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Refined Kato inequalities in Riemannian Geometry

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Abstract

We describe the recent joint work of the author with David M. J. Calderbank and Paul Gauduchon on refined Kato inequalities for sections of vector bundles living in the kernel of natural first-order elliptic operators.

1. Introduction

The Kato inequality is a natural inequality for local (non-vanishing) sections on a Riemannian (or Hermitian) vector bundle E with metric connection ∇ over a Riemannian manifold (M, g). If ξ is such a section,

$$\big|d|\xi|\big| \;\leq\; |\nabla\xi|. \tag{1}$$

This is an easy consequence of the Schwarz inequality. Indeed, one has only to notice that

$$|\xi|\big|d|\xi|\big| \;=\; \frac{1}{2}\big|d\left(|\xi|^2\right)\big| \;=\; \big|\langle\nabla\xi,\xi\rangle\big| \;\leq\; |\xi|\,|\nabla\xi|. \tag{2}$$

The Kato inequality has many applications as a tool to relate vector-valued problems to scalar ones. For instance, if ξ is solution of a Laplace-type equation

$$\Delta\xi + A\xi = 0 \tag{3}$$

(where $\Delta = \nabla^*\nabla$), the Kato inequality yields the useful elliptic inequation

$$\Delta|\xi| \leq |A|\,|\xi|. \tag{4}$$

As more elaborate examples or consequences of the previous estimate, one may quote the approach due to Hess-Schrader and Uhlenbrock of estimates on the heat kernel over complete manifolds [10], K. Uhlenbeck's work on removable singularities

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for Yang-Mills connections [13] or other recent works on spectrum estimates for Riemannian manifolds (see the survey [2] for examples).

It was noticed by some authors that *refined* versions of the Kato inequality were also true in some special circumstances. For instance, R. Schoen, L. Simon and S. T. Yau proved in their work on the Bernstein problem [12] that the second fundamental form h of minimal hypersurface in $\mathbb{R}^{n+1}$ satisfied

$$\big|d|h|\big| \;\leq\; \sqrt{\frac{n}{n+2}}|\nabla h|. \tag{5}$$

More recently, S. Bando, A. Kasue and H. Nakajima [1] proved that the Weyl curvature of a Ricci flat manifold satisfies the inequality

$$\big|d|W|\big| \;\leq\; \sqrt{\frac{n-1}{n+1}}|\nabla W|. \tag{6}$$

To give a last example, one may also quote the work of J. Råde on Yang Mills fields on $\mathbb{R}^4$ [11], where the following inequality for the curvature F of such a field appears:

$$\big|d|F|\big| \;\leq\; \sqrt{\frac{2}{3}}|\nabla F|. \tag{7}$$

In all the works quoted, the refined inequalities were one of the key facts of the proofs in each case : they generally led to much stronger decay properties or spectral estimates than those that could be expected from the classical Kato inequality (1). More precisely, they yielded the following much more stringent inequality for the section ξ under consideration:

$$\Delta\left(|\xi|^{\alpha}\right) \leq |A|\,|\xi|^{\beta} \tag{8}$$

with α and β real numbers depending explicitely of the constant appearing in the right-hand side of the improved Kato inequality and this provides strong estimates on the asymptotic behaviour of $|\xi|$. Moreover, it is important to notice that the knowledge of the optimal constant was in all cases necessary to get the full strength of the results.

These examples hence suggest that it is an interesting question to determine completely the circumstances where a refined inequality can appear. A convenient setting for this problem is provided by the following remark due to Jean Pierre Bourguignon [3]: he noticed that in all the cases quoted above, the sections were in the kernel of a *natural first-order and overdetermined elliptic* differential operator. Indeed, if equality could occur in the classical Kato inequality (1), then from the Cauchy-Schwarz inequlity (2) the covariant derivative of the section ξ of E would be parallel to ξ: there should exist a 1-form α such that

