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Self-similar solutions and Besov spaces for semi-linear Schrödinger and wave equations

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Abstract

We prove that the initial value problem for the semi-linear Schrödinger and wave equations is well-posed in the Besov space $\dot{B}_2^{\frac{n}{2}-\frac{2}{p},\infty}(\mathbf{R}^n)$, when the nonlinearity is of type u^p, for $p \in \mathbf{N}$. This allows us to obtain self-similar solutions, as well as to recover previously known results for the solutions under weaker smallness assumptions on the data.

1. Introduction.

In this introduction we focus on the Schrödinger equation; remarks relevant to the wave equation will be made in the last section. We are interested in the Cauchy problem

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u &= \pm u^p, \\ u(x,0) &= u_0(x), x \in \mathbb{R}^n,\, t \geq 0, \end{cases} \tag{1}$$

where $n \geq 2$. The exact form of the non-linearity is relevant only with respect to the methods which will be used. One can deal with more general non-linearities, but this requires a lot more technicalities which are irrelevant to the equation itself and have to do with the composition of Besov spaces. By restricting ourselves to non-linearities of type $\bar{u}^{p_1}u^{p2}$ where $p1$ and $p2$ are integers, we don't have to worry about further regularity assumptions on the non-linearity, and having an estimate on the non-linearity u^p gives immediatly an estimate on $u^p - v^p$.
The following invariance by scaling of (1) will play an important role

$$\begin{cases} u_0(x) &\longrightarrow & u_{0,\lambda}(x) = \lambda^{\frac{2}{p-1}} u_0(\lambda x) \\ u(x,t) &\longrightarrow & u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t). \end{cases} \tag{2}$$

Let s_p be such that $s_p - \frac{n}{2} = -\frac{2}{p-1}$. The homogeneous Sobolev space $\dot{H}^{s_p}$ is expected to be the "critical" space for well-posedness as its norm is invariant by scaling (2).

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This result is already known, see ([5]), there exists a (weak) solution of (1) which is $C([0,T],\dot{H}^{s_p})$, unique under an additional assumption. Such a solution is global in time, if the $\dot{H}^{s_p}$ norm of the initial data is small. There are of course other results on global well-posedness for appropriate non-linearities, and we refer the reader (without any claim to be exhaustive) to ([8]) or to ([3]) for recent developments.
Our present motivations are of a different nature. They came out of understanding some recent work on self-similar solutions for (1) in ([6, 7, 20, 12]). A self-similar solution is by definition a solution which is invariant by scaling (2). Since this forces the initial data to be homogeneous, such solutions cannot be obtained by the well-posedness results in Sobolev spaces. In [6], under some restrictions on p, the authors introduce a functional space, namely the space of functions u such that

$$\sup_t t^{\beta}\|u(x,t)\|_{p+1} < \infty \tag{3}$$

in which β is to be chosen to preserve the scaling invariance. The authors construct solutions by a fixed point argument in such a space, provided

$$\sup_t t^{\beta}\|e^{it\Delta}u_0(x)\|_{p+1} < \varepsilon_0. \tag{4}$$

By direct computations, one can prove that $u_0(x) = \frac{\tilde{\varepsilon}_0}{|x|^{\frac{2}{p-1}}}$ satisfies (4), thus giving a self-similar solution $u(x,t) = \frac{1}{\sqrt{t}^{\frac{2}{p-1}}}U(\frac{x}{\sqrt{t}})$. More generally, $\tilde{\varepsilon}_0$ could be replaced by a small $C^n(S^n)$ function ([20]).
Our goal will be to draw a connection between such a construction and the usual one in Sobolev spaces. Having this in mind, a natural extension to $\dot{H}^{s_\alpha}$ is the homogeneous Besov space $\dot{B}_2^{s_p,\infty}$, and unlike its Sobolev counterpart, it contains homogeneous functions. Let us recall

$$f(x) \in \dot{H}^{s_\alpha} \Leftrightarrow \int |\xi|^{2s_\alpha}|\hat{f}(\xi)|^2 d\xi \approx \sum_j 2^{2s_\alpha}\int_{2^j<|\xi|<2^{j+1}} |\hat{f}(\xi)|^2 d\xi < +\infty,$$

and one can weaken this requirement to

$$f(x) \in \dot{B}_2^{s_\alpha,\infty} \Leftrightarrow \sup_j 2^{2s_\alpha}\int_{2^j<|\xi|<2^{j+1}} |\hat{f}(\xi)|^2 d\xi < +\infty. \tag{5}$$

