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# Singular Yang-Mills Connections

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by *Johan Råde* at Lund

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First I wish to thank the organizers for inviting me to speak at this conference. I will speak about an intriguing partial differential equation that arises in gauge theory. Gauge theory is mainly concerned with the Yang-Mills equation and related equations, such as the Ginzburg-Landau equation (with a magnetic field), the Yang-Mills-Higgs equation and the Seiberg-Witten equation. I will talk about solutions to the Yang-Mills equation with singularities. In a moment I will write down the Yang-Mills equation, in full detail. First I just want to mention the origin of these singular solutions.

The Yang-Mills equation has mainly been studied by topologists and geometers, in particular in connection with the topology of smooth 4-manifolds. In the early 80's Donaldson showed that Yang-Mills equation could be used as a powerful tool in smooth 4-manifold topology. In particular he defined new invariants for smooth 4-manifolds. These invariants reflect the topology of solution spaces for Yang-Mills equation on the 4-manifold. They are now known as Donaldson polynomials. These developments were a bit of a shock for the 4-manifold topologists. They were suddenly forced to learn about partial differential equations. Many of them did so very succesfully. For a brief introduction to the applications of gauge theory to 4-manifold topology see [L] and for a comprehensive text see [DK]. Both books are masterpieces of mathematical exposition.

The Donaldson polynomials were at first extremely hard to calculate. However, a few years ago Kronheimer and Mrowka discovered a method for calculating them in a large number of cases. The key was to introduce a new type of Donaldson polynomials defined using spaces of singular Yang-Mills connections, [K], [KM1], [KM2], see also [R3]. The purpose of my own work has been to understand these singular Yang-Mills connections from the point of view of partial differential equations.

In October last fall a new equation and new invariants were introduced by Seiberg and Witten. Within a few weeks several famous conjectures about 4-manifolds had been settled. Priority often was a matter of days. An interesting account of these developments is given in [T]. It is not clear if 4-manifold topologists are interested in singular Yang-Mills connections any more.

## §1. The Yang-Mills equation

Recall that if

$$\sigma = \sum_{i_1<\cdots<i_p} \sigma_{i_1\ldots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

is a differential $p$-form, and

$$\omega = \sum_{j_1<\cdots<j_q} \omega_{j_1\ldots j_q} dx_{j_1} \wedge \cdots \wedge dx_{j_q}$$

then the exterior derivative of $\sigma$ is defined to be the $(p+1)$-form

$$d\sigma = \sum_j \sum_{i_1<\cdots<i_p} \frac{\partial}{\partial x_j}\sigma_{i_1\ldots i_p} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}. \tag{1.1}$$

and the wedge product of $\sigma$ and $\omega$ is defined to be the $(p+q)$-form

$$\sigma \wedge \omega = \sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_q}} \sigma_{i_1\ldots i_p}\omega_{j_1\ldots j_q} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}. \tag{1.2}$$

These operations satisfy the identities

$$\begin{aligned} \omega \wedge \sigma &= (-1)^{pq}\sigma \wedge \omega \\ d^2\sigma &= 0 \\ d(\sigma \wedge \omega) &= d\sigma \wedge \omega + (-1)^p \sigma \wedge d\omega. \end{aligned}$$

The adjoint of the exterior derivative (with respect to the Euclidean metric $\sum dx_i^2$) is given by

$$d^*\sigma = \sum_{\nu=1}^{p} \sum_{i_1<\cdots<i_p} (-1)^\nu \frac{\partial}{\partial x_{i_\nu}}\sigma_{i_1\ldots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_{\nu-1}} \wedge dx_{i_{\nu+1}} \wedge \cdots \wedge dx_{i_p}.$$

Now, let $G$ be a compact Lie group. Let $\mathfrak{g}$ be the Lie algebra of $G$. I usually think of $G$ as a group of matrices; that simplifies the notation a good deal. In particular, then the Lie bracket $[X,Y]$ is simply given by $XY - YX$. In fact, I will soon restrict my attention to the case $G = \mathrm{SU}(2)$. This will simplify the notation even further.

