JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1995), p. 1-15
<http://www.numdam.org/item?id=JEDP_1995____A16_0>

Spectral Multipliers and Multiple-parameter Structures on the Heisenberg Group

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My purpose here is to describe some recent results obtained jointly with Detlef Müller and Fulvio Ricci (see [MS], [MRS1], and [MRS2]) regarding an extension of the Marcinkiewicz multiplier theorem to the context of Heisenberg-like groups. I shall concentrate here on the background, the motivations, and a sketch of some of the main ideas of the proofs. The details will be found in the cited literature.

It may be interesting to point out here that the original multiplier theorem of Marcinkiewicz [Ma] had as its motivation the proof of L^p analogues of Schauder estimates, and that these results preceded the Calderón-Zygmund theory [CZ] of singular integrals.

§1. THE CLASSICAL CASE: $\mathbf{R}^n$

The Marcinkiewicz theorem dates from 1939. Later versions are due to Mihlm [Mi], 1957, and Hörmander [H], 1960. We take up first the relevant special case: functions of the Laplacian.

Thus let $$\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

be the Laplacian on $\mathbf{R}^n$, and consider the operator T given as a function $m(-\triangle)$, a "spectral multiplier." We can also write the operator T as a Fourier multiplier, i.e.

$$T(f)^\wedge(\xi) = m(|\xi|^2)\,\hat{f}\,(\xi) \tag{1.1}$$

When m is a bounded function on $(0,\infty)$, then (1.1) gives us a bounded operator on $L^2(\mathbf{R}^n)$. The first question is that of appropriate sufficient conditions that guarantee that T is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$.

Theorem A *We can conclude that T is bounded on L^p, $1 < p < \infty$, if the following assumptions hold:*

(*) $|\,m^{(j)}(\lambda)\,| \le A\,\lambda^{-j}$, for $0 \le j \le k$, if $k > n/2$.

Or more generally:

(**) (*A sharper* L^2 *variant*): $\sup_{t>0} \|\ \chi(\cdot)\,m(t\cdot)\ \|_{L^2_\alpha} < \infty$

if $\alpha > n/2$.

Here χ is a non-zero smooth cut-off function of compact support which vanishes near the origin.

One proof of this theorem goes as follows. We realize T as a convolution operator, $T(f) = f * K$, with $K^\wedge(\xi) = m(|\xi|^2)$. Then the assumptions (*) or (**) imply that T

is a Calderón-Zygmund operator, and the result then follows from the theory of singular integrals. In fact hypothesis (*) implies (at least morally) estimates of the form

$$(1.2) \qquad |K^{(j)}(x)| \leq A\,|x|^{-n-j}$$

while hypothesis (**) guarantees (see [H]) the condition

$$(1.3) \qquad \int_{|x| \geq 2\,|y|} |K(x-y) - K(x)|dx \leq A.$$

Remarks:

(i) The analysis above works as well if the radial multiplier in (1.1) is replaced by a more general non-radial one. The conditions (*) and (**) have obvious analogues.

(ii) The theory has a natural invariance with respect to a one-parameter family of scalings, i.e. "dilations." In fact, if $m(\lambda)$ satisfies the hypotheses (*) or (**) above, then so does $m(\delta\lambda)$, for each $\delta > 0$, uniformly in δ. The corresponding scalings on the ξ-space are $\xi \to \delta^{1/2}\xi$, and in terms of the kernels the dilations are given by $K(x) \to \delta^{n/2}\,K(\delta^{-1/2}x)$.

This dilation-invariance is not only an esthetic adornment of the theory, but is also a basic structural fact. For example, in analyzing estimates such as (1.2) or (1.3), one can decompose the kernel K into a sum over all (dyadic) scales, and then carry out the estimates by rescaling to unit scale via the dilations. This kind of argument occurs in many instances.