$$\nabla\xi \;=\; \alpha\otimes\xi. \tag{9}$$

Moreover, every *natural* first-order operator P is built by projecting (through a linear projection Π) the covariant derivative on a subbundle of $T^*M\otimes E$:

$$P \;=\; \Pi\circ\nabla \tag{10}$$

and P is overdetermined elliptic if and only if the principal symbol map $\sigma_\alpha(P)$ is injective for any non zero cotensor α. Hence, if ξ is in the kernel of P and achieves equality in the Kato inequality, formulas (9-10) above imply one has $\Pi(\alpha \otimes \xi) = 0$. Taking into account the trivial identity between the map $\xi \longmapsto \Pi(\alpha \otimes \xi)$ and the principal symbol map $\xi \longmapsto \sigma_\alpha(P)(\xi)$ (recall the covectors are here denoted by α), this contradicts the ellipticity of P !

In a recent joint work, David M. J. Calderbank (University of Edinburgh), P. Gauduchon (Ecole polytechnique) and the author proved that there is indeed a refined Kato inequality for each section in the kernel of a natural first-order overdetermined elliptic operator and almost completed the task of computing the best constant in each case. These results appeared in [6].

The goal of this short text is to describe the main results of this work as well as a few ideas on their proofs. Another genuinely different method of computing the constants was provided at the same time by T. Branson and appears in [5]. Readers interested may find more detailed information in the original articles as well as in the survey paper [7].

2. The results

Our main goal is to prove existence of a refined inequality and to give a precise value of the best constant for each section in the kernel of a natural first-order overdetermined elliptic operator. This is only feasible for operators which reflect intimately the local geometry and are natural in a strong sense.

Definition 1 *A natural bundle on a Riemannian manifold (M^n, g) is a vector bundle obtained either from the direct orthonormal frame bundle of M with an* irreducible *linear representation $\lambda : SO(n) \longrightarrow End(V)$ or, in case M is spin, from the spinor frame bundle with an* irreducible *linear representation $\lambda : Spin(n) \longrightarrow End(V)$.*

In either case, if G is the group $SO(n)$ or $Spin(n)$ and $\mathcal{P}$ is the corresponding frame bundle, the vector bundle is

$$E \;=\; \mathcal{P} \times_\lambda V \;=\; (\mathcal{P} \times V)\,/\sim \tag{11}$$

where $\sim$ is the equivalence relation $(p, v) \sim (pg, \lambda(g^{-1})v)$ on $\mathcal{P} \times V$. For instance, the tangent bundle is issued from the standard representation $\tau : SO(n) \subset GL(\mathbb{R}^n)$.

Let ∇ a metric connection on E. It then acts by sending sections of E onto sections of the bundle $T^*M \otimes E$ which is a natural bundle asociated to the representation $\tau \otimes \lambda$ on $\mathbb{R}^n \otimes V$.

Definition 2 *Let L be a G-equivariant linear endomorphism of $\mathbb{R}^n \otimes V$. It induces a linear (fibrewise) map, still denoted by L, from $T^*M \otimes V$ to the bundle issued from the image of L, denoted by F. The natural first-order operator associated to L is then the operator*

$$P = L \circ \nabla \tag{12}$$

sending sections of E to sections of F.

We now let $T^*M\otimes E = \otimes_{i=1}^N F_i$ be the splitting induced from the decomposition of $\mathbb{R}^n \otimes V$ into irreducible components W_i. Hence each equivariant map L decomposes as $L = \sum_{i\in I} a_i \Pi_i$ where each Π_i is the projection on the irreducible summand W_i corresponding to F_i, a_i is a (real or complex) number and I is the set of indices of non-zero coefficients $\{a_i\}$.