In gauge theory one considers differential forms $\sigma$ the coefficient $\sigma_{i_1\ldots i_p}$ take values in the Lie algebra $\mathfrak{g}$. These are called $\mathfrak{g}$-valued forms. We can still define the exterior derivative of $\sigma$ by by (1.1). However, the right hand side of (1.2) is quite meaningless if $\sigma_{i_1 \ldots i_p}$ and $\omega_{j_1 \ldots j_q}$ are $\mathfrak{g}$-valued forms. Instead we define the bracket of $\sigma$ and $\omega$ as

$$[\sigma, \omega] = \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q}} [\sigma_{i_1 \ldots i_p}, \omega_{j_1 \ldots j_q}] dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}.$$

Then

$$\begin{aligned} [\omega, \sigma] &= (-1)^{pq+1} [\sigma, \omega] \\ d^2 \sigma &= 0 \\ d[\sigma, \omega] &= [d\sigma, \omega] + (-1)^p [\sigma, d\omega]. \end{aligned}$$

**Gauge transformations.** The Lie group $G$ acts on the Lie algebra $\mathfrak{g}$ by conjugation; for $g \in G$ and $X \in \mathfrak{g}$ we can form $gXg^{-1} \in \mathfrak{g}$. If $\sigma$ is a $\mathfrak{g}$-valued $p$-form and $g$ is a $G$-valued function, then we can form a new $\mathfrak{g}$-valued $p$-form

$$g.\sigma = \sum_{i_1 < \cdots < i_p} g \sigma_{i_1 \ldots i_p} g^{-1} dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

We say that $\sigma$ and $g.\sigma$ are gauge-equivalent. This establishes an equivalence relation on $\mathfrak{g}$-valued $p$-forms.

We can now define gauge theory; it is the study of objects that are invariant under gauge transformations. One example is the commutator of $\mathfrak{g}$-valued forms; it is clear that

$$g.[\sigma, \omega] = [g.\sigma, g.\omega].$$

**Covariant derivatives.** The exterior derivative is not gauge-invariant; we have

$$g.d\sigma = \sum_j \sum_{i_1 < \cdots < i_p} g\left(\frac{\partial}{\partial x_j} \sigma_{i_1 \ldots i_p}\right) g^{-1} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

but

$$d(g.\sigma) = \sum_j \sum_{i_1 < \cdots < i_p} \frac{\partial}{\partial x_j} (g \sigma_{i_1 \ldots i_p} g^{-1}) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

In general these differ by terms that involve the derivatives of $g$. A calculation shows that

$$g.d\sigma = d(g.\sigma) + [A, g.\sigma]$$

where

$$A = -(dg)g^{-1} = -\sum_i \frac{\partial g}{\partial x_i} g^{-1} dx_i.$$

This suggest that we define the covariant exterior derivative of $\sigma$ as

$$d_A = d\sigma + [A,\sigma] = \sum_j \sum_{i_1<\cdots<i_p} \left(\frac{\partial}{\partial x_j}\sigma_{i_1\ldots i_p} + [A_j, \sigma_{i_1\ldots i_p}]\right) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

The covariant derivative depends on the choice of a $\mathfrak{g}$-valued 1-form $A$. We call $A$ a $G$-connection. A short calculation using the Jacobi identity shows that for any connection $A$

$$d_A[\sigma,\tau] = [d_A\sigma] + (-1)^p[\sigma, d_A\omega].$$

Another short calculation shows that exterior covariant derivative is gauge-invariant in the sense that

$$g.d_A\sigma = d_{g.A}(g.\sigma)$$

where

$$g.A = gAg^{-1} - (dg)g^{-1} = \sum_i \left(gA_ig^{-1} - \frac{\partial g}{\partial x_i}g^{-1}\right) dx_i.$$

Note that a connection transforms differently than an ordinary $\mathfrak{g}$-valued 1-form. As before, we say that $A$ and $g.A$ are gauge-equivalent. This establishes an equivalence relation on the set of $G$-connections.

$$d_A = d\sigma + [A,\sigma] = \sum_j \sum_{i_1<\cdots<i_p} \left(\frac{\partial}{\partial x_j}\sigma_{i_1\ldots i_p} + [A_j, \sigma_{i_1\ldots i_p}]\right) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

The adjoint of $d_A$ is given by

$$d_A^*\sigma = \sum_{\nu=1}^{p} \sum_{i_1<\cdots<i_p} (-1)^\nu \left(\frac{\partial}{\partial x_{i_\nu}}\sigma_{i_1\ldots i_p} + [A_{i_\nu}, \sigma_{i_1\ldots i_p}]\right) dx_{i_1} \wedge \cdots \wedge dx_{i_{\nu-1}} \wedge dx_{i_{\nu+1}} \wedge \cdots \wedge dx_{i_p}.$$

**Curvature.** We do not have $d_A^2\sigma = 0$. Instead a short calculation shows that

$$d_A^2\omega = [F_A, \omega]$$

where $F_A$ is the $\mathfrak{g}$-valued 2-form

$$F_A = dA + \tfrac{1}{2}[A,A] = \frac{1}{2}\sum_{ij}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j]\right) dx_i \wedge dx_j.$$

The 2-form $F_A$ is called the curvature of the connection $A$. Another short calculation shows that curvature is gauge-invariant, i.e.

$$g.F_A = F_{g.A}.$$

Yet another short computation shows that

$$d_A F_A = 0.$$

This is known as the Bianchi identity.