(iii) Two examples illustrate the relevance of the sharp restriction $\alpha > n/2$ occurring in the theorem. Consider first the important Bochner-Riesz operators, corresponding to $m(\lambda) = (1-\lambda)_+^{\delta}$. The restriction $\delta > \frac{n-1}{2}$ is equivalent with $m \in L^2_\alpha$, $\alpha > n/2$, and it gives the "critical index" for L^p sumability, for all p. (Of course there are subtler phenomena corresponding to other restrictions of δ).

(iv) Another example of interest is given by the fractional integrals of maginary order, i.e. $(-\triangle)^{i\gamma}$, γ real.

Here the corresponding kernel is given by a distribution kernel which away from the origin is

$$K(x) = C_\gamma\, |x|^{-n+2i\gamma}, \qquad \gamma \neq 0$$

where C_γ (up to harmless factors) equals $\frac{\Gamma(n/2+i\gamma)}{\Gamma(-i\gamma)}$.

The Γ quotient is of order $|\gamma|^{n/2}$ as $\gamma \to \infty$, which corresponds $n/2$ derivatives of the multiplier $m(\lambda) = \lambda^{-i\gamma}$, and which again shows that L^p inequalities (in fact weak-type L^1) cannot be expected if $\alpha < n/2$.

§2. Multi-parameter (Product) Theory in $\mathbf{R}^n$

Instead of dealing only with functions of the Laplacians we can consider the general Fourier multiplier T,

$$(2.1) \qquad (Tf)^\wedge(\xi) = m(\xi)\,\hat{f}(\xi),$$

a function of the n commuting self-adjoint operators,

$$\frac{1}{i}\frac{\partial}{\partial x_1}, \frac{1}{i}\frac{\partial}{\partial x_2}, \dots \frac{1}{i}\frac{\partial}{\partial x_n}, \qquad \text{i.e.} \qquad T = m\left(\frac{1}{i}\frac{\partial}{\partial x_i}, \frac{1}{i}\frac{\partial}{\partial x_2}, \dots \frac{\partial}{i\partial x_n}\right).$$

¿From this point of view it becomes of interest to stress the n-parameter family of dilations of $\mathbf{R}^n$,

$$(2.2) \qquad (\xi_1, \xi_2, \dots \xi_n) \to (\delta_1\xi_1, \delta_2\xi_2, \dots \delta_n\xi_n), \quad \delta_j > 0,$$

in place of the one-parameter family considered in the previous section. For us the main distinction between the original Marcinkiewicz theorem and versions like Theorem A is that the former enjoys an invariance with respect to the multi-parameter dilations (2.2).

We formulate a variant of the theorem

Theorem B *Suppose*

(2.3) $$\left| \prod_{j=1}^{n} \left(\xi_j \frac{\partial}{\partial \xi_j} \right)^{\epsilon_j} m(\xi) \right| \leq A$$

with $\epsilon_j = 0$ *or* 1. *Then* T *given by (2.1) is bounded on* L^p, $1 < p < \infty$.

Remarks:

(1) The differential inequalities on m can be relaxed to appropriate L^1 analogues.

(2) The regularity and decay properties of m envisaged by theorems A and B can be compared as follows: take e.g. $n = 2$; certain characteristics singularities of m are allowed in Theorem B along the "cross" (the ξ_1 and ξ_2 axes); while Theorem A (see Remark (2)) allows only similar singularities for m at the origin.

(3) The "product structure" displayed by the multipliers above is also clearly reflected in the properties of the convolution kernel corresponding to T, i.e. for which $T(f) = f * K$. Again, when $n = 2$, if we take conditions like (2.3) (but with ϵ_j allowed to be sufficiently large) then one has $|K(x_1, x_2)| \leq A|x_1|^{-1} \cdot |x_2|^{-1}$ and more generally $|\partial_{x_1}^a \partial_{x_2}^b K(x_1, x_2)| \leq A|x_1|^{-1-a}|x_2|^{-1-b}$ for appropriate a and b.

However, for kernels that possess this product structure there is no straightforward adaptation of the Calderón-Zygmund theory. Thus a proof of Theorem B has to be quite different from that outlined for Theorem A.

