JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

JON JOHNSEN

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Journées Équations aux dérivées partielles (1995), p. 1-10

<http://www.numdam.org/item?id=JEDP_1995____A14_0>

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1. Summary

For simplicity's sake the following two model problems are considered on a bounded open set $\Omega \subset \mathbb{R}^n$, where $n \geq 2$ and $\Gamma := \partial\Omega$ is C^∞-smooth: first there is the Dirichlét problem

$$\begin{aligned} -\Delta\, u + u\partial_{x_1} u &= f \quad \text{in } \Omega, \\ \gamma_0 u &= \varphi \quad \text{on } \Gamma. \end{aligned} \tag{1.1}$$

Here $\gamma_0 u = u|_\Gamma$ and $-\Delta\, u = -(\partial_{x_1}^2 + \cdots + \partial_{x_n}^2)u$. Secondly there is the corresponding Neumann problem

$$\begin{aligned} -\Delta\, u + u\partial_{x_1} u &= f \quad \text{in } \Omega, \\ \gamma_1 u &= \varphi \quad \text{on } \Gamma, \end{aligned} \tag{1.2}$$

where $\gamma_1 u = \gamma_0(\vec{n} \cdot \operatorname{grad} u)$ with $\vec{n}$ denoting the unit outward normal vector-field near Γ. For the stationary Navier–Stokes equations and other problems, see Theorem 1.3 and Section 6.

The regularity of the solution $u(x)$ is studied here together with the question of carrying over weak solutions to other spaces. To obtain a unified treatment of various well-known scales of function spaces, the Besov spaces $B^s_{p,q}$ are considered together with the Triebel–Lizorkin spaces $F^s_{p,q}$; hereby $s \in \mathbb{R}$ and p and $q \in]0,\infty]$ in general, although $p < \infty$ is required throughout for the $F^s_{p,q}$ spaces.

Among the various identifications, recall eg that $B^s_{\infty,\infty} = C^s_*$ for $s > 0$ (the Hölder–Zygmund spaces); $B^s_{p,p} = W^s_p$ for $s \in \mathbb{R}_+ \setminus \mathrm{N}$, $1 < p < \infty$ (Sobolev–Slobodetskiĭ); $F^s_{p,2} = H^s_p$ for $s \in \mathbb{R}$, $1 < p < \infty$ (Bessel-potentials) so in particular this encompasses the W^k_p and L_p; $F^0_{p,2} = h_p$ for $0 < p < \infty$ (local Hardy space). The scales coincide when $p = q$, so $B^s_{2,2} = F^s_{2,2} = H^s$ is the usual Sobolev space for $s \in \mathbb{R}$.

On $\mathbb{R}^n$ the spaces are defined by means of Littlewood–Paley decompositions, $B^s_{p,q}(\overline{\Omega}) = r_\Omega B^s_{p,q}(\mathbb{R}^n)$ etc denotes the restriction to Ω; on Γ local coordinates are used. A concise review of the definition and the properties of the Besov and Triebel–Lizorkin spaces is given in [**8**], so details are omitted here; for a proper exposition the reader is referred to the books of H. Triebel [**14, 15**] and to Theorems 3.6 and 3.7 in M. Yamazaki's article [**16**].

Supported by the Danish Natural Sciences Research Counsil, no. 11–1221–1.

For the Dirichlét problem above there is the following result:

THEOREM 1.1. *Let $u(x)$ in $F^s_{p,q}(\overline{\Omega})$ be a solution of* (1.1) *for data $f(x)$ in $F^{t-2}_{r,o}(\overline{\Omega})$ and $\varphi(x)$ in $B^{t-\frac{1}{r}}_{r,r}(\Gamma)$, and suppose that*

$$s > \max(\tfrac{1}{2}, \tfrac{n}{p} - 1 + \tfrac{1}{2}\delta_{n2}), \tag{1.3}$$

$$t > \max(\tfrac{1}{2}, \tfrac{n}{r} - 1 + \tfrac{1}{2}\delta_{n2}). \tag{1.4}$$

Then $u(x)$ is also an element of $F^t_{r,o}(\overline{\Omega})$.

Analogously, if $u \in B^s_{p,q}(\overline{\Omega})$, $f \in B^{t-2}_{r,o}(\overline{\Omega})$ and $\varphi \in B^{t-\frac{1}{r}}_{r,o}(\Gamma)$, then (1.3)–(1.4) *imply that $u \in B^t_{r,o}(\overline{\Omega})$.*

The conditions (1.3)–(1.4) in the theorem are natural, for both γ_0 and $B(v) := v\partial_1 v$ make sense on $B^s_{p,q}(\overline{\Omega})$ and $F^s_{p,q}(\overline{\Omega})$ when (1.3) holds. Actually $B(\cdot)$ is even 'better behaved' on these spaces than $-\Delta$ then; this is made precise below by taking a specific $\delta = \delta(s,p)$ such that $\delta > 0$.

