourhoods may be taken arbitrarily small, if we choose $|\mu'|$ sufficiently small.

Now we put

$$(5.4) \quad \begin{aligned} u_{++} &:= U u_+^0 & u_{--} &:= U u_-^0 \\ u_{+-} &:= U w_+^0 & u_{-+} &:= U w_-^0. \end{aligned}$$

We know that these $u_{\pm\pm}$ are solutions of

$$(5.5) \quad (P - \mu)u = 0,$$

microlocally defined in a neighbourhood Ω of $(0,0)$. The equation (5.5) is valid uniformly with respect to μ (small enough) in the sense that after applying an FBI-transform we get an analogue of (5.5), valid locally and with a uniformly exponentially decreasing error. Furthermore the u_{++} , u_{-+} , u_{--} and u_{+-} are microlocally concentrated to small neighbourhoods of $\gamma_{++}^{\Omega} \cup \gamma_{-+}^{\Omega} \cup \gamma_{+-}^{\Omega}$, $\gamma_{-+}^{\Omega} \cup \gamma_{--}^{\Omega} \cup \gamma_{+-}^{\Omega}$, $\gamma_{--}^{\Omega} \cup \gamma_{+-}^{\Omega} \cup \gamma_{++}^{\Omega}$ and $\gamma_{+-}^{\Omega} \cup \gamma_{++}^{\Omega} \cup \gamma_{--}^{\Omega}$ respectively, where $\gamma_{\pm\pm}^{\Omega} := \{(x,\xi) \in p^{-1}(0) \cap \Omega | \pm x \geq 0, \pm \xi \geq 0\}$ is one of the four bicharacteristic segments going out from $(0,0)$. So the microlocal theory of [Sj 1] tells us that the $u_{\pm\pm}$ are even well-defined as functions on an interval containing 0 in its interior, up to errors $r_{\pm\pm}(x,h)$, which

are uniformly (w.r.t. μ) of exponential decrease in some complex neighbourhood of 0, and satisfying (5.5) up to errors of the same type.

Now we will study $u_{\pm\pm}$ more closely outside 0. Using the definition of A_0, B_0 , (4.8), (5.1) and that $\mathcal{F}_h^2 u_+^0(x) = u_+^0(-x) = u_-^0(x)$, we obtain

$$(5.6) \quad u_{-+} = \Gamma u_{--}, \quad u_{+-} = \Gamma u_{++}.$$

Hence it is enough to study u_{++} and u_{--} .

According to the definition of u_+^0 and the expression (4.9) defining U , we write formally

$$(5.7) \quad u_{++}(x) = 2^{\frac{1}{4}} e^{i(\frac{\pi}{8} + \lambda)} (2\pi h)^{-\frac{1}{2}} \int_0^\infty e^{\frac{i}{h}(\psi(x,y) + \mu'_0 \log y)} \sigma_{(x,y;h)} e^{\frac{i}{h}(\mu' - \mu'_0) \log y} |y|^{-\frac{1}{2}} dy.$$

The critical point $y_c(x, \mu)$ of the phase $y \mapsto \psi(x, y) + \mu'_0 \log y$ is uniquely determined and holomorphic near $x_0 > 0$, $\mu = 0$, and the critical value $\varphi(x, \mu)$ satisfies the eiconal equation

$$(5.8) \quad p(x, \varphi'_x) - \mu = 0.$$

Near $x_0 > 0$ we can decompose u_{++} into the sum of two functions microlocally concentrated near γ_{++}^Ω and γ_{+-}^Ω respectively. Since $(x, \varphi'_x(x, \mu))$ lies on γ_{++}^Ω , the first one is precisely that one, we obtain by writing down the stationary phase expansion of (5.7) associated to the critical point $y_c(x, \mu)$. Thus near $x_0 > 0$ the contribution to u_{++} from a neighbourhood of $\gamma_{++}^\Omega \cap \Pi_x^{-1}(x_0)$ (where Π_x is the projection $(x, \xi) \mapsto x$) is of WKB-form:

$$(5.9) \quad u_{++}(x) = e^{\frac{i}{h} \varphi(x, \mu)} b(x, \mu; h)$$

with an c.a.s. b (in the (x, μ) -space) of order 0, satisfying $\arg b_0 = -\frac{\pi}{8}$. In the same manner we get

$$(5.10) \quad u_{--}(x) = e^{\frac{i}{h} \varphi(x, \mu)} d(x, \mu; h)$$

near $\gamma_{--}^\Omega \cap \Pi_x^{-1}(-x_0)$ with a c.a.s. d of order 0, satisfying $\arg d_0 = -\frac{\pi}{8}$. Here φ is another solution of (5.8), that (like φ) can be written down explicitly.

(5.9) and (5.10) extend to $\gamma_{++}^\Omega \cap \Pi_x^{-1}(i_\mu)$, $\gamma_{--}^\Omega \cap \Pi_x^{-1}(i_\mu - 2\pi)$ respectively, for each of the transport equations, determining the b_j (resp. d_j), can be solved over the whole interior of the corresponding well.

6. Computation of the Translation matrix

First we remark that u_{++} and u_{--} are independent in the sense that the Wronskian satisfies:

$$(6.1) \quad |W(u_{++}, u_{--})| \geq \frac{1}{C_\varepsilon} e^{-\frac{1}{h}(\eta(\mu) + \varepsilon)},$$

uniformly on a neighbourhood of $[0, 2\pi]$ for every $\varepsilon > 0$, where η is a continuous function with $\eta(0) = 0$. So it makes sense to compute the translation matrix with respect to u_{++} , u_{--} , which describes the exact operator of translation acting on the solution space of $(P - \mu)u = 0$ up to an exponentially small error.

Since u_{++} and $\tau_{2\pi}u_{-+}$ are WKB-solutions along γ_{++} , we have there

$$(6.2) \quad u_{++} = t \tau_{2\pi} u_{-+} \quad \text{with} \quad t = e^{id(\mu)/h} s(\mu;h),$$

where s is an analytic symbol of order 0 and $d(\mu)$ a real valued function. By a normalization argument of [He,Sj 2] (see also [Sj 2]) we can prove that

$$(6.3) \quad |t| = 1.$$

Fixing some $x_0 \in]0, 2\pi[$, we get (for $|\mu|$ small enough):

$$(6.4) \quad \begin{aligned} \arg t &= \frac{1}{h} (\varphi(x_0, \mu) + \varphi(x_0 - 2\pi, \mu)) + \arg b(x_0, \mu) + \arg d(x_0, \mu) \\ &= \frac{1}{h} (\varphi(x_0, \mu) + \varphi(x_0 - 2\pi, \mu)) - \frac{\pi}{4} + hr(\mu;h), \\ &= \frac{1}{h} (C(\mu) + \mu'_0 (\log |\mu'_0| - 1)) - \frac{\pi}{4} + hr(\mu;h), \end{aligned}$$

where $C(\mu)$ is given by (2.7) and $r(\mu;h)$ is a c.a.s. of order 0.

Recalling (5.2) and the fact that the matrix B is symmetric, we get

$$(6.5) \quad u_{++} = b_{11} u_{-+} + b_{12} u_{+-}.$$

So microlocally near $\gamma_{+-} \cap \Pi_x^{-1}(\dot{i}_\mu)$ we have

$$(6.6) \quad u_{++} = b_{12} u_{+-} = b_{12} \overline{u_{++}},$$

where in the last member we think of u_{++} as defined microlocally near $\gamma_{++} \cap \Pi_x^{-1}(\dot{i}_\mu)$.