From this definition, we obtain immediately that $\frac{1}{|x|^{\frac{2}{\alpha}}} \in \dot{B}_2^{s_\alpha,\infty}$. Thus solving the Cauchy problem (1) in such a space will, among other things, give self-similar solutions.
In the next section, we will treat the Schrödinger equation, and in the last one, the wave equation for which an equivalent analysis can be carried. To end this section let us recall the definition of Besov spaces, their characterizations via frequency localization, and some useful results on Besov spaces.

DEFINITION 1
Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\phi} \equiv 1$ in $B(0,1)$ and $\widehat{\phi} \equiv 0$ in $B(0,2)^c$, $\phi_j(x) = 2^{nj}\phi(2^j x)$, $S_j = \phi_j * \cdot$, $\Delta_j = S_{j+1} - S_j$. Let f be in $\mathcal{S}'(\mathbb{R}^n)$.

- *If $s < \frac{n}{p}$, or if $s = \frac{n}{p}$ and $q = 1$, f belongs to $\dot{B}_p^{s,q}$ if and only if the following two conditions are satisfied*
  - *The partial sum $\sum_{-m}^{m} \Delta_j(f)$ converge to f as a tempered distribution.*
  - *The sequence $\epsilon_j = 2^{js}\|\Delta_j(f)\|_{L^p}$ belongs to ℓ^q.*
- *If $s > \frac{n}{p}$, or $s = \frac{n}{p}$ and $q = 1$, let us denote $m = E(s - \frac{n}{p})$. Then $\dot{B}_p^{s,q}$ is the space of distributions f, modulo polynomials of degree less than $m + 1$, such that*
  - *We have $f = \sum_{-\infty}^{\infty} \Delta_j(f)$ for the quotient topology.*
  - *The sequence $\epsilon_j = 2^{js}\|\Delta_j(f)\|_{L^p}$ belongs to ℓ^q.*

Note that the choice of L^p as the "base" space is in no way an obligation. We will later use more general Besov spaces, with L^p replaced by the Lorentz space $L^{p,r}$. We will denote such a modified space as $\dot{B}^{s,q}_{(p,r)}$. We refer the reader to [1, 14] for the definition and detailed properties of Lorentz spaces.
Another type of space will also be of help

DEFINITION 2
Let $u(x,t) \in \mathcal{S}$. We will say that $u \in \mathcal{L}_t^{\rho}(\dot{B}^{s,q}_{(p,r)})$ iff

$$2^{js}\|\Delta_j u\|_{L_t^{\rho}(L_x^{p,r})} = \varepsilon_j \in \ell^q. \tag{6}$$

Lastly, we recall two lemmas, which allow for an easy characterization of Besov spaces, depending on the sign of s.

LEMMA 1
Let $s > 0$, E a Banach functional space, $q \in [0, +\infty]$, and define

$$f \in \dot{B}_E^{s,q} \equiv 2^{js}\|\Delta_j f\|_E = \varepsilon_j \in \ell^q$$

Then, if $f = \sum_j f_j$, where supp$\hat{f}_j \in B(0, 2^j)$ *and $(2^{js}\|f_j\|_E)_j \in \ell^q$, we have $f \in \dot{B}_E^{s,q}$.*

Its counterpart for $s < 0$ reads

LEMMA 2
Let $s < 0$, $\dot{B}_E^{s,q}$ defined as in lemma 1. Then, an equivalent characterization of $f \in \dot{B}_E^{s,q}$ is

$$2^{js}\|S_j u\|_E = \varepsilon_j \in \ell^q. \tag{7}$$

We omit both proofs, which involve summation over large or small frequencies along with Young inequality for discrete sequences.