**Yang-Mills equation.** Let $A$ be a connection in a domain $\Omega$ in $\mathbb{R}^n$. One defines the energy of the connection $A$ as

$$\mathfrak{YM}(A) = \frac{1}{2}\int_\Omega |F_A|^2 dx = \frac{1}{2}\sum_{ij}\int_\Omega \left|\frac{\partial A}{\partial x_i} - \frac{\partial A}{\partial x_j} + [A_i, A_j]\right|^2 dx.$$

A short calculation shows that the Euler-Lagrange equation for this energy functional is

$$d_A^* F_A = 0.$$

This equation is known as the Yang-Mills equation. A connection $A$ that satisfies Yang-Mills equation is called a Yang-Mills connection. If we write out the Yang-Mills equation fully we get

$$\sum_{j=1}^{n}\left(\frac{\partial^2 A_i}{\partial x_j^2} - \frac{\partial^2 A_j}{\partial x_i \partial x_j} + \left[\frac{\partial A_j}{\partial x_j}, A_i\right] + \left[\frac{\partial A_j}{\partial x_i}, A_j\right] - 2\left[\frac{\partial A_i}{\partial x_j}, A_j\right] + [A_j, [A_j, A_i]]\right) = 0$$

for $i = 1, \ldots, n$. The most convenient way to write the equation is

$$d^* dA + \{A \otimes \nabla A\} + \{A \otimes A \otimes A\} = 0.$$

Here we write $\{A \otimes \nabla A\}$ for terms that are linear in $A_i$ and $\partial A_i/\partial x_j$ et.c.

The Yang-Mills energy is gauge invariant, i.e.

$$\mathfrak{YM}(g.A) = \mathfrak{YM}(A).$$

Hence the Yang-Mills equation is gauge-invariant. In particular, if $A$ is a Yang-Mills connection, then $g.A$ is also a Yang-Mills connection.

To define the Yang-Mills energy and the Yang-Mills equation on a manifold, we need to choose a Riemannian metric. It is easy to verify that in four dimensions the Yang-Mills energy, and hence the Yang-Mills equation, are conformally invariant.

## §2. A regularity theorem for Yang-Mills connections

The pricipal term in Yang-Mills eqaution is $d^*dA$. The operator $d^*d$ is not elliptic. Thus we can not expect solutions to be smooth. This is also clear from the gauge invariance. Given a smooth solution we can manufacture a non-smooth solution by applying a suitable non-smooth gauge transformation. Conversely, the best wecould hope for is that any solution to Yang-Mills equation is gauge-equivalent to a smooth solution. Such a result was proven by K. Uhlenbeck.

Before discussing her theorem, I want to review a classical geometric result. A connection $A$ is said be trivial if it gauge-equivalent to 0. A connection $A$ is said to be flat if $F_A = 0$. Clearly any trivial connection is flat.

**Lemma 2.1.** *If $A$ is a connection defined in a simply connected domain $\Omega$ and $A$ is flat, then $A$ is trivial.*

*Proof.* A connection $A$ is trivial if we can solve the equation $g.A = 0$ for $g$. Fully written out, this equation takes the form

$$\frac{\partial g}{\partial x_i} = gA_i. \tag{2.1}$$

This implies

$$\frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{\partial g}{\partial x_j} A_i + g\frac{\partial A}{\partial x_j} = gA_jA_i + g\frac{\partial A}{\partial x_j}.$$

The identity

$$\frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{\partial^2 g}{\partial x_j \partial x_i}$$

gives rise to the integrability condition

$$A_jA_i + \frac{\partial A_i}{\partial x_j} = A_iA_j + \frac{\partial A_j}{\partial x_i},$$

which is equivalent to

$$F_A = 0.$$

This condition is clearly necessary for the existence of a solution $g$. By Frobenius theorem it is also sufficient, as long as $\Omega$ is simply connected. □

We will not actually use this Lemma. It only serves as a motivation for Uhlenbeck's good gauge theorem. In fact, Uhlenbeck's theorem can be viewed as an analyst's version of Lemma 2.1; it says that if $A$ is a connection, on the unit ball, with small curvature, then there exists a gauge transformation $g$ such that $g.A$ is small.