§3. Heisenberg Group

Having summarized the theory in $\mathbf{R}^n$, we pass to the Heisenberg group. Let $\mathbf{H}^n = \{(z,t) \in \mathbf{C}^n \times \mathbf{R}\}$, with group low $(z,t) \cdot (z',t') = (z + z', t + t' + 2\,Im\,z \cdot \bar{z}')$. We recall the automorphic dilations: $(z,t) \to (\delta z, \delta^2 t)$, $\delta > 0$.

A basis of the left-invariant lie algebraic is given by X_k $1 \leq k \leq 2n$, and T, with

$$X_j = \frac{\partial}{\partial x_j} + 2j, \frac{\partial}{jt}, \qquad X_{j+n} = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{jt}$$

$$z_j = x_j + iy_j, \qquad 1 \leq j \leq n, \; T = \frac{\partial}{\partial t}.$$

These vector fields satisfy the commutation relations

$$[X_{j+n}, X_j] = 4T$$

The basic analogue of the Laplacian is the sub-Laplacian $\mathcal{L}$, and its variants $\mathcal{L}_\alpha$ given by

$$\mathcal{L} = -\frac{1}{4} \sum_{j=1}^{2n} X_j^2, \qquad \mathcal{L}_\alpha = \mathcal{L} + i\alpha T \tag{3.1}$$

We also recall the homogeneous .... $|\cdot|$ on $\mathbf{H}^n$ given by $|x| = (|z|^4 + t^2)^{1/4} \approx |z| + |t|^{1/2}$ if $x = (z,t)$, and note that $|\delta x| = \delta |x|$ if $\delta \cdot x$ denotes the automorphic dilation of x. Finally, Q will denote the homogeneous dimension of $\mathbf{H}^n$, $Q = 2n + 2$.

With these notations we can now state the analogue of Theorem A in this context. It is due in various forms to DeMichele, Mauceri, Hulanicki, the author, Jenkins, Meda, and Christ (see[C]).

Theorem C

(a) *Suppose* $|m^{(j)}(\lambda)| \leq A\lambda^{-j}$ *for* $0 \leq j \leq N$, *with* N *sufficiently large. Then* $m(\mathcal{L})$ *is bounded on* $L^p(\mathbf{H}^n)$, $1 < p < \infty$. *Moreover* $m(\mathcal{L})f = f * K$,

with $|K(x)| \leq A|x|^{-Q}$ *and*

$$|X_j^a K(x)| \leq A|\lambda|^{-Q-a} \quad \textit{for appropriate } a.$$

(b) *The* L^p *boundedness holds under the less restrictive* (*analogous to* (**) *in Theorem A*) *that*

$$\underline{\textit{assumption}} \quad \sup_{t>0} \| \chi(\cdot)\, m(t\,\cdot\,) \|_{L^2_\alpha} < \infty,$$

$$\underline{\textit{if}} \qquad \alpha > Q/2.$$

§4. Problems and Motivations

The theory for $\mathbf{R}^n$ and the above initial result for $\mathbf{H}^n$ leads us to raise a series of questions regarding spectral multipliers on the Heisenberg group. We shall formulate these as three problems, whose solution will then be described below.

Problem 1: Is $Q/2$ the right sharp condition in Theorem C, or should it be rather half the Euclidean dimension, $n + 1/2$?

Problem 2: Consider now functions of $\mathcal{L}$ and $\frac{1}{i}T$, $m(\mathcal{L}, \frac{1}{i}T)$. (Note that $\mathcal{L}$ and T commute.) Are there analogues of the Marcinkiewicz theorem guaranteeing that $m(\mathcal{L}, \frac{1}{i}T)$ is bounded on L^p?

Problem 3: Suppose K is the convolution kernel corresponding to $m(\mathcal{L}, \frac{1}{i}T)$, i.e. $m(\mathcal{L}, \frac{1}{i}T)f = f * K$. Does K display a "product" structure? How does one characterize the K in terms of the m?

The motivation for dealing with such problems comes not only from the understandable desire to generalize the $\mathbf{R}^n$ theory, but arises also (as in much of previous research on the Heisenberg group) from the intimate connection with several complex variables. I will now describe this.

