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KINETIC APPROACH TO SYSTEMS OF CONSERVATION LAWS

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I. INTRODUCTION

We survey various relations between hyperbolic conservation laws arising in gas dynamics and Boltzmann formalism. Recent results show that these two formalisms are equivalent in the case of a single conservation law and in the case of the 2x2 system of isentropic gas dynamics. For the 3x3 system of gas dynamics, so precise results do not hold, but the kinetic formalism is still interesting, at least for numerical applications.

Of course the original motivation is to understand the Euler limit of the Boltzmann equation (see C. Cercignani [Ce]), but it appears that it can also be useful to derive new mathematical results using the tools of kinetic equations (L^p a priori estimates, compactness, approximation methods). This requires to know a large family of entropies, and it turns out that the kinetic formulation of scalar conservation laws or isentropic gas dynamics equations, as stated in P.L. Lions, B. Perthame, E. Tadmor [LPT1, LPT2] is a way to represent by a single equation all the family of entropies.

II. SCALAR CONSERVATION LAWS

The following results are due to P.L. Lions, B. Perthame and E. Tadmor [LPT1].

We consider a multidimensional scalar conservation law

$$(1) \qquad \frac{\partial u}{\partial t} + \sum_{i=1}^{n} \frac{\partial A_i(u)}{\partial x_i} = o \quad , \quad t \geq o \quad , \quad x \in \mathbb{R}^n$$

and we require that the solution satisfies the additional entropy conditions

$$(2) \quad \frac{\partial S(u)}{\partial t} + \sum_{i=1}^{n} \frac{\partial \eta_i(u)}{\partial x_i} \leq o \qquad \text{in } \mathcal{D}'$$

for all convex functions $S(\circ)$ and with

$$(3) \qquad \eta_i'(\circ) = S'(\circ)\, a_i(\circ) \ , \quad a_i(\circ) = A_i'(\circ) \in C^1(\mathbb{R}) \ .$$

It is wellknown (see C. Bardos [Ba], J. Smoller [S] for instance) that, even for smooth initial data, solutions to (1) have discontinuities. This prevents (2) to hold as an equality. S.N. Kruzkov [K] has shown that, adding the family of inequalities (2), the problem (1) - (3) has a unique solution u $\in L^\infty(\mathbb{R}_t^+ \, ; \, L^1(\mathbb{R}^n))$ for an initial data $u(x,t=o) \in L^1(\mathbb{R}^n)$.

II.1. Kinetic formulation

Let us introduce an additonal real parameter v and set

$$\chi(u;v) = \begin{cases} +1 & \text{if } o \leq v \leq u \ , \\ -1 & \text{if } u \leq v \leq o \ , \\ \ 0 & \text{otherwise} \end{cases}$$

Then, we say that the function u(x,t) satisfies the kinetic formulation of (1) if

(4) There is a bounded non-negative measure m on $\mathbb{R}^n_x \times \mathbb{R}_v \times \mathbb{R}^+_t$ such that

$$\frac{\partial \chi(u(x,t);v)}{\partial t} + a(v)\cdot\nabla_x \chi(u(x,t);v) = \frac{\partial m}{\partial v} \text{ in } \mathcal{D}'(\mathbb{R}^{n+1} \times \mathbb{R}^+),$$

where the vector field a(v) is defined in (3).

<u>**Theorem II.1.**</u> Let $u \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^n))$, then u satisfies (1) - (3) if and only if u satisfies (4).

<u>**Proof of theorem II.1.**</u> Being given u as in the statement of theorem II.1., define the distribution m by

(5) $$m := \frac{\partial}{\partial t}\int_0^v \chi(u(x,t)\ ;\ w)dw + \sum_{i=1}^n \frac{\partial}{\partial x_i}\int_0^v a_i(w)\ \chi(u(x,t)\ ;\ w)\ dw,$$

or equivalently

(6) $$\frac{\partial}{\partial t}\chi(u,v) + a(v)\cdot\nabla_x\chi(u,v) = \partial_v m.$$