Combining this with (6.2), we see that microlocally near $\gamma_{+-} \cap \Pi_x^{-1}(\dot{i}_\mu)$

$$(6.7) \quad u_{++} = b_{12} \bar{t} \tau_{2\pi} u_{--}.$$

The next work to do is to extend u_{++} further to the right, to a neighbourhood of 2π . Such an extension should be of the form

$$(6.8) \quad u_{++} = t \tau_{2\pi} u_{-+} + \tilde{t} \tau_{2\pi} u_{+-} = s \tau_{2\pi} u_{++} + \tilde{s} \tau_{2\pi} u_{--} \quad (\text{near } 2\pi).$$

Here the coefficient t is imposed by (6.2), and since from (6.6) $\tilde{s} = b_{12} t$, we get by (5.2)

$$(6.9) \quad s = \frac{t - b_{12}^2 \bar{t}}{b_{11}}.$$

The same considerations give

$$(6.10) \quad u_{--} = -b_{12} \bar{t} \tau_{2\pi} u_{++} + b_{11} \bar{t} \tau_{2\pi} u_{--},$$

such that the corresponding translation matrix is determined as:

$$(6.11) \quad \tilde{T}(\mu;h) = \begin{pmatrix} \frac{t - b_{12}^2 \bar{t}}{b_{11}} & -b_{12} \bar{t} \\ b_{12} \bar{t} & b_{11} \bar{t} \end{pmatrix}.$$

Taking into account the properties of B we easily find

$$(6.12) \quad \tilde{\tau}(\mu;h) = \frac{1}{2} \text{trace } \tilde{T}(\mu;h) = \text{Re} \left(\frac{t}{b_{11}} \right).$$

Inserting finally (6.4) and (5.3) we verify (3.1).

References

- [Gé,Gr] C. Gérard, A. Grigis: Precise Estimates of Tunneling and Eigenvalues near a Potential Barrier; *J. of Diff. Eq.* 72, 149-177 (1988)
- [Ha] E.M. Harrell: The Band Structure of a One-dimensional Periodic System in a Scaling Limit; *Ann. Physics* 119, 351-369 (1979)
- [He,Sj 1] B. Helffer, J. Sjöstrand: Multiple Wells in the Semiclassical Limit I, *Comm. PDE*, 9(4), 337-408 (1984)
- [He,Sj 2] B. Helffer, J. Sjöstrand: Semiclassical Analysis of Harper's Equation III; *Bull. de la S.M.F., Mémoire*, to appear (1990)
- [Hor] W. Horn: Semiclassical Approximation for Tunneling near the Top of a Potential Barrier and its Applications to Solid State Physics; Thesis, Univ. of California, Los Angeles (1989)
- [Ly,Ke] R. Lynn, J.B. Keller: Uniform Asymptotic Solutions of Second Order Linear Ordinary Differential Equations with Turning Points; *Comm. Pure Appl. Math.* 23, 379-408 (1970)
- [Mz] C. März: Spectral Asymptotics for Hill's Equation near the Potential Maximum; Thesis, Université de Paris Sud, (April 1990)
- [Ou] A. Outassourt: Comportement Semi-classique pour l'Opérateur de Schrödinger à Potentiel Périodique; *J. of Funct. Anal.* 72, 65-93 (1987)
- [Re,Si] M. Reed, B. Simon: *Methods of Modern Mathematical Physics IV*; Academic Press (1978)
- [Si] B. Simon: Semiclassical Analysis of Low Lying Eigenvalues III: Width of the Ground State Band in Strongly Coupled Solids; *Ann. Physics* 158, 415-420 (1984)
- [Sj 1] J. Sjöstrand: Singularités analytiques microlocales; *Astérisque* N° 95 (1982)
- [Sj 2] J. Sjöstrand: Density of States Oscillations for Magnetic Schrödinger Operators; Preprint (1990)
- [Sk] M.M. Skriganov: Geometric and Arithmetic Methods in the Spectral Theory of Multidimensional Periodic Operators; *Proc. of the Steklov Inst. of Math.* 171 (1987 (2))
- [We,Ke] M.I. Weinstein, J.B. Keller: Asymptotic Behavior of Stability Regions for Hill's Equation; *SIAM J. Appl. Math.* 47(5), 941-958 (1987)