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ON LOCAL AND GLOBAL ANALYTIC AND GEVREY HYPOELLIPTICITY

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Introduction.

This article summarizes recent progress in the investigation of analytic hypoellipticity of linear partial differential operators having analytic[1] coefficients. Results and examples previously known will first be recalled. The notion of global analytic hypoellipticity will be introduced in §2. Our first main result is then a counterexample to global analytic hypoellipticity in dimension three.

The simpler case of partial differential operators with multiple characteristics in $\mathbb{R}^2$ will be discussed in detail in §3. For sums of squares of vector fields, a conjectured necessary and sufficient condition for analytic hypoellipticity will be stated. A geometric invariant q will be introduced, in terms of which a more refined conjecture on the optimal exponent for hypoellipticity in Gevrey classes will be formulated. A number of partial results supporting the conjecture will be adduced.

The analysis depends on certain nonlinear eigenvalue problems. These are the subject of §4, where a third conjecture will be put forward. No indications of proofs will be given.

1. Background.

Suppose that $L = \sum_j X_j^2$ is a sum of squares of n real, C^ω vector fields X_j on some real analytic manifold M of dimension N, which locally will be regarded as an open subset of $\mathbb{R}^N$. We assume always the bracket hypothesis of Hörmander, which asserts that the Lie algebra generated by the vector fields spans the tangent space to the ambient manifold at every point. L is said to be analytic hypoelliptic (in an open set V) if for every open $V' \subset V$ and every $u \in \mathcal{D}'(V')$ such that $Lu \in C^\omega(V')$, necessarily $u \in C^\omega(V')$. The bracket hypothesis ensures C^∞ hypoellipticity [H2].

Denote by $\Sigma \subset T^*M \backslash \{(x,\xi) : \xi = 0\}$ the characteristic variety of L, that is, the set where the principal symbol of L vanishes. Denoting by $\pi : T^*M \mapsto M$ the natural projection, L is said to be symplectic at a point $p \in M$ is said to be a symplectic point if for some small neighborhood U of p, $\Sigma \cap \pi^{-1}(U)$ is a symplectic submanifold of T^*U.

Consider the special case where the vector fields X_j are linearly independent at p and $N = n+1$. Fix a nonzero cotangent vector $\omega \in T_p^*M$ that annihilates the span V of the X_j at p. Define the skew symmetric quadratic form Q_p on V by $Q_p(Y,Y') = \langle \omega, [Y,Y'](p)\rangle$, where the bracket denotes the pairing between cotangent and tangent vectors. Then p is a symplectic point if and only if Q_p is a nondegenerate quadratic form.

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[1]The terms "analytic" and "real analytic" are synonymous in this paper.

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The fundamental theorem concerning analytic hypoellipticity for these operators, due independently to Treves [Tr1] and Tartakoff [Ta1][Ta2], states simply that L is analytic hypoelliptic in a neighborhood of any point where it is symplectic.[2]

At the opposite extreme is a theorem of Métivier [M1] asserting under certain auxiliary hypotheses that if *no* point of an open set U is symplectic, then L is not analytic hypoelliptic in U. A simple example is [BG] $\partial_x^2 + x^2\partial_t^2 + \partial_y^2$ in $\mathbb{R}^3$.

Our motivation comes from complex analysis in several variables, where one encounters operators similar to sums of squares, especially in the simplest case of $\mathbb{C}^2$ [K].[3] If $\Omega \subset \mathbb{C}^2$ is a bounded pseudoconvex domain with C^ω boundary, then $\partial\Omega$ is a CR manifold on which is defined a Cauchy-Riemann operator $\bar\partial_b$. $\bar\partial_b \circ \bar\partial_b{}^*$ may be expressed in local coordinates as $(X + iY) \circ (-X + iY)$, modulo insignificant lower-order terms, and the bracket hypothesis holds. The set of nonsymplectic (that is, weakly pseudoconvex) points is either empty, in which case the theorem of Treves applies,[4] or is a real analytic subvariety of positive codimension in $\partial\Omega$. The everywhere degenerate situation of [M1] does not arise.