If one denotes $\mathcal{A}_D = \left(\begin{smallmatrix} -\Delta \\ \gamma_0 \end{smallmatrix}\right)$ and $\mathcal{B}(u) = \left(\begin{smallmatrix} B(u) \\ 0 \end{smallmatrix}\right)$, problem (1.1) becomes $\mathcal{A}_D u + \mathcal{B}(u) = \left(\begin{smallmatrix} f \\ \varphi \end{smallmatrix}\right)$. Then, if $\delta(s,p) > 0$ and $\delta(t,r) > 0$, ie if $\mathcal{B}(\cdot)$ respects the *direct* regularity properties of $\mathcal{A}_D$ at (s,p,q) and (t,r,o), the theorem asserts that $\mathcal{B}(\cdot)$ also respects the *inverse* regularity properties of $\mathcal{A}_D$ at these two parameters. Moreover, this holds for both of the $B^s_{p,q}$ and $F^s_{p,q}$ scales.

For the Neumann problem $\mathcal{A}_N u + \mathcal{B}(u) = \left(\begin{smallmatrix} f \\ \varphi \end{smallmatrix}\right)$, where $\mathcal{A}_N = \left(\begin{smallmatrix} -\Delta \\ \gamma_1 \end{smallmatrix}\right)$, there is

THEOREM 1.2. *Let $u(x)$ in $F^s_{p,q}(\overline{\Omega})$ be a solution of* (1.2) *for data $f \in F^{t-2}_{r,o}(\overline{\Omega})$ and $\varphi \in B^{t-1-\frac{1}{r}}_{r,r}(\Gamma)$, and suppose that*

$$s > \max(\tfrac{1}{p} + 1, \tfrac{n}{p} - 1 + \tfrac{1}{2}\delta_{n2}), \quad t > \max(\tfrac{1}{r} + 1, \tfrac{n}{r} - 1 + \tfrac{1}{2}\delta_{n2}). \tag{1.5}$$

Then $u(x)$ belongs to $F^t_{r,o}(\overline{\Omega})$. The analogous result holds in the $B^s_{p,q}(\overline{\Omega})$ spaces.

It turns out that Theorem 1.2 is rather more complicated to prove than Theorem 1.1. The reason for this is that the requirement $s > \frac{1}{2}$ is replaced by $s > \frac{1}{p} + 1$ (because of γ_1), which is 'bad' in its p-dependence. Roughly speaking, this means that if $F^s_{p,q}(\overline{\Omega}) + F^t_{r,o}(\overline{\Omega}) \subset F^{s_1}_{p_1,q_1}(\overline{\Omega})$, then (s_1, p_1, q_1) need not satisfy (1.5) even if both (s,p,q) and (t,r,o) do so. As outlined in Section 5 below the fine theory of pointwise multiplication provides estimates of $B(\cdot)$, that may be used to overcome the difficulties.

Instead of the model problems above, the methods may be applied to eg the stationary Navier–Stokes equations. For each of the five boundary conditions considered in [7] one finds regularity results for the solutions that correspond to either Theorem 1.1 or Theorem 1.2. See [**11**, Thm. 5.5.5] or [**10**] for this.

In addition the existence of weak solutions of the Dirichlét problem may be carried over to the $B^s_{p,q}$ and $F^s_{p,q}$ spaces in this way. In more details the problem is:

$$\begin{aligned} -\Delta\, u + \sum_{j=1}^{n} u\partial_j u + \operatorname{grad} \mathbf{p} &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= g \quad \text{in } \Omega, \\ \gamma_0 u &= \varphi \quad \text{on } \Gamma. \end{aligned} \tag{1.6}$$

Here the solution $(u, \mathbf{p})$ and the data (f, g, φ) are sought such that

$$\begin{aligned} &u \in B^s_{p,q}(\overline{\Omega})^n, \qquad \mathbf{p} \in B^{s-1}_{p,q}(\overline{\Omega}) \\ &f \in B^{s-2}_{p,q}(\overline{\Omega})^n, \qquad g \in B^{s-1}_{p,q}(\overline{\Omega}), \qquad \varphi \in B^{s-\frac{1}{p}}_{p,q}(\Gamma)^n, \end{aligned} \tag{1.7}$$

for $s > \max(\frac{1}{2}, \frac{n}{p} - 1 + \frac{1}{2}\delta_{n2})$; observe that the problem in (1.1) may serve as a model problem for (1.6). For the $F^s_{p,q}$ spaces the requirement is the same, but again $B^{s-\frac{1}{p}}_{p,q}(\Gamma)$ should be replaced by $B^{s-\frac{1}{p}}_{p,p}(\Gamma)$.

Concerning the existence of solutions when $g = 0$ there is:

THEOREM 1.3. *Let $\Omega \subset \mathbb{R}^n$, where $n = 2$ or 3, be a C^∞-smooth open bounded set, and let Ω be connected with finitely many components of Γ, ie $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_N$.*

Suppose that the data $(f, 0, \varphi)$ belong to the spaces indicated in (1.7) for a parameter (s, p, q) satisfying one of the following conditions:

(1) $s > \max(1, \frac{n}{p} + 1 - \frac{n}{2})$;
(2) $s > 1$, $s = \frac{n}{p} + 1 - \frac{n}{2}$ *and* $q \le 2$;
(3) $s = 1$ *and* $p \ge 2 \ge q$.