We start with the unit ball in $\mathbf{C}^{n+1}$. In its unbounded realization it is given by the domain $\Omega = \{Im\, w > |z_1|^2 + \ldots |z_n|^2\}$. The boundary of $\Omega, b\Omega$, is in a natural way identifiable with $\mathbf{H}^n$. We let $\rho = Im\, w - |z_1|^2 \ldots - |z_n|^2$ be the "height" function with respect to the boundary, then (x, ρ), $x \in \mathbf{H}^n$, $\rho \in \mathbf{R}$ will be useful (non-halomorphic) coordinates for what follows.

The $\bar{\partial}$ complex, and its boundary analogue the $\bar{\partial}_b$ complex lead one to two kinds of Laplacians which in the present case of the model domain Ω, have simple expressions.

For the boundary complex one has the Kohn-Laplacian $\Box_b^{(q)}$ on $(0,q)$ forms, which is essentially $\mathcal{L}_\alpha = \mathcal{L} + i\alpha T$ with $\alpha = n - 2q$. (Recall that $T = \frac{\partial}{\partial t}$).

For the $\bar\partial$ complex, except for some minor inaccuracies, the corresponding Laplacian is

$$Lap \equiv \mathcal{L} - T^2 - \frac{\partial^2}{\partial \rho^2}.$$

The $\bar\partial$-Neumann problem then consists of

$$\begin{cases} Lap(u) = f & \text{in } \Omega \\ \left(\frac{\partial}{\partial t} + i\,\frac{\partial}{\partial p}\right) u\,|_{b\Omega} = 0 & \end{cases}$$

This elliptic equation (with non-elliptic boundary condition) can be solved by inverting an appropriate boundary operator. These are in fact a pair of such operators $\Box_\pm$: $(\mathcal{L} - T^2)^{1/2} \pm iT$, one for the upper-half space, and the other for the lower half space. (One should note that remarkably $\Box_+\,\Box_- = \mathcal{L}$, and that to be more precise, $\mathcal{L}$ needs to be replaced by $\mathcal{L}_\alpha$). In any case, these considerations lead me to consider the following functions of $\mathcal{L}$ and $\frac{1}{i}T$, among others: $T\,(\mathcal{L} - T^2)^{-1/2}$, $\mathcal{L}^{1/2}\,(\mathcal{L} - T^2)^{-1/2}$, $\mathcal{L}\,(\mathcal{L} - T^2)^{-1}$, ..., which are of the form $m(\mathcal{L}, \frac{1}{i}T)$ with $m(\lambda, \mu) = \mu\,(\lambda + \mu^2)^{-1/2}$, $\lambda^{1/2}\,(\lambda + \mu^2)^{-1/2}$, $\lambda\,(\lambda + \mu^2)^{-1}$, ..., for $\lambda > 0$.

Kernels of some of these operators were computed by Phong and the author in [PS]. Other kernels, displaying some similar features, arose earlier in the work of G. Henkin (ca. 1970) in writing integral representations for solutions of $\bar\partial u = f$ in terms of Cauchy-Fantappié formulas. All of these kernels have the common feature that they are (sums) of products of parts having different kinds of homogeneities. The two kinds of homogeneities are

(a) isotropic: $(z,t) \to (\delta z, \delta t), \quad \delta > 0$

(b) non-isotropic: $(z,t) \to (\delta z, \delta^2 t), \quad \delta > 0$

That parts of the kernels should have isotropic homogeneity is natural given the elliptic nature of the operator *Lap*. Unfortunately, this homogeneity gives unacceptable scalings, since these are not automorphisms of $\mathbf{H}^n$. That the non-isotropic homogeneity is represented is of course due to the nature of the domain Ω and the group $\mathbf{H}^n$.