2. The semi-linear Schrödinger equation.

In the introduction we did not set any restriction (other than being an integer) on the value of p. However, well-posedness in H^{s_p} holds only if $p \geq p^\star = 1+4/n$, which amounts to dealing with a situation where $s_p \geq 0$. For the critical value $p^\star$, the equation (1) is invariant by the pseudo-conformal transformation, and well-posed in L^2. We will have to restrict ourselves to strictly positive regularity, namely $s_p > 0$, and thus $p > p^\star$, for technical reasons which will be clear from the proof. Such a restriction doesn't appear in [20], where one can go up to $p > 1+2/n$. However the corresponding Cauchy problem in the appropriate Sobolev space H^{s_p} is not known to be well-posed, as $s_p < 0$, and our approach fails for such cases. Let us state our main result

THEOREM 1
Let $n \geq 2$, $p \in \mathbb{N}$, $p > p^\star$, $u_0 \in \dot{B}_2^{s_p,\infty}$, such that $\|u_0\|_{\dot{B}_2^{s_p,\infty}} < \epsilon_0(p,n)$. Then there exists a global solution of (1) such that

$$u(x,t) \in L_t^\infty(\dot{B}_2^{s_p,\infty}), \tag{8}$$

$$u(x,t) \underset{t\to 0}{\longrightarrow} u_0(x) \quad \text{weakly}. \tag{9}$$

Moreover, this solution is unique under the condition

$$\|u(x,t)\|_{\mathcal{L}_t^2(\dot{B}^{s_p,\infty}_{(\frac{2n}{n-2},2)})} < \epsilon_1, \tag{10}$$

for $n \geq 3$ and

$$\|u(x,t)\|_{L_t^{\frac{p^2-1}{2},\infty}(L_x^{p+1,\infty})} < \tilde{\epsilon}_1, \tag{11}$$

for $n = 2$.

The uniqueness conditions (10) and (11), as customary in problems for which solutions are obtained by a fixed-point argument, are related to the auxiliary spaces needed for such an argument. We refer to [15] for a more detailed discussion on this issue, as there are several ways to chose such an auxiliary space. The condition (9) relates to the Besov spaces we consider (see [4, 17] for discussions on such problems). Indeed strong continuity at $t = 0$ is forbidden, and therefore we obtain a somewhat weaker result than what is usually meant for "well-posedness". However, if one has some additional regularity on the initial data, then this regularity is preserved for the solution, namely we obtain

THEOREM 2
Let $u_0 \in \dot{H}^{s_p}$ verify the hypothesis of Theorem 1. Then the global solution obtained by Theorem 1 is such that

$$u(x,t) \in C_t(\dot{H}^{s_p}). \tag{12}$$

This result can be viewed as an extension of global well-posedness in Sobolev spaces for small data, for one can construct initial data with an arbitrary norm in the Sobolev space, but a small one in the Besov space. We could as well construct a local in time theory for $\dot{B}_2^{s_\alpha,q}$, where $q < \infty$. In the following, we only sketch the proof and refer to [15] for details. In particular, we will only deal with $n \geq 3$, which allows for a very simple argument, since we have the so-called "endpoint" Strichartz estimate ([9]). We simply recall the estimates we need

THEOREM 3
Let $S(t) = e^{it\Delta}$. Then

$$\|S(t)u_0(x)\|_{L_t^2(L_x^{\frac{2n}{n-2},2})} \lesssim \|u_0\|_2, \tag{13}$$

$$\|\int_{s<t} S(t-s)f(x,s)ds\|_{L_t^\infty(L_x^2)} \lesssim \|f(x,t)\|_{L_t^2(L_x^{\frac{2n}{n+2},2})} \tag{14}$$

$$\|\int_{s<t} S(t-s)f(x,s)ds\|_{L_t^2(L_x^{\frac{2n}{n-2},2})} \lesssim \|f(x,t)\|_{L_t^2(L_x^{\frac{2n}{n+2},2})} \tag{15}$$

where $\lesssim$ denotes the presence of a constant.