Assume in addition that $\int_{\Gamma_j} \vec{n} \cdot \varphi = 0$ for $j = 1, \dots, N$.

Then there exists a solution $(u, \mathbf{p})$ of (1.6) as in (1.7) above.

For the $F^s_{p,q}$ spaces the analogous result holds (for any $q \in \,]0, \infty]$ in (2)).

The special case with $(s, p, q) = (1, 2, 2)$ is identical to the classical result on weak solutions, cf [**13**]. As a particular case the theorem gives a solvability theory in the Hölder–Zygmund spaces $C^s_*(\overline{\Omega})$ for $s > 1$.

In addition solutions may be constructed by successive approximations for any $s > \max(\frac{1}{2}, \frac{n}{p} - 1 + \frac{1}{2}\delta_{n2})$ in (1.7) provided only that $\int_\Omega g = \int_\Gamma \vec{n} \cdot \varphi$ and that the norms of the data are small enough; for the present spaces, this is elaborated in [**11**]. In comparison Theorem 1.3 asserts that when $g = 0$ and s is sufficiently large (plus some stricter conditions on Ω and φ), then solutions exist for arbitrarily large data.

In view of this even the C^s_* result should be new.

The purpose of this paper is only to indicate the proofs of the theorems; a detailed exposition is in preparation [**10**]. The results are based on [**8, 9**].

2. The pseudo-differential boundary operators

For an efficient treatment of the problems in (1.1), (1.2) and (1.6) one can utilise the calculus of pseudo-differential boundary operators of L. Boutet de Monvel [**1**] for the linear parts. An extension of this calculus to the $B^s_{p,q}$ and $F^s_{p,q}$ scales may be found in [**8, 11**] (with the results of J. Franke (partially contained) in [**2**] and the H^s_p and $B^s_{p,p}$ versions of G. Grubb [**4**] as forerunners).

Introductions to the calculus may be found in [**5**, Sect. 2] and [**3**, Sect. 1.1 ff], or [**4**, Sect. 4], so here it is recalled that the generic object to study is a Green operator

$$\mathcal{A} = \begin{pmatrix} P_\Omega + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(\overline{\Omega})^N \\ \oplus \\ C^\infty(\Gamma)^M \end{matrix} \to \begin{matrix} C^\infty(\overline{\Omega})^{N'} \\ \oplus \\ C^\infty(\Gamma)^{M'} \end{matrix}, \tag{2.1}$$

whereby $P_\Omega = r_\Omega P e_\Omega$ denotes the truncation to Ω of a pseudo-differential operator on $\mathbb{R}^n$; T is a trace operator, K a Poisson operator and S is a pseudo-differential operator on Γ; finally G is a singular Green operator.

To assure that $P_\Omega(C^\infty(\overline{\Omega})^N) \subset C^\infty(\overline{\Omega})^{N'}$ the so-called transmission condition at Γ is imposed on P (cf the elementary exposition in [**5**, Sect. 1]). More precisely, the results in [**8**] have been established for the space-uniformly estimated calculus, for which the Hörmander class $S^d_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is the basic symbol class on $\mathbb{R}^n$; this version of the calculus has been introduced systematically in [**6**]. Hence P is required to satisfy the uniform two-sided transmission condition at Γ, and for $\mathcal{A}$ of the described kind the main result in [**8**] is:

THEOREM 2.1. *Suppose all entries in $\mathcal{A}$ have order $d \in \mathbb{Z}$ and that both T and $P_\Omega + G$ are of class $r \in \mathbb{Z}$. Then there is continuity of*

$$\mathcal{A}\colon \begin{matrix} B^s_{p,q}(\overline{\Omega})^N \\ \oplus \\ B^{s-\frac{1}{p}}_{p,q}(\Gamma)^M \end{matrix} \to \begin{matrix} B^{s-d}_{p,q}(\overline{\Omega})^{N'} \\ \oplus \\ B^{s-d-\frac{1}{p}}_{p,q}(\Gamma)^{M'} \end{matrix}, \tag{2.2}$$

$$\mathcal{A}\colon \begin{matrix} F^s_{p,q}(\overline{\Omega})^N \\ \oplus \\ B^{s-\frac{1}{p}}_{p,p}(\Gamma)^M \end{matrix} \to \begin{matrix} F^{s-d}_{p,q}(\overline{\Omega})^{N'} \\ \oplus \\ B^{s-d-\frac{1}{p}}_{p,p}(\Gamma)^{M'} \end{matrix}, \tag{2.3}$$

for $s > r + \max(\frac{1}{p} - 1, \frac{n}{p} - n)$. In both cases, boundedness can only hold for $s < r + \max(\dots)$ if both $\operatorname{class}(T)$ *and* $\operatorname{class}(P_\Omega + G)$ *are $< r$.*

When all symbols are poly-homogeneous and $\mathcal{A}$ is elliptic, the theorem applies also to any parametrix $\widetilde{\mathcal{A}}$, and it was shown in [**4**, Thm. 5.4] that $\widetilde{\mathcal{A}}$ can be taken of class $r - d$. In general this result is best possible because $\mathcal{A}$ is a parametrix of $\widetilde{\mathcal{A}}$. (An exception is when $\mathcal{A}$ itself only contains a negligible part of class r.)