Example. One may forget the geometric nature of the base manifold under consideration and look at an open set Ω in euclidean space $\mathbb{R}^n$ with a smooth (varying with x) metric $g_{ij}(x)$. One then considers functions on Ω with values in any vector space V that is an irreducible representation of $SO(n)$ (*e.g.* forms, or vectors, or tracefree symmetric bilinear forms, or p-vectors, etc...) Whereas this opens up a lot of possibilities, the operators under consideration are constrained. The choice of $g_{ij}(x)$ yields a smooth family of subgroups G_x (each of them isomorphic to $SO(n)$) of $GL(n)$ that varies with the point. The admissible operators are those of *geometric content*, hence those given by the Levi-Civita connection of g_{ij} followed with some linear map $L_x : \mathbb{R}^n \otimes V \longrightarrow \mathbb{R}^n \otimes V$ which varies with x and commutes pointwise with the action of G_x at each point.

If one chooses linear 1-forms ($V = \mathbb{R}^n$), for instance, then the number of irreducible summands is $N = 3$. The possible (*i.e.* natural) operators are of the type $P = a_S S + a_d d + a_\delta \delta$. Letting Γ_{ij}^k be the Christoffel symbols of the metric g_{ij}, the elementary operators S, d and δ are the tracefree symmetrized covariant derivative:

$$(S\xi)_{ij} = \partial_i\xi_j + \partial_j\xi_i - 2\Gamma_{ij}^k\xi_k - \frac{2}{n}\sum_{k,\ell} g^{k\ell}\left(\partial_k\xi_\ell - 2\sum_m \Gamma_{k\ell}^m\xi_m\right) g_{ij}\ , \tag{13}$$

the exterior differentiation of forms:

$$(d\xi)_{ij} = \partial_i\xi_j - \partial_j\xi_i\ , \tag{14}$$

and (some suspension of) the divergence:

$$(\delta\xi)_{ij} = \sum_{k,\ell} g^{k\ell}\left(\partial_k\xi_\ell - 2\sum_m \Gamma_{k\ell}^m\xi_m\right) g_{ij}. \tag{15}$$

We will denote by Π_S, Π_d and Π_δ the associated projections and we shall come back to this example later on.

Definition 3 *The operator P above is overdetermined elliptic if and only if L is non-vanishing on each simple tensor, i.e.*

$$L(\alpha\otimes v) = 0 \Longrightarrow \alpha\otimes v = 0. \tag{16}$$

Note that the map $L(\alpha\otimes\cdot)$ is the principal symbol $\sigma_\alpha(P)$ of the operator.

Since each projection ends in a different summand, ellipticity only depends on the subset $I \subset \{1,\dots,N\}$. Hence we may (and we will) forget the coefficients $\{a_i\}$ and speak of *the operator defined by the subset I*, denoted by P_I.

Example (continued). In our situation above, if one excludes the trivial operator $S + d + \delta$, there are four overdetermined elliptic operators: S, $S + d$, $S + \delta$, $d + \delta$. This may be easily checked by looking at the symbols.

Our results will be expressed in terms of the conformal weights $w_i = w(E, W_i)$ of the summands W_i. These are real numbers attached to pairs (V, W_i) consisting of an irreducible representation V and an irreducible component W_i of $\mathbb{R}^N \otimes V$. They are computed from representation theoretic data of the Lie algebra $\mathfrak{so}(n)$. Each weight can be computed very easily from the knowledge of the pair (V, W_i) and may be found in standard textbooks on representation theory such as [8], hence our choice to express the results with respect to them. For the purposes of this text, we only notice that conformal weights corresponding to the same V but two different summands W_i are always distinct, except if the two summands are exchanged by a change of orientation; in that case we consider these as only one summand. We may now order the family of summands

$$T^*M \otimes E = \oplus_{i=1}^N F_i \tag{17}$$

by decreasing conformal weights: $w_1 > \cdots > w_N$.

Example (continued). The images of the projections Π_S, Π_d and Π_δ of the example above were already given in right order. One then has $w_s > w_d > w_\delta$.

To state our main result, it is necessary to know which are the (overdetermined) elliptic operators among all natural first-order operators. Our definition of ellipticity implies that if P_I is elliptic and $I \subset J$, then P_J is elliptic, since each projection ends in a different summand. Hence there exists a set of *minimal elliptic* operators.