Multiplying this equality by S'(v) where S is a C^2 function, we get

(7) $$\frac{\partial}{\partial t}\int S'(v)\ \chi(u,v)dv + \sum_{i=1}^n \frac{\partial}{\partial x_i}\int a_i(v)\ S'(v)\ \chi(u,v)dv$$

$$= \langle S'(v), \partial_v m \rangle = - \langle S''(v), m \rangle .$$

Notice that, since m is expected to be bounded, we are allowed in (7) to take S with bounded second derivatives. Finally, (7) can be written

$$\frac{\partial}{\partial t} S(u(x,t)) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \eta_i(u(x,t)) = - < S''(v), m >$$

and thus the set of inequalities (2) for any $S'' \geq o$ is equivalent to the non-negativity of the distribution m, which is therefore a measure.

The following bounds are obtained choosing successively $S'' = 1$, $\delta_w(v)$

(8) $$\int_{\mathbb{R}^+ \times \mathbb{R}^{n+1}} dm(x,v,t) \leq \frac{1}{2} \|u_o\|^2_{L^2(\mathbb{R}^n)} ,$$

(9) $$\int_{\mathbb{R}^+ \times \mathbb{R}^n} dm(x,v,t) \leq \|u_o\|_{L^1(\mathbb{R}^n)} , \ \forall v \in \mathbb{R} ,$$

(10) $$m(\circ,v,\circ) = o \quad \text{if } v \notin [\inf u_o, \sup u_o] ,$$

where $u_o(x) = u(x,t = o)$ is the initial data for the scalar conservation law.

Remarks : A similar formalism has been used for numerical purposes by Brenier [Br] who relates the entropy inequalities a Gibb's variational principle. For any non-decreasing function Σ

(11) $$\inf \left\{ \int_{\mathbb{R}} \Sigma(v) f(v) dv \ ; \ -1 \leq f(v) \leq + 1, \ vf(v) \geq o, \text{ and } \int_{\mathbb{R}} f(v) dv = u \right\}$$

is attained for the function $f = \chi(u;v)$.

II.2. Applications

This kinetic formulation can be used for several purposes. A first possibility is the construction of a semilinear hyperbolic approximation to the quasilinear hyperbolic equation (2), (3). This was achieved in B. Perthame, E. Tadmor [PT] using the model

$$
(12) \qquad \begin{cases} \frac{\partial}{\partial t} f_\varepsilon + a(v).\nabla_x f_\varepsilon + (f_\varepsilon - \chi(u_\varepsilon, v))/\varepsilon = o \\ f_\varepsilon(x,v,t = o) = \chi(u_o(x),v), \ u_\varepsilon(x,t) = \int_{\mathbb{R}} f_\varepsilon(x,v,t)dv \ . \end{cases}
$$

this non-linear equation has a unique solution which convergest to the solution of (1)-(3) as ε tends to o. A clear relation with (4) is described in [LPT1]. Indeed

$$
m_\varepsilon(x,v,t) \ := \ \int_o^v \Big(\chi(u_\varepsilon(x,t),w) \ - \ f_\varepsilon \ (x,w,t) \ \Big) / \varepsilon \ dw
$$

is a non-negative function, applying for instance the variational principle (11).

A second kind of applications developed in [LPT1], is the derivation of $L^p_{loc}(\mathbb{R}^n \times \mathbb{R}^+)$ estimates for the solution of (1)-(2) with an initial data in L^2, for some $p > 2$ depending on the non-linearity of the field A(u). This kind of result (variations are possible) relies on the moments lemma which provides the integrability of moments in v, locally in space for the solution of kinetic equations (see B. Perthame [P], P.L. Lion, B. Perthame [LP]). This is related to the dispersion effect of the variable a(v) in (4).