A particular kernel of interest which well represents the situation is

$$K(z,t) = \frac{\Omega(z)}{(|z|^2 + t^2)^n} \cdot \frac{1}{|z|^2 + it}$$

where Ω is smooth for $z \neq 0$, and homogeneous of degree 0, and satisfies the cancellation property

$$\int_{|z|=1} \Omega(z) = 0$$

§5. THE MAIN RESULTS

After this discussion of the background and motivation for Problems 1-3 we come to the main results. These are contained in the papers [MS], [MRS1], and [MRS2], and can be summarized in the following four theorems.

Theorem 1 *Suppose $m(\lambda,\mu)$ is given on $\mathbf{R}^+ \times \mathbf{R}$ and satisfies*

$$|(\mu\partial_\mu)^b (\lambda\partial_\lambda)^a m(\lambda,\mu)| \leq A_{a,b}, \quad \textit{all } a \textit{ and } b.$$

Then

$$m(\mathcal{L}, \tfrac{1}{i}T) \textit{ is bounded on } L^p, \quad 1 < p < \infty.$$

A more precise version of the conditions on m is in Theorem 4 below.

Theorem 2 *An operator $m(\mathcal{L}, \frac{1}{i}T)$ is of the above kind if and only if $m(\mathcal{L}, \frac{1}{i}T) f = f * K$, with K a distribution which is smooth when $z \neq 0$ and satisfies*

$$(1) \qquad \begin{aligned} &|K(z,t)| \leq A|z|^{-2n}\,||z|^2 + it|^{-1} \\ &|\partial_t^b \partial_z^a K(z,t)| \leq A_{a,b}|z|^{-2n-a}\,||z|^2 + it|^{-1-b} \end{aligned}$$

(2) *A cancellation property.*

(3) *It is radial in* z, i.e. *invariant under the action of the unitary group* $U(n)$ *on* $\mathbf{H}^n$.

The cancellation property can be stated as the requirement that $\sup_{\delta} |K(\eta_\delta)| \leq A$, where $\eta_\delta(z,t) = \eta(\delta_1 z, \delta_2 t)$, and η runs over a "normalized" family at "bump" functions; together with similar requirements involving only z (or t) integration.

Theorem 3 *Suppose* K *is a kernel which satisfies* (1) *and* (2) *above, but not necessarily* (3). *Then* $f \to f * K$ *is bounded on* L^p, $1 < p < \infty$.

Theorem 4

(i) *For* $m(\mathcal{L})$ *in Theorem C* (§3) *we need only* $\alpha > n + 1/2$.

(ii) *For* $m(\mathcal{L}, \frac{1}{i}T)$ *it suffices to have*

$$\sup_{t_1, t_2 > 0} \| \chi(\cdot,\cdot)\, m(t_1 \cdot, t_2 \cdot) \|_{L^2_{\alpha,\beta}} < \infty$$

where $\alpha > n$, $\beta > 1/2$.

(iii) *these requirements are best possible*

The mixed Sobolev space $L^2_{\alpha,\beta}$ *is given by*

$$\| f(\lambda,\mu) \|_{L^2_{\alpha,\beta}} = \| (1 + |\partial_\lambda|)^\alpha (1 + |\partial_\lambda| + |\partial_\mu|)^\beta f \|_{L^2}.$$

The result in (i) was proved independently by W. Hebisch, [He], by a different method than the one we outline below.

§6. Ideas of the Proofs

We shall emphasize three ideas which play a central role.

(1) "Freeing" or "lifting." One proceeds by adding variables so as to introduce a homogeneity which was not present (or realizable) at the start. This approach, previously used in the context of analysis of vector fields, finds a simple expression here via the technique of "transference" (for which see [CW]).

(2) Square functions. The idea is to compare $S\left(m\left(\mathcal{L},\, \frac{1}{i}T\right) f\right)$ with $S\,(f)$ for appropriate Littlewood-Paley-type square functions S. For the background of this see [Ma], [S], [FS].

(3) Harmonic analysis on $\mathbf{H}^n$ and in particular the explicit formulae for the Fourier transform of radial functions via the Laguerre formalism. The latter was developed in [P] and [G].

__Remarks:__ The conclusions based on the techniques (1) and (2) have wide generalizations, but those that also need (3) are restricted to Heisenberg-like groups.