Another very interesting example [M2] is $L = \partial_x^2 + (x^2 + t^2)\partial_t^2$ in $\mathbb{R}^2$. This is a sum of squares of three vector fields, modulo an unimportant lower order term. It is elliptic except at a single point, namely the origin, where it still satisfies the bracket hypothesis, yet is not analytic hypoelliptic. Consider now $L' = \partial_x^2 + x^2\partial_t^2$. L' is essentially weaker than L, for instance in the sense that $\langle -Lf, f\rangle > \langle -L'f, f\rangle$ for all $f \neq 0$ supported sufficiently near 0. Yet L' is symplectic and hence analytic hypoelliptic.

Concerning the intermediate situation, only one result of even a mild degree of generality[5] has been obtained. Given a two-dimensional subbundle T of $T\mathbb{R}^3$, a curve $\gamma : (-\varepsilon, \varepsilon) \mapsto \mathbb{R}^3$ is said to be subordinate to T if $\dot\gamma(s)$ belongs to T for each s; we assume always that $\dot\gamma \neq 0$.

Theorem 1. [C2] *Let X, Y be linearly independent C^ω real vector fields in an open subset $U \subset \mathbb{R}^3$, satisfying the bracket hypothesis, and let $L = X^2 + Y^2$. A necessary condition for analytic hypoellipticity of L is that there exist no curve γ in U subordinate to the subbundle of $T\mathbb{R}^3$ spanned by X, Y with the additional property that $\gamma(s)$ is a nonsymplectic point for every s.*

This is a special case of a much more general conjecture of Treves [Tr1]. The two-dimensional example above suggests that this necessary condition is not sufficient, but to date no example in $\mathbb{R}^3$ having only an isolated nonsymplectic point has been proved to lack analytic hypoellipticity.[6]

The hypothesis of subordinary cannot be omitted. In $\mathbb{R}^3$ set $X = \partial_x$, $Y = \partial_y + a(x,y)\partial_t$ with $a(x,y) = x^{1+k_1} + xy^{k_2}$ where k_j are strictly positive, even integers, and take $L = X^2 + Y^2$. Then $s \mapsto (0,0,s)$ parametrizes a curve consisting entirely of nonsymplectic points, yet L is analytic hypoelliptic [GS].

Another class of examples is $X = \partial_x$, $Y = \partial_y + x^{m-1}\partial_t 1$ in $\mathbb{R}^3$ with coordinates (x, y, t), where $m \geq 2$ is a positive integer. The case $m = 2$ is symplectic, but Theorem 1 asserts that analytic

[2]The results cited are actually formulated much more generally.

[3]In order to avoid complicating the exposition with inessential technicalities, we restrict attention in this article for the most part to sums of squares.

[4]Actually it applies only microlocally, in one half of the characteristic variety of $\bar\partial_b\bar\partial_b{}^*$; analytic hypoellipticity always fails to hold in the other half, but that region turns out not to be relevant for the questions arising in complex analysis.

[5]The case of linear partial differential operators of principal type, in contrast to those having multiple characteristics, is completely understood through work of Trepreau [Tp] and of Treves [Tr2].

[6]It is this author's firm belief that such examples do exist, and work in this direction is underway.

hypoellipticity does not hold for $m \geq 3$.[7]

2. Global Regularity.

Suppose L to be defined on a compact manifold M without boundary. L is said to be globally analytic hypoelliptic if $Lu \in C^\omega(M)$ implies $u \in C^\omega(M)$. Analytic hypoellipticity in the local sense implies it in the global sense, but not conversely. For example, consider any C^∞ hypoelliptic operator L with constant coefficients, regarded as acting on functions defined on the torus $\mathbb{T}^n$ rather than on $\mathbb{R}^n$. Then L is globally analytic hypoelliptic, but is so in the local sense only if it is elliptic.

Modify the example two paragraphs above by replacing x^{m-1} by $\sin^{m-1}(x)$, so that $L = X^2 + Y^2$ is defined on the torus $\mathbb{T}^3$. Then Theorem 1 still guarantees that L is not analytic hypoelliptic in the local sense, yet it is so in the global sense [CH],[C3]. Since these examples are prototypical for the situation of Theorem 1, and since global hypoellipticity is a far weaker property than local hypoellipticity, it was hoped that global analytic hypoellipticity might always hold (for sums of squares, under the bracket hypothesis).

