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TRANSPARENT POTENTIALS AND HAMILTONIAN SYSTEMS

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This report is based an the joint work of J.-P.FRANCOISE and R.G.NOVIKOV (see [1]).

Theorems 1 and 2 of this report are variations of some results from [1].

Consider the equation

$$(i\partial_y + \partial_x^2 + v(x,y))\psi = 0 \tag{1}$$

where $v(x,y)$ is smooth rapidly decreasing potential. Consider also the scattering amplitude $f(k,\ell)$, $k \in \mathbb{R}$, $\ell \in \mathbb{R}$ for this equation. The scattering amplitude $f(k,\ell)$ can be defined using the solutions of (1) with the asymptotic behavior

$$\psi(x,y,k) = e^{i(kx-k^2y)} + \Theta(y)\,\frac{e^{ix^2/y}}{\sqrt{y}}\,f(k,\frac{x}{y}) + \bar{o}(\frac{1}{\sqrt{y}}) ,$$

$$x^2 + y^2 \to \infty \quad ; \; x/y = \ell$$

(where $\Theta(y)$ is the Heaviside function).

Consider the following problem. To describe and to construct all transparent potentials (i.e. potentials with zero scattering amplitude) for eq. (1) .

In order to present some results about this problem let us introduce into consideration the hamiltonian Calogero-Moser system (see [2,3])

$$\dot{x} = \frac{H_g}{P} \quad , \quad \dot{p} = -\frac{\partial H_g}{\partial x} \tag{2}$$

$$H_g(x,p) = \sum_{n=1}^{N} p_n^2 + g^2 \sum_{n \neq m}\sum (x_n - x_m)^{-2} , \text{ where } n,m = 1 \ldots N$$

For this report it is important to remind that

$$H_g(x,P) = \operatorname{tr} L^2 , \tag{3}$$

where

$$L_{nm} = p_n\,\delta_{nm} + (x_n - x_m)^{-1}\, ig(1 - \delta_{nm}) \tag{4}$$

Let $x_n(t)$ $(n = 1 \ldots N)$ be a solution of the Calogero-Moser system $(g^2 > 0)$. Write down its asymptotics for $t \to -\infty$ in the following form

$$x_n(t) \approx \xi_n t + \eta_n \quad \text{for } t \to -\infty \tag{5}$$

We shall consider solutions of (2) with additional symmetry

$$\xi_{2j} = \overline{\xi_{2j-1}} \quad , \quad \eta_{2j} = \overline{\eta_{2j-1}} \quad , \tag{6}$$

where $j = 1 \ldots M$, $N = 2M$. We assume also that

$$\xi_n \neq \xi_m \quad \text{if } n \neq m_o \tag{7}$$

we are sure that the following result is valid.

Hypothesis 1. If $x_n(t)$ $(n = 1 \dots 2M)$ is a solution of the Calogero-Moser system $(g^2 > 0)$ with the properties (5,6,7), then

$$Im(x_n(t)) \neq 0 \quad \text{for any} \quad n = 1 \dots 2M \quad \text{and any} \quad t \in \mathbb{R} \qquad (8)$$

[This hypothesis can be also formulated in the following way. If $x(t)$ is a solution of the Calogero-Moser system $(g^2 > 0)$ with the symmetric with respect real axis Cauchy data, i.e. $x_{2j}(0) = \overline{x_{2j-1}(0)}$, $\dot{x}_{2j}(0) = \overline{\dot{x}_{2j-1}(0)}$, and $x_n(0) \neq x_m(0)$ if $n \neq m$, then $Im(x_n(t)) \neq 0$ for any $n = 1 \dots 2M$ and any $t \in \mathbb{R}$].

It is clear (for us) how to prove the hypothesis 1 if $Re(\xi_n) = Re(\eta_n) = 0$ for $n = 1 \dots 2M$. It is quite possible, however, that somebody knowes how to prove this result in the general case.

Following result gives unexpected relation between transparent potentials for eq. (1) and the Calogero-Moser system.

Theorem 1. Let $x(t)$ is a solution of the Calogero-Moser system $(g^2 = 1)$ with the properties (5,6,7) then

$$v(x,y) = - \sum_{n=1}^{2M} (x - x_n(y))^{-2}, \quad \text{where} \quad x,y \in \mathbb{R} \qquad (9)$$

is a transparent potential for eq. (1); this potential is real, rational, decreasing at infinity function. Besides, if (8) is valid, then this potential is bounded Using numbers ξ_n and η_n from (5), this potential can be written in the following form

$$v(x,y) = 2\partial_x^2 \ln \det \Theta, \qquad (10)$$

where

$$\Theta_{nn} = -x + \xi_n y + \eta_n, \quad \Theta_{nm} = 2i / (\xi_n - \xi_m) \quad \text{for} \quad n \neq m.$$

Theorem 1 arises from the Krichever and Fokas-Ablowitz results [4,5].