For simplicity we now restrict out attention to 4-dimensions. Let $B_1$ denote the unit ball in $\mathbb{R}^4$. Let $\nu$ denote the outward unit normal of $\partial B_1$. Let $L^{p,k}(B_1)$ denote the Sobolev space of functions with $k$ derivatives in $L^p$. We say that a form or a connection is in $L^{p,k}(B_1)$ if all its components are in $L^{p,k}(B_1)$. It is natural to consider connection $A \in L^{2,1}(B_1)$. It follows from the Sobolev embedding $L^{2,1} \to L^4$ that if $A \in L^{2,1}$ then $F_A \in L^2$ and $\mathfrak{YM}(A) < \infty$.

**Theorem 2.2.** [U1] *There exists $\varepsilon > 0$ such that if $A$ is a connection in $L^{2,1}(B_1)$ with*

$$\|F_A\|_{L^2(B_1)} \leq \varepsilon$$

*then there exists a gauge transformation $g$ in $L^{2,2}(B_1)$ such that*

$$\text{(2.6)} \qquad \begin{cases} \nu \lrcorner (g.A) = \sum_i x_i (g.A)_i = 0 & \text{on } \partial B_1 \\ d^*(g.A) = \sum_i \dfrac{\partial}{\partial x_i}(g.A)_i = 0 & \text{on } B_1 \end{cases}$$

*and*

$$\|g.A\|_{L^{2,1}(B_1)} \leq c \|F_A\|_{L^2(B_1)}.$$

The conditions (2.6) are called gauge conditions.

The theorem is proven as follows. Assume that $A$ satsifies the gauge conditions. Let $A+b$ be a small perturbation of $A$. We want to show that $A+b$ can be transformed to a connection that satisfies the gauge conditions. This amounts to solving the non-linear boundary value problem

$$\begin{cases} d^*(g.(A+b)) = 0 & \text{on } B_1 \\ \nu \lrcorner (g.(A+b)) = 0 & \text{on } \partial B_1. \end{cases}$$

for $g$. If we let $g = \exp \varphi$ and linearize around $\varphi = 0$ and $b = 0$ then we get the linear boundary value problem

$$\text{(2.4)} \qquad \begin{cases} \Delta \varphi + \sum_i \left[ A_i, \dfrac{\partial \varphi}{\partial x_i} \right] = -d^* b & \text{on } B_1 \\ \nu \lrcorner d\varphi = -\nu \lrcorner b & \text{on } \partial B_1. \end{cases}$$

This system can clearly be solved if $A$ is small enough; then it is a small perturbation of the Neumann problem for the Laplace operator. It then follows from the implicit function theorem that the non-linear boundary value problem can be solved if $b$ is small enough. The theorem can then be proven by the continuity method. See [U1] for more details.

**Theorem 2.3.** [U1] *There exist constants $c_k$ such that if $A$ in addition to the assumptions in Theorem 2.2 satisfies Yang-Mills equations, then $g.A$ is smooth on the interior of $B_1$ and*

$$\|g.A\|_{C^k(B_1/2)} \leq c_k \|F_A\|_{L^2(B_1)}.$$

This is seen as follows. Assume that $A$ is Yang-Mills and $d^*A = 0$. We now have that $\Delta A = dd^*A + d^*dA$. Hence it follows that

$$\Delta A + \{A \otimes \nabla A\} + \{A \otimes A \otimes A\} = 0. \tag{2.5}$$

This is a semi-linear elliptic equation. If $A \in L^{2,1}$, then we can estimate higher derivatives of $A$ by bootstrapping. In the first iteration step we have to use to usual trick of estimating the difference quotient of $A$.

This siuation is common in gauge theory. In order to prove regularity for an equation, one has to supplement it with gauge conditions. Thus, when facing a new equation, the first question is, what is the right gauge condition.

## §3. Singular connections

According to a theorem by K. Uhlenbeck, point singularities of finite energy connections are removable. The precise statement is as follows:

**Theorem 3.1.** [U2], [U3] *If $A$ is a connection in $L^{2,1}_{\text{loc}}(B_1 \setminus \{0\})$ and $F_A \in L^2(B_1 \setminus \{0\}) = L^2(B_1)$, then there exists a gauge transformation $g \in L^{2,2}(B_1)$ such that $g.A \in L^{2,1}(B_1)$.*

This theorem was originally proven under the extra assumption that $A$ be Yang-Mills, [U2]. Later it was discovered that finite energy sufficed, [U3].

According to a theorem of mine, singularities along embedded curves are removable. It suffices to consider the connections on $B_1 \setminus L_1$ where $L_1 = \{\, (x_1, 0, 0, 0) \,|\, |x_1| \leq 1 \,\}$.