These were completely classified by T. Branson in [4]. It turns out that for a given bundle E, the list of minimal elliptic operators acting on E depends only on the order of the conformal weights, except in some exceptional case where some extra operator has to be added. The exceptional case will cause us problems since there are not enough non-elliptic operators. For sake of simplicity, we will forget here the exceptional situation and argue only for the general case. The interested reader will find information on the exceptional case, its occurrences and its consequences in [6, 7].

As there exists a set of minimal elliptic operators, there exists a set of maximal non-elliptic operators, *i.e* the set of operators P_I which are non-elliptic and I has a maximal number of elements. ¿From [4], we may state

Lemma 4 *In the general case, the set $\mathcal{N}$ of I's corresponding to maximal non-elliptic operators is the set whose elements are obtained by choosing exactly one index in each of the sets $\{j, N+2-j\}$ for each j with $2 \leq j \leq \nu$ if $N = 2\nu$ (giving $2^{\nu-1}$ elements) and for each j with $2 \leq j \leq \nu+1$ if $N = 2\nu+1$ (giving 2^ν elements).*

In the exceptional case, we also argue with the set above but it unfortunately contains exactly one elliptic operator. This will be the cause for some non-sharp computations below.

Example (continued). In the case of 1-forms studied above, the minimal elliptic operators are obviously the operators S and $d+\delta$. Moreover, it can be checked that the exceptional case cannot appear (this involves some elementary representation theory).

For sake of simplicity in the formulas, we denote by $\hat{I}$ the complement of I in $\{1,\dots,N\}$ and we let $\widetilde{w}_i = w_i + \frac{n-2}{2}$ and $\varepsilon_i(J)$ be 0 if i belongs to the subset J of $\{1,\dots,N\}$, 1 if not.

Main Theorem *Let I a subset of $\{1,\dots,N\}$ corresponding to an elliptic operator P_I acting on E. Then a refined Kato inequality $\big|d|\xi|\big| \leq k_I|\nabla\xi|$ holds for any section ξ in the kernel of P_I, outside the zero set of ξ.*

If N is odd, then

$$k_I^2 = 1 - \inf_{J\in\mathcal{N}} \left(\sum_{i\in I} \frac{\prod_{j\in J\setminus\{i\}}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j\in \hat{J}\setminus\{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \varepsilon_i(J) \right). \tag{18}$$

These results are sharp except if n is odd where the exceptional case may *sometimes appear.*

If N is even, then

$$k_I^2 = 1 - \inf_{J\in\mathcal{N}} \left(\sum_{i\in I} \left(\widetilde{w}_i - \frac{1}{2}\right) \frac{\prod_{j\in J\setminus\{i\}}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j\in \hat{J}\setminus\{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \varepsilon_i(J) \right). \tag{19}$$

This result is always sharp.

Example (continued). For the case already described above of natural operators acting on 1-forms, the weights are as follows : $w_S = 1$, $w_d = -1$ and $w_\delta = 1-n$. The constants one finds with this procedure are

(i) $k^2 = \frac{1}{2}$ if $S\xi = 0$ (forms dual to conformal vector fields) or $(S+\delta)\xi = 0$ (forms dual to Killing fields);

(ii) $k^2 = \frac{n-1}{n}$ if $(d+\delta)\xi = 0$ (harmonic forms);

(iii) $k^2 = \frac{1}{n}$ if $(S+d)\xi = 0$.

3. The proofs

3.1. Starting point

We shall obtain refined Kato inequalities from purely algebraic refined Schwarz inequalities of the form

$$\frac{|\langle\Phi, v\rangle|}{|v|} \leq k|\Phi|, \tag{20}$$

where $\Phi \in \mathbb{R}^n \otimes V$ and $v \in V$.