Consider $L = X^2 + Y^2$ on $\mathbb{T}^2$, with periodic coordinates (x, t) (so that functions on $\mathbb{T}^2$ are identified with periodic functions on $\mathbb{R}^2$). Assume that $X \equiv \partial_x$ and $Y = \theta(x,t)\partial_t$ for some C^ω real coefficient θ, and that the bracket hypothesis is satisfied.

Theorem 2. [C6] *Suppose that the Taylor expansion of $\theta(x,t)$ at 0 is of the form $\theta(x,t) = c_1 x^{m-1} + c_2 t^k$ plus higher order terms, where $k > 0$, $m \geq 3$, and $c_1, c_2 \neq 0$. Suppose also that the range of L contains $L^2(\mathbb{T}^2)$. Then L is not globally analytic hypoelliptic.*

By higher order terms we mean all monomials $x^\alpha t^\beta$ satisfying $\alpha/(m-1) + \beta/k > 1$. The assumption $m \geq 3$ means that 0 is not a symplectic point.

Thus certain behavior of a finite part of the Taylor expansion of a coefficient at a single point is enough to preclude global regularity. The term t^k acts as a perturbation of the situation where θ depends on x alone. There is then a rotational symmetry with respect to t, and global analytic hypoellipticity holds quite generally in the presence of such a symmetry [C3]. Much work has been done on symmetric special cases, which Theorem 2 now reveals to be atypical.

Three-dimensional counterexamples are constructed directly from the two-dimensional situation by replacing $\theta(x,t)\partial_t$ by $\partial_y + \theta\partial_t$, and considering functions on $\mathbb{T}^3$ independent of the y variable. Analogous analysis then leads to the following counterexample.

Theorem 3. [C4] *There exist a bounded, pseudoconvex domain $\Omega \subset \mathbb{C}^2$ with C^ω boundary and a function $f \in C^\omega(\partial\Omega)$, whose Szegö projection does not belong to $C^\omega(\partial\Omega)$.*

3. The Two-Dimensional Case.

The simplest case of all is that of a sum of squares $L = X^2 + Y^2$ of two vector fields in an open subset of $\mathbb{R}^2$. The bracket hypothesis implies that at every point, at least one of X, Y is nonzero. In general there will be some points at which L is elliptic, others at which it is nonelliptic but symplectic (that is, X, Y are dependent at p but $X, Y, [X, Y]$ span the tangent space at p), and yet others at which it is neither. Define m to be the smallest integer such that the vector space spanned by X, Y and all of their iterated Lie brackets with m or fewer factors equals the whole tangent space at p.[8] Then p is said to be a point of type $m = m(p)$. Type 1 means elliptic, type 2 symplectic.

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[7]These examples were treated earlier in a series of papers [He],[PR],[HH],[C5].

[8]For this purpose X, Y themselves are considered to be Lie brackets with 1 factor.

In this section we discuss only hypoellipticity in the local sense. Fixing a local coordinate system, X, Y may be regarded as the two columns of a square matrix, and we define $\Theta(p)$ to be the determinant of that matrix, evaluated at p. Changing the coordinates has the effect only of multiplying Θ by a nowhere vanishing factor; the same goes if the pair X, Y is replaced by a second pair represented as an invertible linear combination, with analytic coefficients, of X, Y.[9]

The invariant m alone does not govern analytic hypoellipticity. Shortly we will introduce a second geometric invariant, $q \in (0, \infty]$. Like m, q is determined by the Taylor expansion of the coefficients of X, Y at p. For our immediate purpose it suffices to know that if p is a point of type $m \geq 2$, then $q = q(p)$ equals ∞ if and only if there exist coordinates (x, t) with respect to which $p = 0$ and the span of X, Y equals the span of ∂_x, $x^{m-1}\partial_t$ in a neighborhood of 0.