From these estimates (and their generalization to Sobolev spaces, by adding fractional derivatives, which commute with the Schrödinger group), one can prove well-posedness in H^{s_p} for (1). Essentially, if $u \in C_t(H^{s_p}) \cap L_t^2(H^{s_p}_{\frac{2n}{n-2}})$, then $u^{p-1} \in L_t^\infty(L^{\frac{n}{2}})$ by Sobolev embedding and using the Leibnitz rule for fractional derivative together with Kato-Ponce type estimates one gets $u^p \in L_t^2(H^{s_p}_{\frac{2n}{n+2}})$ and the result follows from the Strichartz estimates. Now, one would like to extend this scheme to the Besov spaces. For this purpose, the extra bit of information provided by the presence of a Lorentz space in the Strichartz estimate will be of importance, as by Sobolev embedding we only get $u^p \in L_t^\infty(L^{\frac{n}{2},\infty})$, and the Lorentz space will compensate for this loss of integrability when dealing with the non-linearity. We will set up a fixed point argument in the intersection of two spaces,

$$E = L_t^\infty(\dot{B}_2^{s_p,\infty}),$$

and

$$F = \mathcal{L}_t^2(\dot{B}^{s_p,\infty}_{(\frac{2n}{n-2},2)}) = \{u(x,t) \mid \sup_j 2^{js_p}\|\Delta_j u\|_{L_t^2(L_x^{\frac{2n}{n-2},2})} < +\infty\}.$$

Then, the uniqueness condition we left aside will be $\|u\|_{E\cap F} < \varepsilon_1$. Note that F appears naturally if one consider the linear part $S(t)u_0$, which belongs to F thanks to the Strichartz estimate (15), localized in frequency. We aim at proving the following

PROPOSITION 1
Let $u \in E \cap F$. Then

$$u^p \in F' = \mathcal{L}_t^2(\dot{B}^{s_p,\infty}_{(\frac{2n}{n+2},2)}) = \{v(x,t) \mid \sup_j 2^{js_p}\|\Delta_j v\|_{L_t^2(L_x^{\frac{2n}{n+2},2})} < +\infty\}.$$

and

$$\|u^p\|_{F'} \lesssim \|u\|_E^{p-1}\|u\|_F. \tag{16}$$

In order to prove the proposition, we will make use of lemma 1. Namely, writing u as a telescopic sum,

$$u^p = \sum_j (S_{j+1}u)^p - (S_j u)^p = \sum_j \Delta_j u((S_{j+1}u)^{p-1} + \cdots + (S_j u)^{p-1}) \tag{17}$$

we are left to consider p pieces, each being an infinite sum of functions localized in frequencies $|\xi| \lesssim 2^j$. All terms are essentially the same up to shifts in indices, and we will only deal with the last one,

$$v = \sum_j \Delta_j u(S_j u)^{p-1} = \sum_j v_j.$$

Note that such a decomposition is a paraproduct type formula, see ([2]), in its most simple version. Now, it suffices to prove $v \in F'$, knowing $u \in E \cap F$. From the Sobolev type embedding , see ([1]),

$$\dot{B}_2^{s_p,\infty} \hookrightarrow L^{\frac{n(p-1)}{2},\infty},$$

we have

$$u \in L_t^\infty(L_x^{\frac{n(p-1)}{2},\infty})$$

and since the operators S_j are continuous on Lorentz spaces, we get

$$S_j u \in L_t^\infty(L_x^{\frac{n(p-1)}{2},\infty})$$

uniformly in j. On the other end, $u \in F$ reads

$$2^{js_p}\Delta_j u \in L_t^2(L_x^{\frac{2n}{n-2},2})$$

and by the generalized Hölder inequality ([11]), since $\frac{2+n}{2n} = \frac{n-2}{2n} + (p-1)\frac{2}{(p-1)n}$, we get

$$\begin{aligned} \|v_j\|_{L_t^2(L_x^{\frac{2n}{n+2},2})} &\lesssim \|\Delta_j u\|_{L_t^2(L_x^{\frac{2n}{n-2},2})} \|S_j u\|^{p-1}_{L_t^\infty(L_x^{\frac{n(p-1)}{2},\infty})} \\ \|v\|_{F'} &\lesssim \|u\|_F \|u\|_E^{p-1}. \end{aligned}$$