The most spectacular application of this formulation is still a regularizing effect proved in [LPT1]. Under suitable assumptions on the non-linearity A(u), the solution of (1), (2) with an initial data in $L^1 \cap L^\infty (\mathbb{R}^n)$, belongs to a Sobolev space in x and t. More precisely for any $\varepsilon > o$

(13) $\|u\|_{W^{s,1}((\varepsilon,1/\varepsilon)\times\mathbb{R}^n)} \le C(s,\varepsilon,M,K)$ with $s = \frac{\alpha}{2+\alpha}$

where $M = \|u_o\|_\infty$, $K = \|u_o\|_1$ and α is given by the non degeneracy condition

(14) $\sup\ [\mathrm{mes}\{|v| \le M,\ |\tau + a(v)\cdot\xi| \le \delta\}\ ;\ |\xi|^2 + |\tau|^2 = 1] \le C\ \delta^\alpha.$

this result is based on the version by R. Di Perna, P.L. Lions, Y. Meyer [DLM] of the averaging lemmas [GLPS]. In one space dimension and for a strictly convex non-linearity A(u), Oleinik's entropy condition gives

$$\frac{\partial}{\partial x}\, a(u) \le C/t$$

(see [S] for instance) which implies a BV estimate showing that the regularity (13) is not optimal. In this case it gives indeed $s = 1/3$ (α=1) while any $s < 1$ works. Finally, let us point out that the condition (14) is a multidimensional extension of the non-degeneracy condition introduced in one dimension by L. Tartar [T] to prove, using compensated compensation, that a family of initial data u_o^ε bounded in $L^2(\mathbb{R}^n)$ gives rise to a compact family of solutions $u^\varepsilon(x,t)$ in $L^2_{loc}(\mathbb{R}^n \times \mathbb{R}^+)$.

III. ISENTROPIC GAS DYNAMICS

The above results are due to P.L. Lions, B. Perthame, E. Tadmor [LPT2].

We now consider the 2x2 system of isentropic gas dynamics in one space dimension

$$(15)\qquad \begin{cases} \dfrac{\partial}{\partial t}\,\rho + \dfrac{\partial}{\partial x}\,\rho u = o\ , \\ \dfrac{\partial}{\partial t}\,\rho u + \dfrac{\partial}{\partial x}(\rho u^2 + p) = o\ , \qquad t \ge o,\ x \in \mathbb{R} \\ p(\rho) = \kappa\rho^\gamma,\ \gamma > 1,\ \kappa = \dfrac{(\gamma-1)^2}{4\gamma}\ . \end{cases}$$

The unknowns are the density $\rho(x,t)$ and the momentum $q = \rho u(x,t)$ which are conservative quantities. Following the classification introduced by P.D. Lax [L], (15) is a hyperbolic system which eigenvalues are $u \pm c$, $c = \sqrt{p'(\rho)}$. Thus they are distinct except when $\rho = o$.

III.1. Entropy inequalities

Let us seek the additonal conservation laws that can be deduced of (15) for smooth solutions i.e. the couples (η,F) such that

(16) $$\frac{\partial}{\partial t}\,\eta(\rho,u) + \frac{\partial}{\partial x}\,F(\rho,u) = o \ .$$

The natural couple (η,F) is given by the energy

$$\eta = \frac{1}{2}\,\rho u^2 + \frac{\kappa}{\gamma-1}\,\rho^\gamma \ .$$

Lemma III.1. (16) is satisfied iff

(17) $$\eta_{\rho\rho} = \frac{p'(\rho)}{\rho^2}\,\eta_{uu} \ ,$$

(18) $$F_\rho = u\eta_\rho + \frac{p'(\rho)}{\rho}\,\eta_u \ , \ F_u = \rho\eta_\rho + u\eta_u \ .$$

Proof of Lemma III.1. Notice that the equation of conservation of momentum can be written

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \frac{p'(\rho)}{\rho}\,\partial_x\,\rho = o$$

and multiply it by η_u . Adding the result to the equation of conservation of mass multiplied by η_ρ just yields (16) as soon as (18) is satisfied. Finally (18) is solvable iff (17) holds, thanks to Schwartz' equality $F_{\rho u} = F_{u\rho}$ and Poincaré theorem.

We are going to consider the so called weak entropies satisfying

(19) $\eta(\rho = o, u) = o$, $\eta_\rho(\rho = o, u) = g(u)$,

for smooth function u. We say that $(\rho, \rho u)$ is an entropy solution of (15) if

(20) $\frac{\partial}{\partial t}\eta + \frac{\partial}{\partial x} F \leq o$ in $\mathcal{D}'$ $(\mathbb{R}\times\mathbb{R}^+)$,

for any (η, F) solution of (17)-(20) which is convex in $(\rho, \rho u)$ (see [Ba, L, S] for motivations).