With obvious modifications the theorem also holds for multi-order and multi-class operators (of the Douglis–Nirenberg type) or when either M or $M' = 0$; see [**8**, Thm. 5.2]. As examples there are then $\mathcal{A}_D$ and $\mathcal{A}_N$; throughout

$$\widetilde{\mathcal{A}}_D := \begin{pmatrix} R_D & K_D \end{pmatrix} = \mathcal{A}_D^{-1} \tag{2.4}$$

will serve as a special choice of parametrix of $\mathcal{A}_D$.

For convenience $\mathbb{D}_k$, with $k \in \mathbb{Z}$, will denote the admissible parameters (s, p, q) for which the inequality

$$s > k + \max(\tfrac{1}{p} - 1, \tfrac{n}{p} - n) \tag{2.5}$$

holds. Equivalently this means that $s > k - 1 + \frac{1}{p} + (n-1)(\frac{1}{p} - 1)_+$.

3. Product estimates

The bilinear operator $B(v,w) = v\partial_1 w$, that has been used above with $B(v) := B(v,v)$, is analysed as the composite

$$(v,w) \mapsto (v, \partial_1 w) \mapsto \pi(v, \partial_1 w), \tag{3.1}$$

where $\pi(f,g)$ denotes $f(x) \cdot g(x)$. More precisely, $\pi(\cdot,\cdot)$ is the following generalisation, that eg allows $s > \frac{1}{2}$ (instead of $s > 1$) in (1.3):

DEFINITION 3.1. For u and $v \in \mathcal{S}(\mathbb{R}^n)$ let, with $\psi_k(\xi) = \psi(2^{-k}\xi)$,

$$\pi(u,v) = \lim_{k\to\infty} \mathcal{F}^{-1}(\psi_k \hat{u}) \cdot \mathcal{F}^{-1}(\psi_k \hat{v}), \tag{3.2}$$

whenever the limit, calculated in $\mathcal{D}'(\mathbb{R}^n)$: (i) exists for each $\psi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 near 0, and (ii) is independent of such ψ's.

This product has been studied in [**9**], where it is shown that it fills a part of the gap between two immediate meanings of 'pointwise multiplication': $\pi(f,u) = fu$ for $f \in \mathcal{O}_M$ and $u \in \mathcal{S}'$, and $\pi(f_0, f_1) = f_0 \cdot f_1$ when the f_j lie in $L_{p_j}^{\mathrm{loc}} \cap \mathcal{S}'$ such that $0 \le \frac{1}{p_0} + \frac{1}{p_1} \le 1$ (so that $f_0 \cdot f_1 \in L_1^{\mathrm{loc}}$).

Moreover, for an open set $\Omega \subset \mathbb{R}^n$ there is a restriction to Ω defined as

$$\pi_\Omega(u,v) = \lim_{k\to\infty} r_\Omega(\mathcal{F}^{-1}(\psi_k \hat{u}_1)\mathcal{F}^{-1}(\psi_k \hat{v}_1)), \tag{3.3}$$

when the limit exists in $\mathcal{D}'(\Omega)$ and satisfies (i) and (ii) for some u_1 and v_1 in $\mathcal{S}'$ such that $r_\Omega u_1 = u$ and $r_\Omega v_1 = v$. (The existence of such a pair (u_1, v_1) implies that the limit exists, equals $\pi_\Omega(u,v)$ and fulfils (i) and (ii) for any other pair restricting to (u,v).)

Perhaps more importantly, the continuity properties of $\pi(\cdot,\cdot)$ may be obtained by para-multiplication. For spaces over Ω the definition in (3.3) allows one to carry boundedness over from $\pi(\cdot,\cdot)$ to $\pi_\Omega(\cdot,\cdot)$, cf [**9**, Thm. 7.2].

For simplicity's sake only the needed $F^s_{p,q}$ results will be recalled. For the Besov spaces it is necessary with a stricter control over the sum-exponents q, but in the end this does not affect the results in Theorems 1.1–1.3; hence these technicalities are omitted here.