Applying lemma 1 we get $v \in F'$ and the appropriate norm control. This concludes the proof of proposition 1. The same argument can be used for a product of p functions, leading to the estimate

$$\|\Pi_1^p f_k\|_{F'} \lesssim \sum_1^p \|f_1\|_E \cdot \|f_k\|_F \cdots \|f_p\|_E.$$

¿From such an estimate, one can easily estimate $u^p - v^p = (u-v)(u^{p-1} + \cdot + v^{p-1})$, and setting up the fixed point argument for the integral equation

$$u(x,t) = S(t)u_0(x) + \pm \int_0^t S(t-s)u^p(x,s)ds = S(t)u_0 + \Gamma(u) \tag{18}$$

is essentially straightforward and will be omitted. Uniqueness in a ball of $E \cap F$ follows from the fixed-point. Lastly, we refer to [15] for the weak continuity in zero. This will finish the proof of theorem 1.
Let us now briefly indicate the proof of theorem 2. By theorem 1 we have a solution which is $L^\infty_t(\dot{B}_2^{s_p,\infty})$. However, since $u_0 \in \dot{H}^{s_p}$, from (13) we get

$$2^{js_p}\|\Delta_j S(t)u_0\|_{L^2_t(L_x^{\frac{2n}{n-2},2})} = \eta_j \in \ell^2$$

and from the argument of proposition 1,

$$2^{js_p}\|v_j\|_{L^2_t(L_x^{\frac{2n}{n+22},2})} \lesssim \|u\|_E^{p-1}\eta_j,$$

which in turn implies by (15)

$$\|2^{js_p}\Delta_j\Gamma(u)\|_{L^2_t(L_x^{\frac{2n}{n-2},2})} \in \ell^2.$$

Since all constants are the same as in the proof of theorem 1, in fact all iterates u_n from the fixed-point verify this last estimate, uniformly in n. Therefore the solution itself verifies the estimate, and from (14) we get $2^{js_p}\Delta_j u \in L^\infty_t(L^2_x)$, which is equivalent to $u \in L^\infty_t(\dot{H}^{s_p})$. Strong continuity can be carried along the iterates in the same manner. In the context of self-similar solutions, many interesting additional properties on the profile of such solutions can be derived from the existence construction, by getting rid of the time variable and interpolating between E and F. We refer to [15] and [18] for details and possible applications of such estimates.

3. The semi-linear wave equation.

In this section we deal with the same equation as in the previous sections, where the Schrödinger operator has been replaced by the D'Alembertian operator. Hence, the numerology associated with the equation is changed, where essentially every occurrence of n is replaced with $n-1$. Then, well-posedness in H^{s_p} holds only if $p \geq p^\star = \frac{n+3}{n-1}$, or $s_p \geq \frac{1}{2}$. Again, the conformal exponent $p^\star$ for which the equation is be well-posed in $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$ will be excluded in our analysis. Therefore, we are interested in

$$\left\{ \begin{array}{rcl} \Box u &=& \pm u^p, \\ u(x,0) &=& u_0(x), \\ \partial_t u(x,0) &=& u_1(x), \end{array} \right. \tag{19}$$

for $n \geq 2$. Such an equation is well-posed for initial data $(u_0, u_1) \in H^{s_p} \times H^{s_p-1}$ ([10]), for $p \geq p^\star$. Below $p^\star$, concentration effects take over scaling, and (19) is ill-posed below some critical value above s_p (see [10] or [23] for recent results). It should be noted that for radially symmetric data well-posedness holds up to the scaling, but we won't try to generalize such results here. The natural scaling associated to the equation is

$$(20)\qquad \begin{cases} u_0(x) & \longrightarrow \quad u_{0,\lambda}(x) = \lambda^{\frac{2}{p-1}} u_0(\lambda x) \\ u_1(x) & \longrightarrow \quad u_{1,\lambda}(x) = \lambda^{\frac{2}{p-1}+1} u_1(\lambda x) \\ u(x,t) & \longrightarrow \quad u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda t) \end{cases}$$