Conjecture 1. *$L = X^2 + Y^2$ in $\mathbb{R}^2$ is analytic hypoelliptic in some neighborhood of a point p if and only either $m(p) = 1$ or $q(p) = \infty$.*

When $m(p) = 2$ then q is always ∞. An example where $q < \infty$ is $X = \partial_x$ and $Y = [x^{m-1} + t^k]\partial_t$, for any $m \geq 3$ and $k \geq 1$.

In general, q is defined as follows. Where $m = 1$, q is simply defined to be ∞. Assume henceforth that $m(p) \geq 2$. It is possible to choose coordinates (x, t) in which $p = 0$, together with vector fields $\tilde{X}, \tilde{Y}$ having everywhere the same span as X, Y, such that $\tilde{X} \equiv \partial_x$, $\tilde{Y} = \theta(x,t)\partial_t$, $\theta(x,t) = x^{m-1} + \sum_{j=0}^{m-3} \beta_j(t)x^j$, and each coefficient β_j vanishes where $t = 0$. q is defined to be ∞ if and only if each β_j vanishes identically. Otherwise define τ_j to be the order of vanishing of β_j at $t = 0$ and set

$$q = \min_j \tau_j/(m-1-j).$$

This quantity can be shown to be independent of all choices made.[10]

The basic example is $\theta(x,t) = x^{m-1} + t^\ell x^{k-1}$ where $1 \leq k \leq m-2$ and $\ell > 0$. Then $q = \ell/(m-k)$. Thus q is rational, and $(m-1)^{-1} \leq q < \infty$.

In those situations where q is finite, define the exponent s_0 by the relation $1 - s_0^{-1} = (mq)^{-1}$. Then $1 < s_0 \leq m$, since $q \geq (m-1)^{-1}$. Given $m \geq 3$, the set of possible values for s_0 is a certain infinite set of rational numbers in the interval $(1, m]$.

Denote by G^s the Gevrey class of order $s \in [1, \infty)$. Recall that $G^s \subset G^t$ whenever $s < t$, and that $G^1 = C^\omega$. A partial differential operator L is said to be G^s hypoelliptic if each distribution u belongs to G^s in any open set in which $Lu \in G^s$. Under a mild hypothesis always satisfied by sums of squares of vector fields satisfying the bracket condition, G^s hypoellipticity implies G^t hypoellipticity for any $t > s$ [M1].

Let X, Y be as in Conjecture 1.

Conjecture 2. *Assume that $m(p) \geq 3$ and $q(p) < \infty$. Then in every sufficiently small neighborhood of p, $L = X^2 + Y^2$ is G^s hypoelliptic if and only if $s \geq s_0$.*

Here $s_0 = s_0(p)$. Any sum of squares operator is G^s hypoelliptic for all $s \geq m$ [GS], but $s_0 < m$ unless $q = (m-1)^{-1}$, the minimum possible value for q.

Recall [RS],[H2] that if p is a point of type m and Lu belongs to some Sobolev space H^s $(s \geq 0)$ in a neighborhood of p, then $u \in H^{s+2/m}$ in some neighborhood of p, and that the exponent $s + 2m^{-1}$

[9] All our results depend only on the span of X, Y, rather than on the vector fields themselves.

[10] It is essential in the definition that the coefficient of x^{m-2} vanish identically. When $m = 2$, there are no terms $\beta_j(t)x^j$ at all, so that $q = \infty$.

is best possible in all cases. Thus m alone suffices to determine the regularity properties of L in the Sobolev scale.

Theorem 4. [C6] *If $q = \infty$ then L is analytic hypoelliptic. If $q < \infty$ then L is G^s hypoelliptic for all $s \geq s_0$.*

Typical examples where $q = \infty$ are $\partial_x^2 + [a(x,t)x^{m-1}\partial_t]^2$, where $a \neq 0$. In the next theorem we assume that $m \geq 3$, $1 \leq k \leq m-2$, and $\ell > 0$.