III.2. Kinetic formulation.

[LPT2] proposes to use the kinetic equilibrium given by

(21) $\chi(\rho \; ; \; v-u) = (\rho^{\gamma-1} - (v-u)^2 \;)_+^\lambda$,

where x_+ denotes the positive part of x, $x_+ = \sup(o,x)$, and

(22) $\lambda = \frac{3-\gamma}{2(\gamma-1)}$.

Finally set

(23) $\theta = \frac{\gamma-1}{2}$.

The kinetic formulation of (15) is given, setting $\chi = \chi(\rho(x,t) \; ; \; v-u(x,t)\;)$, by

(24) there is a non-positive bounded measure m on $\mathbb{R}^n_x \times \mathbb{R}_v \times \mathbb{R}^+_t$ such that

$$\frac{\partial}{\partial t} \chi + \frac{\partial}{\partial x} \{ [\theta v + (1-\theta)u] \chi \} = \partial_{vv} m \text{ in } \mathcal{D}'(\mathbb{R}^2 \times \mathbb{R}^+) .$$

Theorem III.2. [LPT2]. Let $(\rho,\rho u) \in L^\infty(\mathbb{R}^+ ; L^1(\mathbb{R}))$ have finite energy i.e. $\frac{1}{2} \rho u^2 + \rho^\gamma \in L^\infty(\mathbb{R}^+ ; L^1(\mathbb{R}))$. Then $(\rho,\rho u)$ is an entropy solution to (15) if and only if (24) holds.

Proof of theorem III.2. The solutions to (17), (19) are given by

$$(25) \qquad \eta(\rho,u) = \int_{\mathbb{R}} g(v)\, \chi(\rho,v-u)dv ,$$

$$(26) \qquad F(\rho,u) = \int_{\mathbb{R}} g(v)[\theta v + (1-\theta)u]\, \chi(\rho ; v-u)dv ,$$

(see R. J. Di Perna [DP], Chen [ch] or [LPT2]). Moreover it is proved in [LPT2] that η is convex in $(\rho,\rho u)$ if $g(v)$ is a convex function. Now (24) is equivalent to

$$(27) \qquad \partial_t \int_{\mathbb{R}} g(v)\, \chi dv + \partial_x \int_{\mathbb{R}} [\dots]\, g(v)\, \chi\, dv = \int_{\mathbb{R}} g''(v)\, dm$$

for any $g \in C^2(\mathbb{R})$ with subquadratic growth so that, by the assumption on the energy, the two integrals on the ℓ.h.s. of (27) are well defined. Finally (27) is equivalent to (20) iff m is a non-positive measure.

III.3. Remarks.

1. The energy is recovered using $g(v) = v^2/2$ and gives the estimate

$$(28) \qquad o \le \int_{\mathbb{R}^+ \times \mathbb{R}^2} dm(x,v,t) \le \int_{\mathbb{R}^2} \left(\frac{1}{2} \rho_o u_o^2 + \frac{\kappa}{\gamma-1} \rho_o^\gamma \right) dx$$

where $\rho_o(x)$, $u_o(x)$ denote the initial data $\rho(x,t=o)$, $u(x, t=o)$.

2. As in the scalar case, one can easily see that $m(x,v,t) = o$ for $(x,t) \in \mathcal{O}$, $v \in \mathbb{R}$ if (ρ,u) is a C^1 solution to (15) on $\mathcal{O}$.

3. The support of χ is given by

$$v \in \left[u-\rho^{\frac{\gamma-1}{2}} , u + \rho^{\frac{\gamma-1}{2}} \right] ,$$

then, setting $\lambda = \theta v + (1-\theta)u$, we have

$$\lambda \in [u-c, u+c]$$

which coincides with the natural speeds of propagation in (15).

4. Multiplying (24) by $(1,v)$ and integrating dv, we recover the two equations of (25). This does not seem to be a general feature (see the other example given in [LPT2].