Regarding (1): Suppose we wrote $G = G_1 \times G_2$, where $G_1 = \mathbf{H}^n = \{(z,t)\}$, and $G_2 = \mathbf{R} = \{u\}$. Let $\mathcal{L}_1$ be the sub-Laplacian on G_1 and $\mathcal{L}_2 = \frac{1}{i}\frac{\partial}{\partial u}$. Let N be the sub-group of $G = \{(z,t,u) : z = 0,\, t = u\}$. Then $G/N \approx \mathbf{H}^n$. The crucial observation is that if we can prove that $m\,(\mathcal{L}_1,\, \mathcal{L}_2)$ is bounded on $L^p\,(G)$, then it follows by transference that $m\,(\mathcal{L},\, \frac{1}{i}T)$ is bounded on $L^p\,(\mathbf{H}^n)$. Note that on $G = G_1 \times G_2$ we have a two-parameter family of (automorphic) dilations, in distinction to what happens on $\mathbf{H}^n$.

The following (formal) identity is also relevant: if $K^{\#}\,(z,t,u)$ is the convolution kernel on G, then

$$K\,(z,t) = \int_{-\infty}^{\infty} K^{\#}\,(z, t-u,\, u)\, du \tag{6.1}$$

is the corresponding kernel on $\mathbf{H}^n$.

Regarding square functions, we indicate how these can be used to prove that $m\,(\mathcal{L}_1,\, \mathcal{L}_2)$ is bounded on $L^p\,(G)$.

We write $\quad S(f) = \left(\sum_I |\Pi_I(f)|^2\right)^{1/2},$

where $I = (i,j)$ is a double-index set, with $\Pi_I(f) = \varphi(2^{-2i}\mathcal{L}_1)\,\varphi(2^{-j}\mathcal{L}_2)$, where φ is an appropriate smooth function of compact support, vanishing near the origin. Because of the product situation, it follows that $\| f \|_{L^p(G)} \approx \| S(f) \|_{L^p(G)}$, $1 < p < \infty$. One writes $\Pi_I(f) = f * \gamma_I$, and $m(\mathcal{L}_1, \mathcal{L}_2) f = f * K$. Also, $\gamma_I = \gamma'_I * \gamma'_I$, with the γ'_I similar to the γ_I.

Now, $\quad \Pi_I(m(\mathcal{L}_1, \mathcal{L}_2) f) = f * K * \gamma_I = f * K * \gamma'_I * \gamma'_I$
$= f_I * K_I$, where $f_I = f * \gamma'_I$ and $K_I = K * \gamma'_I$

However, one prove that $|g * K_I| \leq C\,M(g)$ for some (strong) maximal function. This is done by rescaling to unit scale with the available two-parameter family of dilations. Thus one gets that

$$\| m(\mathcal{L}_1, \mathcal{L}_2) f \|_{L^p} \leq C \left\| \left(\sum_I (\mathbf{M}(f_\Gamma))^2\right)^{1/2} \right\|_{L^p},$$

and the proof is concluded by appealing to the vector-valued maximal inequality.

To prove that the kernel of the operator $m(\mathcal{L}, \frac{1}{i}T)$ satisfies conclusion (1) and (2) of Theorem 2, we consider first the kernel $K^{\#}$ of the corresponding operator $m(\mathcal{L}_1, \mathcal{L}_2)$ in G. One shows, using a Littlewood-Paley decomposition as above, that, e.g.

$$|K^{\#}(z,t,u)| \leq A\,||z|^2 + it|^{-n-1}\,|u|^{-2}$$

with corresponding estimates on derivatives. The result for $K(z,t)$ is then a consequence of the formula (6.1).

We next give a few indications about the proof of Theorem 3. Starting with a kernel K, that satisfies (1) and (2) we lift it to a kernel $K^{\#}$ on G via the formula

$$K^{\#}(z,t,u) = |z|^{-2}\chi(t/|z|^2)\,K(z,t+u),$$

where χ is smooth, supported in $[1/2, 1]$, and $\int \chi(u)\,du = 1$.