For $k = 1$, this holds for any Φ and nonzero v, with equality if $\Phi = \alpha \otimes v$ for some $\alpha \in \mathbb{R}^n$. Recall that the classical Kato inequality (1) is obtained from this by lifting it to the associated bundles and putting $v = \xi$, $\Phi = \nabla\xi$ for a section ξ of E. If ξ lies in the kernel of the operator P_I then $\Phi = \nabla\xi$ is an element of $\ker \Pi_I = W_{\widehat{I}}$ (recall $\widehat{I}$ is the complement of I in $\{1, \ldots, N\}$ hence $W_{\widehat{I}}$ denotes the image of $\Pi_{\widehat{I}}$). Thus to obtain a Kato inequality for the operator P_I, we only need an estimate of the form (20) for $\Phi \in W_{\widehat{I}}$ and $v \in V$.

The supremum, over all nonzero v, of the left hand side of (20) is the *operator norm* $|\Phi|_{op}$ of $\langle \Phi, . \rangle$, viewed as a linear map from V to $\mathbb{R}^n$. Now observe that for any $\Phi \in W_{\widehat{I}}$, we have:

$$\begin{aligned} |\Phi|_{op} &= \sup_{|v|=1} |\langle \Phi, v\rangle| = \sup_{|\alpha|=|v|=1} |\langle \Phi, \alpha\otimes v\rangle| = \sup_{|\alpha|=|v|=1} |\langle \Phi, \Pi_{\widehat{I}}(\alpha \otimes v)\rangle \\ &\leq \left(\sup_{|\alpha|=|v|=1} |\Pi_{\widehat{I}}(\alpha \otimes v)|\right) |\Phi|. \end{aligned}$$

This gives a refined Schwarz inequality with $k = \sup_{|\alpha|=|v|=1} |\Pi_{\widehat{I}}(\alpha \otimes v)|$:

$$\begin{aligned} \frac{|\langle \Phi, v\rangle|}{|v|} &= \frac{|\langle \Phi, \alpha_0 \otimes v\rangle|}{|v|} = \frac{|\langle \Phi, \Pi_{\widehat{I}}(\alpha_0 \otimes v)\rangle|}{|v|} \\ &\leq \frac{|\Pi_{\widehat{I}}(\alpha_0 \otimes v)|}{|v|} |\Phi| \leq \left(\sup_{|\alpha|=|v|=1} |\Pi_{\widehat{I}}(\alpha \otimes v)|\right) |\Phi|, \end{aligned}$$

where α_0 is any unit 1-form such that $\langle \Phi, v\rangle = c\,\alpha_0$ for some $c \in \mathbb{R}$. We have then proved:

Lemma 5 *For any section ξ on the kernel of P_I, and at any point where ξ does not vanish, we have:*

$$\big|d|\xi|\big| \leq k_I |\nabla\xi|,$$

where the constant k_I is defined by

$$k_I = \sup_{|\alpha|=|v|=1} |\Pi_{\widehat{I}}(\alpha \otimes v)| = \sqrt{1 - \inf_{|\alpha|=|v|=1} |\Pi_I(\alpha \otimes v)|^2}.$$

Furthermore equality holds at a point if and only if $\nabla\xi = \Pi_{\widehat{I}}(\alpha \otimes \xi)$ for a 1-form α at that point such that $|\Pi_{\widehat{I}}(\alpha \otimes \xi)| = k_I |\alpha \otimes \xi|$.

Lemma 5 is sharp. Equality in the refined Kato inequality is attained by a suitable affine section of E on flat euclidean space. We shall try below to give an explicit expression of the constant k_I. Sharpness is then sometimes lost in these computations when the exceptional case occurs.

3.2. Some hints on the proof

The proof of the main result above is rather technical. We shall try below to give a very rough idea of the strategy we used and the techniques employed. A more detailed version appears in [7] and full information is of course given in the original paper [6].