5. Notice that χ is the fundamental solution of the "wave" equation (17). Indeed

$$\chi(o,v-u) = o , \qquad \chi_\rho(o,v-u) = \delta_o .$$

6. The main difficulty in using this kinetic formulation is that the advection in (24) is not purely kinetic. Nevertheless various applications are given in [LPT2]. For example it is easy to recover the invariant regions (see [S]) giving a priori L^∞ bounds on ρ, u depending on $\|\rho_o, u_o\|_{L^\infty}$. A new estimate is also deduced, from the version in [LP] of **moments lemma**,

$$(29) \qquad \int_o^{+\infty} \left[\rho(y,t) \ |u(x,t)|^3 + \rho^{\frac{3\gamma-1}{2}}(y,t) \right] dt$$

$$\leq C \int \left(\rho_o \ u_o^2 + \rho_o^\gamma \right) dx \ , \ \forall y \in \mathbb{R} \ ,$$

which holds for any entropy solution of (15). Finally, the weak $* - L^\infty$ stability is obtained for $\gamma \geq 3$ using compensated compactness, thus completing the range of γ for which global existence is proved (R.J. Di Perna [DP] did it for any $\gamma = \frac{N+2}{N}$, $N > 3$ and Chen [Ch] extended its proof to any $1 < \gamma \leq 5/3$].

REFERENCES

[Ba] C. BARDOS. Introduction aux problèmes hyperboliques. In CIME Lesson, LN 1047, Springer Verlag, Berlin. H.B. Da Veiga Editor.

[Br] Y. BRENIER, Résolution d'équations d'évolution quasilinéaires. J. Diff. Eq. 50(3), (1986), 375-390.

[Ce] C. CERCIGNANI. The Boltzmann Equation and its applications. Applied Math. Sc. 67, Springer Verlag, Berlin (1988).

[Ch] C.Q. CHEN. The compensated compactness method and the system of isentropic gas dynamics. Preprint MSRI - 00527-91, Mathematical Sciences Research Institute, Berkeley (1990).

[DP] R.J. DI PERNA. Convergence of approximate solutions to conservation laws. Arch. Rat. Mech. Anal. 82 (1983) pp. 27-70.

[DLM] R. DI PERNA, P.L. LIONS, Y. MEYER, L^p regularity of velocity averages. A Paraître dans Ann. IHP Anal. Non Lin., 1991.

[GLPS] F. GOLSE, P.L. LIONS, B. PERTHAME, R. SENTIS, Regularity of the moments of the solution of a transport equation. J. Funct. Anal. 76(1), (1988), 11°-125.

[K] S. KRUZKOV, First order quasi-linear equations with several space variables. Math. USSR Sb. 10(1970), 217-273.

[L] P.D. LAX. Hyperbolic systems of conservation laws and the mathematical theory of shock waves. CBMS-NSF conference n° 11, SIAM, Philadelfia (1973).

[LP] P.L. LIONS, B. PERTHAME. Moments, averaging and dispersion lemmas. C.R. Ac. Sc. Paris (1992) to appear.

[LPT1] P.L. LIONS, B. PERTHAME, E. TADMOR, Kinetic formulation of scalar conservation laws. Note C.R.A.S. t. Série 1 (1991).

[LPT2] P.L. LIONS. B. PERTHAME, E. TADMOR. Kinetic formulation of isentropic gas dynamics in preparation.

[P] B. PERTHAME, Higher moments for kinetic equations ; Applications to Vlasov-Poisson and Fokker-Planck Equations. Math. Methods in the Appl. Sc. 13(1990), 441-452.

[PT] B. PERTHAME, E. TADMOR, A kinetic equation with kinetic entropy functions for scalar conservation laws. A paraître dans Comm. in Math. Phys.

[S] J. SMOLLER, Shock waves and reaction diffusion equations. Springer-Verlag New York, Heidelberg-Berlin, (1982).

[T] L. TARTAR, In Research notes in Mathematics, 39, Henriot-Watt Symp. Vol. 4 Pitman Press Boston, London (1975), 136-211.