Theorem 5. [C6] *$L = \partial_x^2 + \left[(x^{m-1} + t^\ell x^{k-1})\partial_t\right]^2$ fails to be G^s hypoelliptic for all $s < s_0$, except possibly when all of the following conditions hold: $m/(m-k)$ is an integer, m is even, k is odd, $k > 1$, and $m/(m-k)$ is not divisible by 4.*

We believe this restriction on (m,k) to be merely an artifact of an *ad hoc* method of proof.

These examples suffice to demonstrate that the optimal Gevrey exponent need not be an integer, in contrast to all cases previously known to this author.

In $\mathbb{R}^2$ the pair X, Y is said to define a pseudoconvex structure if Θ does not change sign. The characteristic variety Σ of $L = (X + iY) \circ (-X + iY)$ is then a trivial line bundle over the variety of nonelliptic points in the base space. As in the three-dimensional case, it splits as the union of two half-line bundles $\Sigma^\pm$ (depending on the sign of the variable dual to t in the special coordinates (x,t) described above). The natural question for L is whether it is analytic microhypoelliptic, or G^s microhypoelliptic, in some conic neighborhood of Σ^+.[11]

Theorem 6. [C6] *Assume pseudoconvexity and the bracket hypothesis. Then the analogues of Conjectures 1 and 2 hold for $L = (X + iY) \circ (-X + iY)$, in a conic neighborhood of Σ^+, in full generality.*

In §4 we will introduce, for each operator $X^2 + Y^2$ or $(X + iY) \circ (-X + iY)$, an associated nonlinear eigenvalue problem, and will conjecture that this problem has an affirmative solution whenever q is finite.

Theorem 7. [C6] *If Conjecture 3, concerning nonlinear eigenvalue problems, is correct, then Conjectures 1 and 2 hold in full generality.*

More precisely, Conjectures 1 and 2 hold in any particular case for which the unique associated nonlinear eigenvalue problem satisfies Conjecture 3.

It is interesting to contrast these results with the following example in $\mathbb{R}^5$, analyzed by Ching-Chau Yu [Y]. For $m \geq 3$ set $L_m = \partial_{x_1}^2 + (\partial_{y_1} + x_1^{m-1}\partial_t)^2 + \partial_{x_2}^2 + (\partial_{y_2} + x_2\partial_t)^2$. Then the quadratic form Q has rank one where $x_1 = 0$, and full rank elsewhere.

Theorem 8. (Yu) *For any even $m \geq 4$, L_m fails to be analytic hypoelliptic. More precisely, L_m is G^s hypoelliptic if and only if $s \geq 2$.*

The fact that G^2 hypoellipticity holds for all $s \geq 2$ is implied by the theorem of Derridj and Zuily [DZ]. This is the first example known to this author in dimension greater than three for which analytic hypoellipticity is shown to fail, yet Q is not everywhere degenerate. Although L_m becomes more degenerate as m increases, the optimal Gevrey exponent does not change so long as $m \geq 3$.

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[11] $\bar\partial_b{}^*$ is never microlocally Gevrey, analytic, or C^∞ hypoelliptic in any conic neighborhood of Σ^- in this situation, hence neither is $\bar\partial_b \circ \bar\partial_b{}^*$.

4. Nonlinear Eigenvalue Problems.

Suppose that Φ is a homogeneous polynomial of the form

$$\Phi(x,z) = x^{m-1} + \sum_{j=0}^{m-2} \alpha_j z^{m-1-j} x^j$$

with $\alpha_j \in \mathbb{R}$. Suppose further that P is a homogeneous quadratic polynomial in two noncommuting variables w_1, w_2 of the form $P(w) = \left[c_{11}w_1 + c_{12}w_2\right]^2 + \left[c_{12}w_1 + c_{22}w_2\right]^2$, where the coefficients c_{ij} are real and the matrix (c_{ij}) is nonsingular. Define the ordinary differential operator $\mathcal{L}_z = P(d/dx, i\Phi(x,z))$, acting on functions of $x \in \mathbb{R}$ and depending on the parameter $z \in \mathbb{C}$.