**Theorem 3.2.** [R2] *If $A$ is a connection in $L^{2,1}_{\text{loc}}(B_1 \setminus L_1)$ and $F_A \in L^2(B_1 \setminus L_1) = L^2(B_1)$, then there exists a gauge transformation $g \in L^{2,2}_{\text{loc}}(B_1 \setminus L_1)$ such that $g.A \in L^{2,1}(B_1)$.*

The next case is connections on a 4-manifold with singularities along an embedded surface. The local model are then connections on $B_1$ with singularities along $D_1 = \{\, (x_1, x_2, 0, 0) \,|\, x_1^2 + x_2^2 = 1 \,\}$. It is not true that finite energy connections on $B_1 \setminus D_1$ can be extended to connections on $B_1$. Unlike $B_1 \setminus \{0\}$ and $B_1 \setminus L_1$, the domain $B_1 \setminus D_1$ is not simply connected. Hence Lemma 2.1 does not apply to $B_1 \setminus D_1$. Thus, before we attempt to generalize the theorems of §2 and §3 to $B_1 \setminus D_1$ we need to generalize Lemma 2.1 to non-simply-connected domains $\Omega$. This requires the notion of holonomy.

**Holonomy and flat connections.** Let $A$ be a connection in a region $\Omega$ in $\mathbb{R}^4$. Let $x_0 \in \Omega$. Let $\gamma : [0,1] \to \Omega$ be a closed smooth curve in $\Omega$ with $\gamma(0) = \gamma(1) = x_0$. The initial value problem

$$\text{(2.3)} \qquad \begin{cases} \dfrac{\partial h}{\partial t} + h \sum_i A_i \dfrac{d\gamma_i}{dt} = 0. \\ h(0) = 1 \end{cases}$$

has a unique solution. The element $h(1) \in G$ is called the holonomy of $A$ around $\gamma$. This initial value problem is gauge-invariant in the sense that

$$(g.h)(t) = g(x(t))h(t)g(x(t))^{-1}$$

is a solution for $g.A$. Thus the conjugacy class of the holonomy is gauge-invariant.

If the connection is trivial, then (2.1) has a solution $g$ with $g(x_0) = 1$. Then the solution to (2.3) is given by $h(t) = g(\gamma(t))$. It follows that that the holonomy is $h(1) = g(\gamma(1)) = g(x_0) = g(\gamma(0)) = h(0) = 1$. Thus we get another condition for a connection to be trivial; the holonomy around each loop has to be the identity.

One can show that if $A$ is flat, then the holonomy of $A$ is invariant under smooth deformations of $\gamma$. Thus the holonomy only depends the homotopy class of $\gamma$. Hence it gives a map $\pi_1(\Omega, x_0) \to G$. Here $\pi_1(\Omega, x_0)$ denotes the fundamental group of $\Omega$ with base point $x_0$. It is easily seen that that this map is a homomorphism. If we apply a gauge transformation $g$ to $A$ or if we change the base point, then this homomorphism gets conjugated by an element of $G$.

**Theorem 3.3.** *There is a 1–1 correspondence between gauge equivalence classes of flat $G$-connections on $\Omega$ and conjugacy classes of homomorphisms $\pi_1(\Omega) \to G$.*

The proof is not hard; see for instance [KN] Prop. 9.3.

In our special case of $B_1 \setminus D_1$, the fundamental group is generated by any loop that goes around $D_1$ once. It follows that flat connections are classified by the holonomy around this loop.

**Corollary 3.4.** *There is a 1-1 correspondence between gauge equivalence classes of flat $G$-connections on $B_1 \setminus D_1$ and conjugacy classes in $G$.*

**Limit Holonomy.** As we have seen, flat connections on $B_1 \setminus D_1$ are classified by their holonomy. A non-flat connection does not a uniquely defined holonomy. However, any connection on $B_1 \setminus D_1$ with curvature in $L^2$ has a well-defined limit holonomy.

We introduce cylindrical coordinates $(x_1, x_2, r, \theta)$ on $B_1$, with $x_3 = r\cos\theta$ and $x_4 = r\sin\theta$. In these coordinates $D_1$ is given by $r = 0$.

**Theorem 3.5.** [SS] *If $A$ is a $G$-connection in $L^{2,1}_{\text{loc}}(B_1 \backslash D_1)$ with $F_A \in L^2(B_1 \backslash D_1)$, then the holonomy of $A$ around the loop $\gamma(t) = (x_1, x_2, r\cos(2\pi t), r\sin(2\pi t))$ exists for almost all $x_1$, $x_2$ and $r$. The limit of this holonomy as $r \to 0$ exists for almost all $x_1$ and $x_2$. This limit is independent of $x_1$ and $x_2$ for almost all $x_1$ and $x_2$.*

This unique limit is called the limit holonomy of the connection.