The first step of our minimization procedure is to use the classical Lagrange interpolation process. We let B be the operator on $\mathbb{R}^n \otimes V$ whose eigenvalue on each W_i is exactly the conformal weight w_i and $\widetilde{B} = B + \frac{n-2}{2} Id$ with eigenvalues $\{\widetilde{w}_i\}$. We may write each projection Π_j as

$$\Pi_j = \prod_{k \neq j} \frac{\widetilde{B} - \widetilde{w}_k Id}{\widetilde{w}_j - \widetilde{w}_k} = \frac{\sum_{k=0}^{N} \widetilde{w}_j^{N-1-k} \left(\sum_{\ell=0}^{k} (-1)^\ell \sigma_\ell(w) \widetilde{B}^{k-\ell} \right)}{\prod_{k \neq j} (\widetilde{w}_j - \widetilde{w}_k)}, \tag{21}$$

where $\sigma_i(w)$ denotes the i-th elementary symmetric function of the modified weights (as it will appear below, it is much easier to work with the modified rather than the original weights).

The next step consists in converting this expression into a more practical form. For this we let $\widetilde{A}_k = \sum_{\ell=0}^{k} (-1)^\ell \sigma_\ell(w) \widetilde{B}^{k-\ell}$, and $Q_k = \langle \widetilde{A}_k(\alpha \otimes v), \alpha \otimes v \rangle$ so that

$$|\Pi_j(\alpha \otimes v)|^2 = \left| \langle \Pi_j(\alpha \otimes v), \alpha \otimes v \rangle \right| = \frac{\sum_{k=0}^{N} \widetilde{w}_j^{N-1-k} Q_k}{\prod_{k \neq j} (\widetilde{w}_j - \widetilde{w}_k)}. \tag{22}$$

This reduces the problem (of estimating $\inf |\Pi_I(\alpha \otimes v)|^2$ for I a subset corresponding to an elliptic operator) to minimizing an affine function in the variables $\{Q_k\}$ over the *admissible region* in the affine space: the region consisting in points of coordinates $\{Q_k\}$ such that there exists indeed α and v unit such that $Q_k = \langle \widetilde{A}_k(\alpha \otimes v), \alpha \otimes v \rangle$ for each k.

The task that remains is then to get a better grasp on the admissible region. It turns out that approximately half of the $Q_k's$ can be eliminated. The reason for this is that the operator B above plays indeed a very important part both in *conformal geometry* (hence the name of its eigenvalues, see [9]) and in the representation theory of the special orthogonal groups (or Lie algebras). A quite involved analysis of its properties finally leads to:

Lemma 6 *If N is odd, then $Q_{2j+1} = 0$ for every j. If N is even, then $2\,Q_{2j+1} + Q_{2j} = 0$ for every $j \geq 1$.*

This lemma stands as the main reason for using the modified conformal weights rather than the original weights.

The last step is done as follows: our goal is to find the minimum value of the affine function $\sum_{i \in I} |\Pi_j(\alpha \otimes v)|^2$ in the remaining coordinates $\{Q_k\}$ over the *admissible* region. As each $|\Pi_j(\alpha \otimes v)|^2$ is nonnegative, the admissible set is contained in a compact convex polyhedron in the Q-space, and its vertices can be shown to be admissible points in one-to-one correspondence with the set $\mathcal{N}$ of maximal non-elliptic operators. We then choose to minimize the affine function on this (larger) polyhedral region. The infimum is certainly achieved on the vertices and the output of the whole procedure is our Main Theorem.

Although it would seem the contrary, careful examination of the above proof shows that sharpness is not lost when minimizing on the (larger) polyhedral region rather than on the admissible one, unless the exceptional case appears. This follows from the identification of the vertices of the polyhedron. In the general case, vertices are given by solutions (in the affine space of coordinates $\{Q_k\}$) of the system of equations defined by

$$|\Pi(\alpha \otimes v)|^2 = 0 \quad \forall j \in J \tag{23}$$

where J corresponds to a *maximal non-elliptic* operator. In the exceptional case, the above equations yield all vertices *but one*, which is given by equations with a set J corresponding to an elliptic operator. Hence, in the general case, the infimum on the polyhedral region of the affine function we sought to minimize is achieved at a vertex and since each vertex corresponds to a maximal non elliptic set J, *there exists indeed* α and v unit such that the system of equations (23) above holds. Thus, the vertex is an admissible point. In the exceptional case, it may happen that the infimum is achieved at the vertex corresponding to the set J which is elliptic, hence which is not admissible and the result is not sharp.