Given a family $\{\mathcal{L}_z : z \in \mathbb{C}\}$ of ordinary differential operators, we say that $z \in \mathbb{C}$ is a nonlinear eigenvalue if there exists $0 \neq f \in L^\infty(\mathbb{R})$ such that $\mathcal{L}_z f \equiv 0$. In the situation of the preceding paragraph, it is equivalent to ask for $f \in L^2$, or $f \in \mathcal{S}$, rather than $f \in L^\infty$.

Conjecture 3. *Assume $\mathcal{L}_z$ to be a family of ordinary differential operators of the class described. Then either there exists at least one nonlinear eigenvalue, or $\Phi(x,z) = c'(x+cz)^{m-1}$ for some constants c, c'.*

Various problems of this type have been analyzed in [PR],[K],[FS],[C5],[C1]. Yu [Y] has determined the asymptotic distribution of the nonlinear eigenvalues for $-\partial_x^2 + (x^{m-1}+z)^2$.

To an operator $L = X^2 + Y^2$ on $\mathbb{R}^2$ and a point p at which L is not elliptic we assign a family $\mathcal{L}_z$ of the above type by the following procedure. Choose coordinates (x,t) with origin at p as in the definition of q, and determine the function $\theta(x,t)$. Then define a polynomial P by $P(x,z) = x^{m-1} + \sum_j \alpha_j z^{m-j} x^j$ where $\alpha_j = 0$ if β_j vanishes to order $\tau_j > (m-1-j)q$ at $t = 0$, and α_j is the leading-order coefficient in the Taylor expansion $\beta_j(t) = \alpha_j t^{\tau_j} + O(t^{\tau_j+1})$ if $\tau_j = (m-1-j)q$. Unlike Θ and θ, P is independent of all choices made in its construction, modulo multiplication by constants.

There exist analytic real-valued functions $\tilde{c}_{ij}$ such that $X = \tilde{c}_{11}\partial_x + \tilde{c}_{12}\theta\partial_t$, $Y = \tilde{c}_{21}\partial_x + \tilde{c}_{22}\theta\partial_t$, and the matrix $(\tilde{c}_{ij})$ is invertible at p. Set $c_{ij} = \tilde{c}_{ij}(p)$. The family of ordinary differential operators associated to L at p is then

$$\mathcal{L}_z = \left[c_{11}\partial_x + ic_{12}P(x,z)\right]^2 + \left[c_{21}\partial_x + ic_{22}P(x,z)\right]^2 .$$

When $q < \infty$ the polynomial P is never of the exceptional form $c'(x+cz)^{m-1}$, because the coefficient of x^{m-2} for θ vanishes.

Let p be a polynomial satisfying $\partial_x p = P$. If λ is any real constant, then defining $\tilde{L}_z = \exp(-i\lambda p) \circ \mathcal{L}_z \circ \exp(i\lambda p)$, $z \in \mathbb{C}$ is a nonlinear eigenvalue for $\{\mathcal{L}_z\}$ if and only if it is one for $\{\tilde{\mathcal{L}}_z\}$. Therefore the nonlinear eigenvalue problem for $L = X^2 + Y^2$ depends only on the span of X, Y, rather than on the vector fields themselves.

Theorems 5 and 6 are obtained by showing that nonlinear eigenvalues exist for $-\partial_x^2 + (x^{m-1} + z^{m-k}x^{k-1})^2$ and for $(\partial_x + P(x,z)) \circ (-\partial_x + P(x,z))$, respectively. In the latter case there is the pseudoconvexity hypothesis that $\partial P/\partial x \geq 0$ for all $x, z \in \mathbb{R}$.

For partial differential operators with sufficiently many geometric symmetries, such as $\partial_x^2 + (\partial_y + x^{m-1}\partial_t)^2$, the associated nonlinear eigenvalue problems arise directly via separation of variables. One looks for solutions of $Lu = 0$ of the form $u = \exp(i\tau t + i\eta y) f_{\eta,\tau}(x)$. A dilation symmetry allows reduction to the case $\tau = 1$. If $z = \eta$ is a nonlinear eigenvalue for the resulting family of ordinary differential operators, then $u_\tau(x,y,t) = \exp(i\tau t + i\tau^{1/m} zy) f(\tau^{1/m}x)$ defines a one-parameter family