We can now state the correct analog of Theorem 3.1 and Theorem 3.2 for $B_1 \setminus D_1$. Note that if $G$ is connected, then $\exp : \mathfrak{g} \to G$ is surjective. (Proof: On a complete Riemannian manifold any two points can be connected by a geodesic curve. On a Lie group with an invariant metric, in particular any compact Lie group, the geodesic curves through the identity are precisely the 1-parameter subgroups.)

**Theorem 3.6.** [R2] *If $A$ is a $G$-connection in $L^{2,1}_{\text{loc}}(B_1 \setminus D_1)$ with limit holonomy* $\exp(-2\pi X)$, *then there exists a gauge transformation $g \in L^{2,2}_{\text{loc}}(B_1 \setminus D_1)$ such that*

$$g.A = X\, d\theta + a$$

*where $a, \nabla_{X d\theta} a \in L_{X d\theta}(B_1)$.*

Here

$$\nabla_A \sigma = \sum_j \sum_{i_1 < \cdots < i_p} \left( \frac{\partial}{\partial x_j} \sigma_{i_1 \ldots i_p} + [A_j, \sigma_{i_1 \ldots i_p}] \right) dx_j \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Note that the connection $X\, d\theta + a$ has curvature $d_{X d\theta} a + \frac{1}{2}[a, a]$. Hence the condition $a \in L^{2,1}_{X d\theta}$ ensures that the curvature lies in $L^2$.

As a consequence of Thm. 3.6, a singularity along a surface of a finite energy connection is removable is and only if the limit holonomy is trivial.

The Yang-Mills connections used by Kronheimer and Mrowka are Yang-Mills connections on a 4-manifold with singularities along an embedded surface. Near any point of the surface they are of the form $X\, d\theta + a$ with $a \in L^{2,1}_{X d\theta}(B_1)$.

## §4. A regularity theorem for singular Yang-Mills connections

To keep the notation simple, we will now restrict our attention to the Lie group SU(2). This is the group of all unitary $2 \times 2$ matrices with determinant one. These are precisely the matrices

$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}$$

where $z$ and $w$ are complex numbers with $|z|^2 + |w|^2 = 1$.

The corresponding Lie algebra $\mathfrak{su}(2)$ consists of the skew-hermitian $2 \times 2$ matrices with trace zero. These are precisely the matrices

$$\begin{pmatrix} it & z \\ -\overline{z} & -it \end{pmatrix}$$

with $t$ real and $z$ complex.

Each conjugacy class in SU(2) contains exactly one element of the form

$$\begin{pmatrix} \exp(-2\pi i\alpha) & 0 \\ 0 & \exp(2\pi i\alpha) \end{pmatrix}$$

with $0 \le \alpha \le 1/2$. It then follows from Theorem 3.6 that the natural class of connections on $B_1 \setminus D_1$ are connections of the form

$$\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta + a$$

with $0 \le \alpha \le 1/2$. Here I will only discuss the case $0 < \alpha < 1/2$. In the case of $\alpha = 0$, the singularity is removable, and we are back to the case discussed in §2. In the case $\alpha = 1/2$, the singularity is removable as far as the local analysis is concerned; however there can be topological obstructions to removing the singularity globally on a 4-manifold, see [KM1].

If $\sigma$ is an $\mathfrak{su}(2)$-valued $p$-form, then we can decompose $\sigma$ as

$$\sigma = \begin{pmatrix} i\sigma_D & \sigma_T \\ -\overline{\sigma}_T & -i\sigma_D \end{pmatrix}$$

where $\sigma_D$ is a real valued $p$-form and $\sigma_T$ is a complex valued $p$-form. We have

$$\nabla_{\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right) d\theta}\sigma = \nabla\sigma + [\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right) d\theta, \sigma] = \begin{pmatrix} i\nabla\sigma_D & \nabla_{2i\alpha d\theta}\sigma_T \\ -\overline{\nabla_{2i\alpha d\theta}\sigma_T} & -i\nabla\sigma_D \end{pmatrix}$$

Thus $\nabla_{\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right) d\theta}$ acts on $\sigma_D$ as $\nabla$ and on $\sigma_T$ as $\nabla_{2i\alpha d\theta}$. Let $d_{2i\alpha d\theta}$ denote the covariant exterior derivative given by the connection $2\alpha\, d\theta$. Let $d^*_{2i\alpha d\theta}$ denote the adjoint of $d_{2i\alpha d\theta}$.