One then shows that $K^{\#}$ satisfies the estimates expected for the product theory on $G_1 \times G_2$. However, to prove that such kernels lead to bounded operators via the square function inequalities brings up a difficulty we did not need to face until now. It is connected with the commutativity

$$K * \gamma_I' * \gamma_I' = \gamma_I' * K * \gamma_I'$$

that we exploited previously, which commutativity follows because the functions in question are radial. In the present circumstance a substantially more complicated substitute, involving an approximate commutativity, needs to be used.

We now pass to the harmonic analysis on $\mathbf{H}^n$. For each real μ, $\mu \neq 0$, there is associated the usual unitary representation π^{μ} of $\mathbf{H}^n$. If K is any distribution which is radial, then the operator $\pi^{\mu}(K)$ is diagonal in the "Hermite" basis. The diagonal elements $\delta_{(\mu,k)}$ corresponding to K are given by the formula

$$\delta(\mu,k) = \int_{\mathbf{H}^n} K(z,t)\,e^{-i\mu t}\,\ell_k(2|\mu|\,|z|^2)\,dz dt$$

where ℓ_k are the Laguerre functions given by:

$$\ell_k(x) = \frac{x^{n+1}\,e^{x/2}}{n(n+1)\dots(n+k-1)} \cdot \left(\frac{d}{dx}\right)^k (x^{n+k-1}\,e^{-x})$$

For the correspondence $K \mapsto \delta$, a basic fact is that $(|z|^2 - it)K$ corresponds to $\left(\frac{\partial}{\partial\mu} - \frac{1}{\mu}\,j\,\triangle_k\right)\delta(\mu,k)$ of $\mu < 0$ and $n = 1$, with a similar formula for $\mu > 0$ and also for general n.

As a consequence, one can show that if K is the kernel corresponding to $m(\mathcal{L})$, $(m(\mathcal{L})f = f * K)$, with m supported in [0,1], then

(6.2) $$\int_{\mathbf{H}^n} | \, | x |^{\alpha} K(x) |^2 \, dx \leq A \, \| \, m \, \|_{L^2_\beta}^2,$$

wherever $\beta > \alpha - 1/2$, and $\alpha \geq 2$.

This surprising conclusion represents a gain of 1/2 over what can be proved in analogy with $\mathbf{R}^n$, and it yields assertion (i) of Theorem 4.

For conclusion (ii) one uses estimates related to (6.2) to show that

$$\| \, S_1 (m (\mathcal{L}, \frac{1}{i} T) f) \, \|_p \leq A_p \, \| \, S_2 (f) \, \|_p,$$

when $2 \leq p < \infty$, with appropriate square functions S_1 and S_2. Here, because of the minimal smoothness hypotheses, the analysis is done on $\mathbf{H}^n$ and not on $G = \mathbf{H}^n \times \mathbf{R}$.

Finally, a word why the results are sharp. One considers the kernel K_γ of the operator $\mathcal{L}^{i\gamma}$. Now $K_\gamma (x) = | x |^{-Q-2i\gamma} H_\gamma (x)$, where $H_\gamma (x)$ is homogeneous of degree 0 as a function of x. The behavior of H_γ as a function of γ is not simple, but one can show that $| H_\gamma (x) | \geq C | \gamma |^{n+1/2}$ as $\gamma \to \infty$, for x in a neighborhood of the point $(1,0)$. It is a curious fact that when $\bar{x} = (0,1)$ the behavior in question is even worse, but relates explicitly to the ζ-function on the boundary of the critical strip. The formula giving this in [MS] contains an error which we wish to correct. It should read in the case $n = 1$, that $H_\gamma (0,1)$ is a constant multiple of

$$2^{i\gamma} (1 - 2^{i\gamma}) \, \Gamma (2 + i\gamma) \cosh (\gamma\pi/2) \, \zeta (-i\gamma),$$

which is not $O(| \gamma |^2)$ as $\gamma \to \infty$ (γ real).

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