THEOREM 3.2. *The product in* (3.3) *is defined on* $F^{s_0}_{p_0,q_0}(\overline{\Omega}) \times F^{s_1}_{p_1,q_1}(\overline{\Omega})$ *when*

$$s_0 + s_1 > \max(0, \tfrac{n}{p_0} + \tfrac{n}{p_1} - n), \tag{3.4}$$

and then there is boundedness

$$\pi_\Omega(\cdot,\cdot) \colon F^{s_0}_{p_0,q_0}(\overline{\Omega}) \oplus F^{s_1}_{p_1,q_1}(\overline{\Omega}) \to F^{s_2}_{p_2,q_2}(\overline{\Omega}) \tag{3.5}$$

if all of the following conditions are fulfilled:

$$s_2 < \min(s_0, s_1); \tag{3.6}$$

$$s_2 - \tfrac{n}{p_2} \le \min(s_0 - \tfrac{n}{p_0}, s_1 - \tfrac{n}{p_1}, s_0 + s_1 - \tfrac{n}{p_0} - \tfrac{n}{p_1}); \tag{3.7}$$

$$s_2 - \tfrac{n}{p_2} = s_0 - \tfrac{n}{p_0} \quad \text{and} \quad s_1 = \tfrac{n}{p_1} \quad \text{hold only if} \quad p_1 \le 1; \tag{3.8}$$

$$s_2 - \tfrac{n}{p_2} = s_1 - \tfrac{n}{p_1} \quad \text{and} \quad s_0 = \tfrac{n}{p_0} \quad \text{hold only if} \quad p_0 \le 1. \tag{3.9}$$

Here it suffices with $s_2 \le \min(s_0, s_1)$ *in* (3.6) *provided* $q_2 \ge q_j$ *if* $s_2 = s_j$.

For this result the reader is referred to the theorems in [**9**, Sect.s 6 and 7]. Since $\pi_\Omega(\cdot,\cdot)$ is commutative, it may be assumed that $s_0 \geq s_1$, and then the value, p_1^*, of p_2 for which there can be equality in both (3.6) and (3.7) is given by the formula

$$\tfrac{n}{p_1^*} = \tfrac{n}{p_1} + (s_0 - \tfrac{n}{p_0})_- + (s_1 - \tfrac{n}{p_1} - (s_0 - \tfrac{n}{p_0})_+)_+. \tag{3.10}$$

REMARK 3.3. In Theorem 3.2 the receiving spaces $F^{s_2}_{p_2,q_2}$ are determined implicitly by (3.6)–(3.9). But, since Ω is bounded, $F^{s_1}_{p_1^*,q}(\overline{\Omega}) \hookrightarrow F^{s_2}_{p_2,q_2}(\overline{\Omega})$ holds in any case, if $q = q_1$ for $s_0 > s_1$ and if $q = \max(q_0, q_1)$ for $s_0 = s_1$. Thus the receiving space with $(s_2, p_2, q_2) = (s_1, p_1^*, q)$ may be considered as optimal.

4. The Dirichlét model problem

This section concerns the proof of Theorem 1.1. Preference will be given to the Triebel–Lizorkin spaces for simplicity, however, everything holds mutatis mutandem for the Besov spaces as well.

Firstly, for the linear parts of (1.1), there is boundedness of

$$\mathcal{A}_D := \begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix} : F^s_{p,q}(\overline{\Omega}) \to \begin{matrix} F^{s-2}_{p,q}(\overline{\Omega}) \\ \oplus \\ F^{s-\frac{1}{p}}_{p,p}(\Gamma) \end{matrix} \tag{4.1}$$

for each parameter (s,p,q) with $s > 1 + \max(\frac{1}{p} - 1, \frac{n}{p} - n)$, ie in $\mathbb{D}_1$.

For $\mathcal{A}_D$, the calculus asserts that the parametrix $\widetilde{\mathcal{A}}_D$ is bounded in the opposite direction in (4.1) for each parameter $(s,p,q) \in \mathbb{D}_1$.

Secondly, when the non-linear term $u\partial_{x_1}u$ is taken into consideration too, it is found from (1.1) that

$$u = R_D f + K_D\varphi - R_D(u\partial_{x_1}u). \tag{4.2}$$

This turns out to be meaningful when $s > \max(\frac{1}{p}, \frac{1}{2}, \frac{n}{p} - \frac{1}{2})$ for $n = 2$ and for $s > \max(\frac{1}{p}, \frac{1}{2}, \frac{n}{p} - 1)$ when $n \geq 3$, so in general the condition is

$$s > \max(\tfrac{1}{2}, \tfrac{n}{p} - 1 + \tfrac{1}{2}\delta_{n2}). \tag{4.3}$$

To obtain this one can derive from Theorem 3.2 that $u \mapsto (u, \partial_{x_1}u) \mapsto u\partial_{x_1}u$ is bounded, for some $\delta(s,p)$,

$$F^s_{p,q}(\overline{\Omega}) \to F^s_{p,q}(\overline{\Omega}) \oplus F^{s-1}_{p,q}(\overline{\Omega}) \to F^{s-2+\delta}_{p,q}(\overline{\Omega}), \tag{4.4}$$

in general when $s > \max(\frac{1}{2}, \frac{n}{p} - \frac{n-1}{2})$, cf (3.4). Here the deficit $\delta(s,p)$ measures how much the order of $B(\cdot)$ deviates from the order of $-\Delta$.