We then have the following analog of Theorem 2.2.

**Theorem 4.1.** [R1] *For any $\alpha$ with $2\alpha \notin \mathbb{Z}$ there exists $\varepsilon > 0$ such that if $A = \left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right) d\theta + a$ is a connection with $a \in L^{2,1}_{\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right) d\theta}(B_1)$ and*

$$\|F_A\|_{L^2(B_1 \setminus D_1)} \leq \varepsilon,$$

*then there exists a gauge transformation $g \in L^{2,2}_{\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right) d\theta}(B_1)$ such that*

$$g.A = \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta + a'$$

*where*

$$\begin{cases} d^* a'_D = 0 & \text{on } B_1 \\ d^*_{2i\alpha d\theta}(r^{-2} a'_T) = 0 & \text{on } B_1 \setminus D_1 \\ \nu \lrcorner a' = 0 & \text{on } \partial B_1 \end{cases}$$

*and*

$$\|a'\|_{L^{2,1}_{\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right) d\theta}(B_1)} \leq c \|F_A\|_{L^2(B_1)}.$$

We also have the following analog of Theorem 2.3.

**Theorem 4.2.** [R1] *If in additional to the assumptions of Theorem 4.1 the connection is Yang-Mills, then*

$$\begin{cases} |a'_D| + |\nabla a'_D| \leq c \|F_A\|_{L^2(B_1)} & \text{on } B_{1/2} \\ |\nabla^k a'_T| \leq c r^{2\min\{2\alpha, 1-2\alpha\} - k} \|F_A\|_{L^2(B_1)} & \text{on } B_{1/2}\ . \end{cases}$$

These seemingly strange theorems demand an explanation. The key is to understand the function space $L^{2,1}_{\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right) d\theta}(B_1)$ in more detail. It follows from (1.1) that

$$a \in L^{2,1}_{\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right) d\theta}(B_1) \Leftrightarrow \begin{cases} a_D \in L^{2,1}(B_1) \\ a_T \in L^{2,1}_{2i\alpha d\theta}(B_1) \end{cases}$$

Here $a_T \in L^{2,1}_{2i\alpha d\theta}$ means that $a_T, \nabla_{2i\alpha d\theta} a_T \in L^2(B_1)$. Fully written out

$$\|\nabla_{2i\alpha d\theta} \sigma_T\|^2_{L^2(B_1)} = \int_{B_1} \left( \left|\frac{\partial \sigma_T}{\partial x_1}\right|^2 + \left|\frac{\partial \sigma_T}{\partial x_2}\right|^2 + \left|\frac{\partial \sigma_T}{\partial r}\right|^2 + r^{-2} \left|\frac{\partial \sigma_T}{\partial \theta} + 2i\alpha \sigma_T\right|^2 \right) r\, dx_1\, dx_2\, dr\, d\theta$$

Now,

$$\int_{S^1} f^2\, d\theta \leq \left(\min\{2\alpha, 1 - 2\alpha\}\right)^{-1} \int_{S_1} (df/d\theta + 2i\alpha f)^2 d\theta.$$

It follows that

$$\|r^{-1}\sigma_T\|_{L^2(B_1)} \leq c\|\nabla_{2i\alpha d\theta}\sigma_T\|_{L^2(B_1)}.$$

On the other hand, it is clear that

$$\|\nabla_{2i\alpha d\theta}\sigma_T\|_{L^2(B_1)} \leq c(\|\nabla\sigma_T\|_{L^2(B_1)} + \|r^{-1}\sigma_T\|_{L^2(B_1)}).$$

Hence

$$a \in L^{2,1}_{\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right)d\theta}(B_1) \Leftrightarrow \begin{cases} a_D \in L^{2,1}(B_1) \\ \nabla a_T, r^{-1}a_T \in L^2(B_1) \end{cases}$$

So now you think I'm going to talk about analysis on weighted Sobolev spaces with singular weights. I'm not.

As I mentioned before, the finite energy condition and the Yang-Mills equation are conformally invariant in 4 dimensions. Thus we can replace the standard metric

$$\sum_i dx_i^2 = dx_1^2 + dx_2^2 + dr^2 + r^2 d\theta^2$$

with any conformal metric. A natural choice is the metric

$$r^{-2}\sum_i dx_i^2 = r^{-2}(dx_1^2 + dx_2^2 + dr^2) + d\theta^2.$$

With this metric $D_1$ is moved out to infinity. We recognize $r^{-2}(dx_1^2 + dx_2^2 + dr^2)$ as the upper half space model of hyperbolic 3-space. Thus $\mathbb{R}^4$ with the metric $r^{-2}\sum dx_i^2$ is isometric with $H^3 \times S^1$, the cartesian product of hyperbolic 3-space and the unit circle. The unit ball with this metric is isometric with $H^3_+ \times S^1$, the cartesian product of one half of hyperbolic 3-space and the unit circle. Thus we can view $\sigma_T$ as a differntial form on $H^3_+ \times S^1$. A short calculation shows that $r^{-1}\sigma_T, \nabla\sigma_T \in L^{2,1}$ if and only if $\sigma_T \in L^{2,1}(H^3_+ \times S^1)$. Thus