In the same spirit as for the Schödinger equation, self-similar solutions for (19) were constructed in [21, 19, 13]. We will recover and extend such results in the range $p > p^\star$, while these authors allow for smaller value of p for which our analysis fails exactly as noted for the Schrödinger equation. We intend to prove the following theorem

THEOREM 4
Let $p \in \mathbb{N}$, $p > p^\star$, $(u_0, u_1) \in (\dot{B}_2^{s_p,\infty}, \dot{B}_2^{s_p-1,\infty})$, such that $\|u_0\|_{\dot{B}_2^{s_p,\infty}} + \|u_1\|_{\dot{B}_2^{s_p-1,\infty}} < \epsilon_0(p,n)$. Then there exists a global solution of (19) such that

$$(21)\qquad u(x,t) \in L_t^\infty(\dot{B}_2^{s_p,\infty}) \quad \text{and} \quad \partial_t u(x,t) \in L_t^\infty(\dot{B}_2^{s_p-1,\infty})$$

$$(22)\qquad u(x,t) \underset{t\to 0}{\longrightarrow} u_0(x) \quad \text{weakly.}$$

Moreover, this solution is unique under an additional assumption

$$(23)\qquad xx\|u\|_{\mathcal{L}_t^{2\frac{n+1}{n-1}}(\dot{B}^{s_p-\frac{1}{2},\infty}_{2\frac{n+1}{n-1}})} = \sup_j 2^{js_p} \|\Delta_j u\|_{L_{t\times x}^{2\frac{n+1}{n-1}}} < \varepsilon_1.$$

Essentially the same remarks apply to this result as in the previous section. We obtain the same kind or regularity preserving results, as well as local in time results for data which would be in a Besov space with a third index $q < \infty$ (see [16]). Let us introduce a few notations $\alpha = \frac{2(n+1)}{n-1}$, $\beta = \frac{2(n+1)}{n+3}$, $\gamma^+ = \gamma + \varepsilon$ for some ε and $\gamma^- = \gamma - \varepsilon$. Recall the fundamental solution, which gives the solution to the linear equation

$$(24)\qquad v(t) = \dot{W}(t)u_0 + W(t)u_1$$

where $W(t)$ is a Fourier multiplier with symbol $\frac{\sin(t|\xi|)}{|\xi|}$ and $\dot{W}(t)$ with symbol $\cos(t|\xi|)$. We intend to solve by fixed point the integral equation

$$(25)\qquad u(x,t) = v(t) + \int_0^t W(t-s)u^p(x,s)ds.$$

The strategy is the same as for the Schrödinger equation, we set up a fixed point in the appropriate spaces, derived from the ones used for solving the initial problem in Sobolev spaces. Namely, we recall

THEOREM 5

We have the following Strichartz estimate (which is the original estimate from [22])

$$\|u\|_{C_t(\dot H_{\frac12})} + \|u\|_{L^\alpha_{t\times x}} \lesssim \|\Box u\|_{L^\beta_{t\times x}} + \|u_0\|_{\dot H_{\frac12}} + \|u_1\|_{\dot H_{-\frac12}}. \tag{26}$$

One can prove well-posedness in H^{s_p} for (19), letting $u \in C_t(H^{s_p}) \cap L^\alpha_t(H^{s_p}_\alpha)$, then $u^{p-1} \in L^{\frac{n+1}{2}}_{t\times x}$ by interpolation, this leads to $u^p \in L^\beta_t(H^{s_p}_\beta)$, and then the result follows from the Strichartz estimates. We now only give a sketch of the proof of theorem 4. and refer to [16] for a full detailed proof as well as additional results. If we localize in frequency, it seems obvious to require

$$2^{js_p}\Delta_j u \in L^\infty_t(L^2_x) \tag{27}$$

$$2^{j(s_p-\frac12)}\Delta_j u \in L^\alpha_t(L^\alpha_x), \tag{28}$$

as the linear part v verifies these estimates for initial data in the appropriate Besov spaces. We aim to prove

$$2^{j(s_p-\frac12)}\Delta_j(u^p) \in L^\beta_{t\times x} \tag{29}$$

when u verifies (27) and (28). This will be the equivalent of proposition 1 in the previous section,