It is essential that the non-linear term is *more* regular than $\mathcal{A}_D u$ when (4.3) holds. In fact $\delta(s,p)$ equals $1 + \min(0, s - \frac{n}{p})$ — an increasing function of $s - \frac{n}{p}$ — except that $\delta(\frac{n}{p}, p) = 1 - \varepsilon$ for some arbitrary $\varepsilon > 0$. Hence $\delta(s,p) > 0$ as long as $s > \max(\frac{1}{2}, \frac{n}{p} - 1 + \frac{1}{2}\delta_{n2})$.

Thirdly, after these preparations, an iteration yields that $u \in F^t_{r,o}(\overline{\Omega})$: observe that in (4.2) one has, by (4.4) and Theorem 2.1 applied to $\widetilde{\mathcal{A}}_D$, for the summands on the right hand side that

$$R_D f + K_D\varphi \in F^t_{r,o}(\overline{\Omega}), \qquad R_D(u\partial_{x_1}u) \in F^{s+\delta(s,p)}_{p,q}(\overline{\Omega}). \tag{4.5}$$

By determination of $F^{s_1}_{p_1,q_1} \supset F^t_{r,o} + F^{s+\delta(s,p)}_{p,q}$, it follows that $u \in F^{s_1}_{p_1,q_1}$. Application of (4.4) then gives $R_D(u\partial_{x_1}u) \in F^{s_1+\delta(s_1,p_1)}_{p_1,q_1}$ etc.

In the case $r = p$ one may take $p_1 = p$, and, because $\delta(s,p) > 0$, the process ends with the conclusion that $u \in F^s_{p,q}$ in approximately $|t-s|/\delta(s,p)$ steps (as is well known).

For $r \neq p$ the conclusion follows by consideration of four different cases, namely those with the combinations of $s + \delta(s,p) \gtreqless t$ and $t - \frac{n}{r} \gtreqless s - \frac{n}{p}$. The procedure is far easier to sketch with a diagram than with words, so the reader is referred to Figure 1.

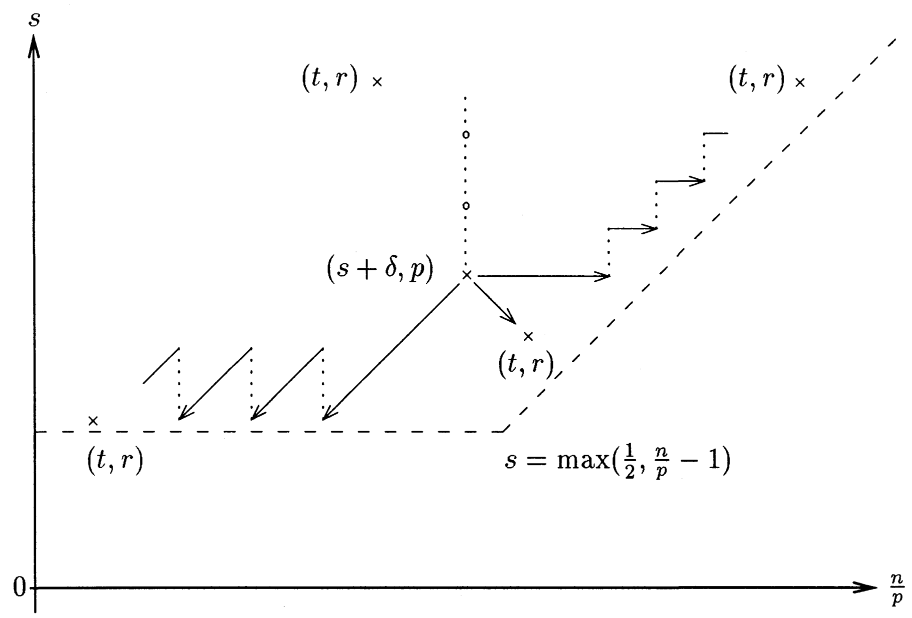

FIGURE 1. The p-dependent iteration (for $n = 3$).

The figure displays the location of $F^{s+\delta(s,p)}_{p,q}$ and four examples of $F^t_{r,o}$ corresponding to the subdivision mentioned above. However, the sum-exponents q and o are not represented. The sector where $\delta(s,p) > 0$, ie $s > \max(\frac{1}{2}, \frac{n}{p} - 1 + \frac{1}{2}\delta_{n2})$, is indicated in dashed line; note that the '$\times$' representing $F^{s+\delta}_{p,q}$ and $F^t_{r,o}$ all lie *inside* the sector.

The arrows in full line indicates embeddings $F^{s+\delta(s,p)}_{p,q} \hookrightarrow F^{s_1}_{p_1,q_1}$ etc: there are Sobolev embeddings down the lines of slope 1, and to the right along horizontal lines because Ω has finite measure. The dotted line indicates the improved knowledge of the non-linear term, ie the spaces $F^{s_1+\delta(s_1,p_1)}_{p_1,q_1}$ etc.