$$a \in L^{2,1}_{\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right)d\theta}(B_1) \Leftrightarrow \begin{cases} a_D \in L^{2,1}(B_1) \\ a_T \in L^{2,1}(H^3_+ \times S^1) \end{cases}$$

*Thus we should view $a_D$ as a differential form on $B_1$ and $a_T$ as a differential form on $H^3_+ \times S^1$.* Let $d^*_{\mathrm{h},2\alpha d\theta}$ denote the adjoint of $d_{2\alpha d\theta}$ with respect to the metric $r^{-1}\sum dx_i^2$. Moreover, a short calculation shows that

$$d^*_{\mathrm{h},2i\alpha d\theta}a_T = r^4 d^*_{2i\alpha d\theta}(r^{-2}a_T).$$

In other words, *the gauge condition says that $a_D$ is coclosed on $B_1$ and $a_T$ is coclosed on $H^3_+ \times S^1$.*

Theorem 4.1 is now proven along the same lines as Thm. 2.1. Instead of the equation (2.4) we get the equations

$$\begin{cases} \Delta\varphi_D + \sum_i (a_T)_i \dfrac{\partial\varphi_T}{\partial x_i} = -d^* b_D & \text{on } B_1 \\ \Delta_{\mathrm{h},2i\alpha d\theta}\varphi_T + \operatorname{Re}\sum_i ((a_T)_i (d\varphi_D)_i + (a_D)_i (d_{2i\alpha d\theta}\varphi_T)_i) & \\ \qquad = -d^*_{\mathrm{h},2i\alpha d\theta} b_T & \text{on } H^3_+ \times S^1. \\ \nu \lrcorner d\varphi_D = \nu \lrcorner b_D & \text{on } \partial B_1 \\ \nu_{\mathrm{h}} \lrcorner d_{2i\alpha d\theta}\varphi_T = \nu_{\mathrm{h}} \lrcorner b_T & \text{on } \partial H^3_+ \times S^1 \end{cases}$$

where $\Delta_{\mathrm{h},2i\alpha d\theta} = d_{2i\alpha d\theta} d^*_{\mathrm{h},2i\alpha d\theta} + d^*_{\mathrm{h},2i\alpha d\theta} d_{2i\alpha d\theta}$ is the covariant Hodge Laplacian for 1-forms on $H^3_+ \times S^1$ given by the connection $2i\alpha\, d\theta$, and $\nu_{\mathrm{h}} = r\nu$ is the outward unit normal of $H^3_+ \times S^1$. Thus we get a small perturbation of the Neumann problem for $\Delta$ on $B_1$ and the Neumann problem for $\Delta_{\mathrm{h},2i\alpha d\theta}$ on $H^3_+ \times S^1$. The theory for the former is well known. The letter is analyzed in [R1] by elementary methods.

Theorem 4.2 is now proven along the same lines as Thm. 2.3. Instead of the equation (2.5) we get the system

$$\begin{cases} \Delta a_D + \{a_T \otimes \nabla_{2i\alpha d\theta} a_T\} + \{a_D \otimes a_T \otimes a_T\} = 0 & \text{on } B_1 \\ \Delta_{\mathrm{h},2i\alpha d\theta} a_T + \{a_D \otimes \nabla_{\mathrm{h},2i\alpha d\theta} a_T\} + \{a_T \otimes \nabla_{\mathrm{h}} a_D\} & \\ \quad +\{a_T \otimes a_T \otimes a_T\} + \{a_D \otimes a_D \otimes a_T\} = 0 & \text{on } H^3_+ \times S^1\ . \end{cases}$$

The 1-form $a$ can now be estimated by a bootstrapping procedure. On the first equation we apply standard elliptic estimates for the usual Laplacian $\Delta$ on $B_1$. On the second equation we apply decay estimates at infinity for the covariant Hodge Laplacian $\Delta_{\mathrm{h},2i\alpha d\theta}$ on $H^3_+ \times S^1$. These decay estimates are derived in [R1] by elementary methods.

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More references on singular Yang-Mills fields can be found in [R1].

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