Example (the end). We shall try here to detail the arguments above in the special case of 1-forms, where the number of components is $N = 3$. For such a small N, it is possible to get the results faster than by following step by step our previous arguments, but we aim here at giving a more concrete view of our method. Introducing the notation $\pi_. = |\Pi_.(\alpha \otimes v)|^2$ for squared norms of projections, Lagrange interpolation for $\widetilde{B}$ may be rewritten as

$$\begin{aligned} \langle \alpha \otimes v, \alpha \otimes v \rangle &= \pi_S + \pi_d + \pi_\delta, \\ \langle \widetilde{B}(\alpha \otimes v), \alpha \otimes v \rangle &= \widetilde{w}_S\, \pi_S + \widetilde{w}_d\, \pi_d + \widetilde{w}_\delta\, \pi_\delta, \\ \langle \widetilde{B}^2(\alpha \otimes v), \alpha \otimes v \rangle &= \widetilde{w}_S^2\, \pi_S + \widetilde{w}_d^2\, \pi_d + \widetilde{w}_\delta^2\, \pi_\delta. \end{aligned} \tag{24}$$

which give in turn (introducing the Q_i's)

$$\begin{aligned} Q_0 &= \pi_S + \pi_d + \pi_\delta, \\ Q_1 - \sigma_1(w) &= \widetilde{w}_S\, \pi_S + \widetilde{w}_d\, \pi_d + \widetilde{w}_\delta\, \pi_\delta, \\ Q_2 - \sigma_1(w)\, Q_1 + \sigma_1(w)^2 - \sigma_2(w) &= \widetilde{w}_S^2\, \pi_S + \widetilde{w}_d^2\, \pi_d + \widetilde{w}_\delta^2\, \pi_\delta. \end{aligned} \tag{25}$$

Since we know that $Q_0 = 1$, $Q_1 = 0$, one may use the first two equations to express two among the squared norms $\pi_.$ in terms of the third and the weights. As we know P_d and P_δ are maximal non-elliptic, we choose to do this procedure twice. We first express everything with π_δ. Plugging this information into the third equation, we end up with an equation of the form

$$Q_2 = f(w) + g(w)\, \pi_\delta, \tag{26}$$

where f and g are rational functions of the weights. Running the same procedure with π_d instead of π_δ gives another equation

$$Q_2 = k(w) + \ell(w)\,\pi_d \tag{27}$$

with other rational functions k and ℓ. The polyhedron defined by nonnegativity of the squared norms $\pi_.$ here reduces to a bounded and closed interval in Q_2 defined by

$$\pi_d \geq 0, \quad \pi_\delta \geq 0 \tag{28}$$

(it is easy to show that the information $\pi_S \geq 0$ is useless here) and this gives an explicit view of the correspondance between vertices of the polyhedron and the maximal non-elliptic operators P_d and P_δ.

We may now look at the refined Kato constants. Select $I \subset \{1, 2, 3\}$ corresponding to an elliptic operator. Then, using the equations (25) above, one may express $\sum_{i \in I} \pi_i$ as a function of Q_2 and the weights only. The desired constant is then obtained by looking at the minimum value of this function in the range defined by (28) hence by looking at the values at both ends of the interval.

3.3. Final remarks: explicit computations

In a number of cases, including for example almost all minimal elliptic operators and the most common operators used in Riemannian Geometry, the result of our Main Theorem can be made completely explicit. As the discussion above shows, the results are sharp, and even when the exceptional case appears, the results are sharp if one is able to show that the sought infimum is not achieved at the exceptional vertex. This explains why it is possible to give optimal constants for a large number of operators.

The interested reader is referred to [6] where explicit constants are given for the case of small number of components N (this covers the most commonly used situations in geometry) and a detailed study of small dimensions.

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