Note that one of the four cases is trivial since $F^{s+\delta}_{p,q} \hookrightarrow F^t_{r,o}$, while another in finitely many steps (indicated by '$\circ$') reduces to this or to one of the cases with either the "sawtooth" or the "staircase" manoeuvres.

Altogether this leads to the proof of Theorem 1.1.

REMARK 4.1. The procedure followed above has been used by S. I. Pohožaev, at least in the case with $s = t$ and $r > p$, cf **[12]**.

5. The Neumann problem

For the Neumann problem in (1.2) the arguments in Section 4 turn out to require more detailed estimates of the non-linear term. The reasons for this will be described in the following.

For the problem in (1.2), one should take $(s,p,q)\in\mathbb{D}_2$, for then

$$\gamma_1\colon F^s_{p,q}(\overline{\Omega})\to F^{s-1-\frac{1}{p}}_{p,p}(\Gamma), \tag{5.1}$$

is bounded. It is important here that γ_1 can not be continuous from $F^s_{p,q}(\overline{\Omega})$ unless $s\ge 2+\max(\frac{1}{p}-1,\frac{n}{p}-n)$ holds, so the restriction for (s,p,q) can not be essentially improved. For the Green operator $\mathcal{A}_N=\begin{pmatrix}-\Delta\\ \gamma_1\end{pmatrix}$ this means that the class is 2.

Since $\mathcal{A}_N$ is elliptic, the Boutet de Monvel calculus asserts that there exists a parametrix $\widetilde{\mathcal{A}}_N=\begin{pmatrix}R_N & K_N\end{pmatrix}$ of class 0 — but not lower — that is bounded

$$\widetilde{\mathcal{A}}_N\colon \begin{matrix}F^{s-2}_{p,q}(\overline{\Omega})\\ \oplus\\ F^{s-1-\frac{1}{p}}_{p,p}(\Gamma)\end{matrix}\to F^s_{p,q}(\overline{\Omega}) \tag{5.2}$$

when $(s,p,q)\in\mathbb{D}_2$. Hence the class of R_N is 0, so R_N can *not* be extended to an operator that is continuous from $F^{s-2}_{p,q}(\overline{\Omega})$ when $s-2<\max(\frac{1}{p}-1,\frac{n}{p}-n)$. Again the restriction on (s,p,q) can not be essentially improved.

Contrary to the Dirichlét case above, $\widetilde{\mathcal{A}}_N$ is not an inverse, but

$$\widetilde{\mathcal{A}}_N\mathcal{A}_N=1-\mathcal{R} \tag{5.3}$$

for an operator $\mathcal{R}$ of order $-\infty$ and class ≤ 2. In fact, this regularising operator may be taken as $\mathcal{R}u=|\Omega|^{-1}\int_\Omega u$ by a specific choice of $\widetilde{\mathcal{A}}_N$.

Hence (4.2) is replaced by $u=R_Nf+K_N\varphi+\mathcal{R}u-R_N(u\partial_{x_1}u)$, where the first three terms belong to $F^t_{r,o}(\overline{\Omega})$.

The conditions that make $u\partial_{x_1}u$ defined remain the same, of course, so the assumption for (s,p,q) in (4.3) is here replaced by

$$s>\max(\tfrac{1}{p}+1,\tfrac{n}{p}-1+\tfrac{1}{2}\delta_{n2}) \tag{5.4}$$

and similarly for (t,r,o). Cf (1.5).

Now the cases with $t<s+\delta(s,p)$ and $t-\frac{n}{r}>s+\delta-\frac{n}{p}$ are rather more complicated than the corresponding cases for the Dirichlét problem. The reason for this is that the space $F^{s_1}_{p_1,q_1}\supset F^{s+\delta}_{p,q}+F^t_{r,o}\ni u$ seemingly may be much too large for an application of the non-linear operator $v\mapsto R_N(v\partial_{x_1}v)$ to it.

Indeed, whilst $t=s_1$ the integral-exponent p_1 is smaller than r, and in many cases $s_1<1+\frac{1}{p_1}$ holds, although $t>1+\frac{1}{r}$. An example is sketched in Figure 2 below, where the sector determined by (5.4) is indicated by dashes. When s_1 is close to $\frac{n}{p_1}-1$, then the deficit $\delta(s_1,p_1)$ is close to 0, and so eventually

$$s_1-2+\delta(s_1,p_1)<\tfrac{1}{p_1}-1. \tag{5.5}$$

In such cases, since $\mathrm{class}(R_N) = 0$, the solution operator R_N simply *does not* make sense on $F^{s_1-2+\delta(s_1,p_1)}_{p_1,q_1}(\overline{\Omega})$, as recalled after (5.2). By comparison with the Dirichlét problem, the iteration is seemingly unable to begin.