PROPOSITION 2

Let $u \in L^\infty_t(\dot B^{s_p,\infty}_2) \cap \mathcal{L}^\alpha_t(\dot B^{s_p-\frac12,\infty}_\alpha)$. *Then* $u^p \in \mathcal{L}^\beta_t(\dot B^{s_p-\frac12,\infty}_\beta)$, *and*

$$\sup_j 2^{j(s_p-\frac12)}\|\Delta_j(u^p)\|_{L^\beta_{t\times x}} \lesssim (\sup_j(2^{js_p}\|\Delta_j u\|_{L^\infty_t(L^2_x)} + 2^{j(s_p-\frac12)}\|\Delta_j u\|_{L^\alpha_{t\times x}}))^p. \tag{30}$$

Exactly as before, since $s_p - \frac12 > 0$, we are left by paraproduct considerations to deal with $w = \sum_j w_j$, where

$$w_j = (S_j u)^{p-1}\Delta_j u.$$

The difficulty here arises from getting an estimate on $S_j u$. Unlike for the Schrödinger equation, one cannot get useful uniform estimates with respect to j, but on the other hand since we know that $\Delta_j u$ has regularity s_p rather than just $s_p - \frac12$, we will be allowed to lose some ε of regularity in estimating the $S_j u$ piece, namely getting an estimate in a Besov space or regularity $-\varepsilon$ and using lemma 2. We first interpolate (27) and (28), to get

$$2^{j(s_p-\frac12)^+}\Delta_j u \in L^{\alpha^+}_t(L^{\alpha^-}_x), \tag{31}$$

and to be able to recover (29) we would like to get (recall that $\frac1\beta = \frac1\alpha + (p-1)\frac{2}{n+1}$)

$$2^{j0^-}S_j u \in L_t^{\frac{(n+1)(p-1)}{2}^-}(L_x^{\frac{(n+1)(p-1)}{2}^+}). \tag{32}$$

In order to get such an estimate, interpolate again between (27) and (28), to get $L_t^{\frac{(n+1)(p-1)}{2}^-}$ as the time space, which is always possible as $\frac{(n+1)(p-1)}{2} > \alpha$

$$2^{j\tilde{s}}\Delta_j u \in L_t^{\frac{(n+1)(p-1)}{2}^-}(L_x^{\tilde{q}}),$$

where

$$\frac{(n+1)(p-1)^-}{2} = \frac{\alpha}{\gamma} \qquad \tilde{s} = s_p - \frac{\gamma}{2} \qquad \frac{1}{\tilde{q}} = \frac{\gamma}{\alpha} + \frac{1}{2} - \frac{\gamma}{2}.$$

Then, using Bernstein inequality

$$\|\Delta_j u\|_{L_x^r} \lesssim 2^{nj(\frac{1}{\tilde{q}}-\frac{1}{r})}\|\Delta_j u\|_{L_x^{\tilde{q}}}$$

with $\tilde{s} - \frac{n}{\tilde{q}} = 0^- - \frac{n}{r}$. Computing r gives $\frac{(n+1)(p-1)}{2}^+$, as expected by scaling considerations. Thus

$$\|\Delta_j u\|_{L_t^{\frac{(n+1)(p-1)}{2}^-}(L_x^{\frac{(n+1)(p-1)}{2}^+})} \lesssim 2^{j0^+},$$

or equivalently $u \in \mathcal{L}_t^{\frac{(n+1)(p-1)}{2}^-}(\dot{B}^{0^-,\infty}_{\frac{(n+1)(p-1)}{2}^+})$and using lemma 2 we obtain the desired estimate (32), ending the proof of proposition 2.

Setting the fixed point is then straightforward as in the previous section and will be omitted. Once again, we stress that the index juggling with $^+$ and $^-$ is nothing but a convenient notation to avoid carrying along various ε parameters, which are all tied by the scaling. In other word the first epsilon (in (31)) has to be set sufficiently small so that subsequent interpolations can be carried out, and all other ε parameters which appear later are chosen in a unique way dictated by scaling.

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