Hypothesis 2. Potentials (9, 10) are dense in the space of all continuous, real, decreasing at infinity, transparent potentials for eq. (1).

Consider now a problem about transparent potentials for the equation

$$(-\partial_x^2 - \partial_y^2 + v(x,y))\psi = \psi, \qquad (11)$$

where $v(x,y)$ is smooth rapidly decreasing potential. It is convenient for us introduce new variables potential. It is convenient for us introduce new variables $z = x + iy$, $\bar{z} = x - iy$ and rewrite eq. (11) as

$$(-4\partial_z \partial_{\bar{z}} + v(z,\bar{z}))\psi = \psi .$$

The scattering amplitude $f(\lambda,\lambda')$, $\lambda \in \mathbb{C}$, $\lambda' \in \mathbb{C}$, $|\lambda| = |\lambda'| = 1$ for eq. (12) can be defined using the solutions of eq. (12) with the asymptotic behavior

$$\psi(z,\bar{z},\lambda) = \exp\left[\frac{i}{2}(\lambda\bar{z} + z/\lambda)\right] + \frac{e^{i|z|}}{\sqrt{|z|}} f\left(\lambda, \frac{z}{|z|}\right) + \bar{o}\left(\frac{1}{\sqrt{|z|}}\right)$$

$|z| \to \infty$.

In order to give an analogy of the theorem 1 for eq. (12) , we have to introduce the lower analogy of the Calogero-Moser system (see [1])

$$\dot{x} = \frac{\partial\tilde{H}_g}{\partial P} , \quad \dot{P} = -\frac{\partial\tilde{H}_g}{\partial x} , \quad \tilde{H}_g = \operatorname{tr} L^{-1}, \tag{13}$$

where L is defined by (4) .

Let $\chi_n(t)$ $(n = 1 \dots N)$ be a solution of the lower Calogero-Moser system $(g^2 > 0)$. Write down its asymptotics for $t \to -\infty$ in the following form

$$\chi_n(t) \approx \zeta_n^{-2} t - \eta_n \quad \text{for} \quad t \to -\infty. \tag{14}$$

We shall consider solutions of (13) $(g^2 > 0)$ with additional symmetries

$$\zeta_{2j} = -\zeta_{2j-1} , \quad \eta_{2j} - \eta_{2j-1} = -2i/\zeta_{2j-1} ; \tag{15}$$

$$\zeta_{4j-1} = -1/\bar{\zeta}_{4j-3} , \quad \zeta_{4j} = -1/\bar{\zeta}_{4j-2} ,$$

$$\bar{\eta}_{4j-3} = 2i\zeta_{4j-1} - \eta_{4j-1}\zeta_{4j-1}^2 , \quad \bar{\eta}_{4j-2} = 2i\zeta_{4j} - \eta_{4j}\zeta_{4j}^2 ,$$

where $j = 1 \dots M$, $N = 4M$. We assume also that

$$\zeta_n \neq \zeta_m \quad \text{if} \quad n \neq m \tag{16}$$

Let us reproduce one Grinevich result from [6] .

If numbers ζ_j , η_j , $j = 1 \dots 4M$ have the properties (15,16), then

$$v(z,\bar{z}) = -8\,\partial_z \partial_{\bar{z}} \ln \det \Theta , \tag{17}$$

where $\Theta_{nn} = \bar{z} - \zeta_n^2 z + \eta_n$, $\Theta_{nm} = 2/i(\zeta_n - \zeta_m)$ for $n \neq m$, is a real, decreasing at infinity, transparent potential eq. (12). Besides, it is stated in [6], that this potential is bounded.

Now it is time to present analogies of the theorem 1 and the hypothesis 2 for the case of eq. (12).

<u>Theorem 2</u>. *If* $\chi(t)$ *is a solution to the hamiltonian system (13)* $(g = 2)$ *with the properties (14,15,16) then*

$$v(z,\bar{z}) = -8 \sum_{n=1}^{4M} \frac{(d/dz)\,\chi_n(z)}{(\bar{z} + \chi_n(z))^2} \tag{18}$$

is a transparent potential for eq. (12).

Using numbers ζ_n *and* η_n *from (14), this potential can be written in the form (17).*

<u>Hypothesis 3</u>. *Potentials (17,18) are dense in the space of all continuous, real, decreasing at infinity, transparent potentials for eq. (12).*

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