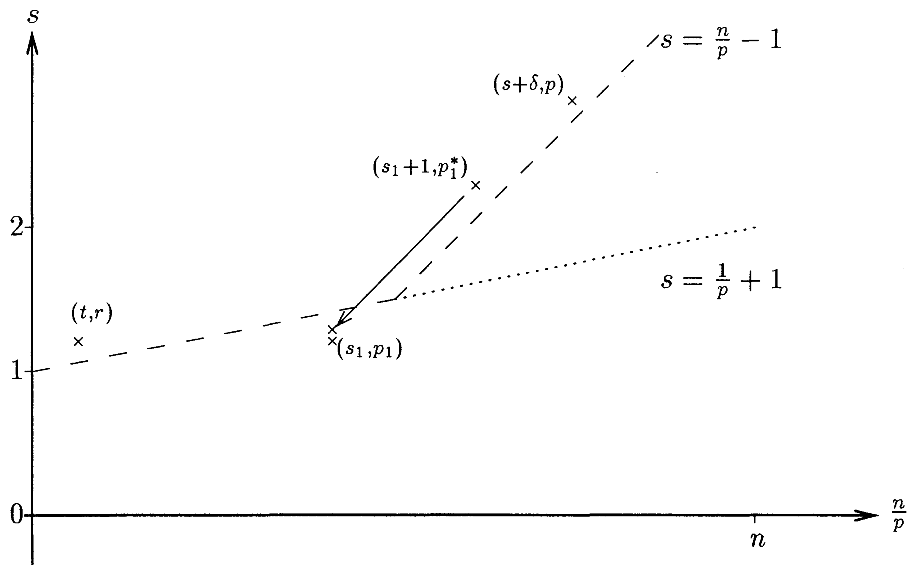

FIGURE 2. The integral-exponent p^*.

At this place the fine theory of pointwise multiplication offers a remedy. In fact, one can do better than regarding the non-linear operator $v \mapsto v\partial_{x_1} v$ as one of order $2 - \delta(s,p)$, as in (4.4) above.

The problem only arises when $\delta(s,p)$ is close to 0, hence only for $s < \frac{n}{p}$, and then $v\partial_1 v$ may be seen to factor through a space with smoothness index $\mathbf{s-1}$. More exactly, with $\frac{n}{p^*} = \frac{n}{p} + (\frac{n}{p} - s)_+$, Theorem 3.2 gives that

$$B(\cdot)\colon F^s_{p,q}(\overline{\Omega}) \to F^{s-1}_{p^*,q}(\overline{\Omega}) \tag{5.6}$$

is a bounded non-linear operator for $s > \max(\frac{1}{2}, \frac{n}{p} - 1 + \frac{1}{2}\delta_{n2})$. With (s,p,q) in the subsector given by $s > \max(1, \frac{n}{p} - 1 + \frac{1}{2}\delta_{n2})$, in which the (s_1,p_1,q_1) above lies, it may be checked that the receiving space $F^{s-1}_{p^*,q}$ in (5.6) lies *above* the critical broken line $s = \max(\frac{1}{p} - 1, \frac{n}{p} - n)$, so that $(s-1,p^*,q) \in \mathbb{D}_0$. See [**10**] or [**11**] for this.

According to (5.2) this assures that R_N may be applied to $F^{s-1}_{p^*,q}(\overline{\Omega})$, so, because $s - 1 - \frac{n}{p^*} = s + \delta(s,p) - \frac{n}{p}$, there is (after all) boundedness of

$$F^s_{p,q}(\overline{\Omega}) \xrightarrow{B(\cdot)} F^{s-1}_{p^*,q}(\overline{\Omega}) \xrightarrow{R_N} F^{s+1}_{p^*,q}(\overline{\Omega}) \hookrightarrow F^{s+\delta(s,p)}_{p,q}(\overline{\Omega}) \tag{5.7}$$

for *all* $s > \max(1, \frac{n}{p} - 1 + \frac{1}{2}\delta_{n2})$. Evidently this last sector is stable under the forming of the intermediate parameters (s_1,p_1,q_1), (s_2,p_2,q_2), ...

In principle also the cases with $t > s+\delta$ and $t - \frac{n}{r} < s + \delta - \frac{n}{p}$ need a special argument, but also here (5.7) may be applied. Altogether the iteration used for the Dirichlét problem applies also to the Neumann problem.

6. Final remarks

(1) To prove Theorem 1.3, notice that each of the conditions (1)–(3) there implies that the data belong to the spaces considered in Theorem 2.1 of [**13**, App. 1]. Hence there is a weak solution to which the regularity results apply. For details, see [**11**, Thm. 5.5.5] or [**10**].

(2) The iteration methods apply also to the von Karman equations for a plate in $\Omega \subset \mathbb{R}^2$, or to problems with a suitable semi-linear perturbation of an injectively elliptic Green operator $\begin{pmatrix} P_\Omega + G & K \\ T & S \end{pmatrix}$ in the calculus.

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MATHEMATICAL INSTITUTE, UNIVERSITY OF COPENHAGEN, UNIVERSITETSPARKEN 5, DK-2100 COPENHAGEN O, DENMARK

E-mail address: `jjohnsen@math.ku.dk`