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Propagation of analyticity of solutions to the Cauchy problem for Kirchhoff type equations

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-Dedicated to Kiyoshi Mochizuki on his 60th birthday-

Abstract

We shall give the local in time existence of the solutions in Gevrey classes to the Cauchy problem for Kirhhoff equations of p-Laplacian type and investigate the propagation of analyticity of solutions for real analytic deta. When p=2, his equation as the global real analytic solution for the real analytic initial data.

1. Introduction

In [6] we have obtained the local existence theorem in Gevrey class of solutions to the Cauchy problem for Kirchhoff equations of p-Laplacian type

$$\partial_t^2 u(t,x) - M(||\nabla_x u(t)||^p_{L^p(R^n)})\Delta_p u(t,x) = f(t,x), t \in (0,T), x \in R^n, \tag{1.1}$$

where Δ_p is defined by

$$\Delta_p u = \nabla_x(|\nabla_x u|^{p-2}\nabla_x u). \tag{1.2}$$

In this note we shall investigate the popagation of analyticity of solutions of the following Cauchy problem which is a generalization of (1.1),

$$\partial_t^2 u(t,x) - M((Au,u)_{L^2})A[u] = f(t,x), \quad t \in (0,T), \quad x \in R^n, \tag{1.3}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in R^n, \tag{1.4}$$

where

$$A[u] = \sum_{j,k=1}^n \partial_{x_j}\{a_{jk}(x, \nabla_x u(t))\partial_{x_k} u(t,x)\} \tag{1.5}$$

and M is a non negative two times continuous differentiable function defined in $[0,\infty)$ and $[a_{jk}(x,y)]$ is a non negative symmetric matrix of which elements are real valued functions defined in $R^n \times R^n$ and satisfies

$$\sum_{j,k=1}^n \{a_{jk}(x,y)\xi_k\xi_j + \sum_{l=1}^n \partial_{y_l} a_{jk}(x,y)\xi_j\xi_l y_k\} \geq 0 \tag{1.6}$$

for $x, \xi \in R^n, y \in R^n$. This assumption (1.6) assures that the equation (1.3) is hyperbolic. We can see easily that p-Laplacian Δ_p is verifies with the condition (1.6). Moreover we assume that the coefficients $\{a_{jk}\}$ satisfy that for a compact set K in R^n there are $C_K, \rho_K > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta a_{jk}(x.y)| \leq C_K \rho_K^{|\alpha|+|\beta|} |\alpha|!^s |\beta|! \tag{1.7}$$

for $x \in R^n, y \in K$. We introduce a functional space as follows,

$$H^l_{\rho,d} = \{u \in L^2(R^n); \langle\xi\rangle_h^l e^{\rho\langle\xi\rangle_h^{1/d}} \hat{u}(\xi) \in L^2\}, \tag{1.8}$$

where $\hat{u}$ stands for a Fourier transform of u and $l \in R^1, d \geq 1, \langle\xi\rangle_h = \sqrt{h^2 + |\xi|^2}$ and $\rho \geq 0$ and denote by $H^l = H^l_{0,d}$ and $L^2_{\rho,d} = H^0_{\rho,d}$. Now we begin to state a local existence theorem for the Cauchy problem (1.3)-(1.4).

Theorem 1.1 *Assume that $M(\eta) \geq 0$ is in $C^2([0,\infty))$, (1.6) is valid, a_{jk} satisfy (1.7) and there is $\rho_0 > 0$ such that the initial data u_0, u_1 belong to $L^2_{\rho_0,d}$ and $f \in C^0([0,T_0]; L^2_{\rho_0,d})$. Then if $s \leq d < 2$ there are $T_0 > T > 0$ and $\rho_1 > 0$ such that the Cauchy problem (1.3) and (1.4) has a unique solution u in $C^2([0,T]; L^2_{\rho_1,d})$.*

It is well known that when $p = 2$ and $d = 1$, the Cauchy problem (1.1) and (1.4) has a global real analytic solution. For example see the article of Bernstein [2] and of Pohozaev [9]. When $M(\eta) \equiv 1$, Theorem 1,1 is proved in [3]. In [7] we proved that Thorem 1,1 holds for (1.1) and (1.4) under the assumption $s \leq d < 4/3$. Moreover we can investigate the propagation of analyticity of solutions (1.3) and (1.4).

Theorem 1.2 *Let $T > 0$ a positive number. Assume $M(\eta) \geq 0$ is in $C^2([0,\infty))$, (1.6) and (1.7) with $s = 1$ are valid and there are $\rho_1 > 0, 1 < d < 2$ such that a function $u \in C^2([0,T]; L^2_{\rho_1,d})$ satisfies (1.3) and (1.4). Then if there is $\rho_0 > 0$ such that the initial data u_0, u_1 belong to $L^2_{\rho_0,1}$ and $f \in C^0([0,T_0]; L^2_{\rho_0,1})$, we have $\rho_2 > 0$ such that u belongs to $C^2([0,T]; L^2_{\rho_2,1})$, that is, $u(t,x)$ is real analytic in a space variable x for any $t \in [0,T]$.*

2. A priori estimates for linearized equations

We shall derive a priori estimate in Gevrey class $H^m_{\rho,d}(R^n)$ for the linearized second order hyperbolic equations of equation (1.3). For $m, \rho, \delta \in R (0 \leq \delta \leq \rho \leq 1)$ denote by $S^m_{\rho,\delta}$ the usual symbol class of oder m of ρ, δ type. For simplicity we write $S^m = S^m_{1,0}$ and introduce the seminorms as follows,

$$|a|_\ell^{(m)} = \sup_{|\alpha+\beta| \leq \ell, x, \xi \in R^n} \frac{|a^{(\alpha)}_{(\beta)}(x,\xi)|}{\langle\xi\rangle_h^{m-|\alpha|}} < \infty, \tag{2.1}$$

where $\langle\xi\rangle_h = (h^2 + |\xi|^2)^{\frac{1}{2}}$ ($h > 0$ is a large parameter). For $d \geq 1, \rho > 0$ and $m \in R$ we define the symbols of Gevrey class $\gamma^d_\rho S^m$. We say that a symbol $a \in S^m$ belongs to $\gamma^d_\rho S^m$, if a satisfies

$$|a|^{(m)}_{\rho,d.\ell} = \sup_{x,\xi \in R^n, |\alpha+\beta| \leq \ell, \delta, \gamma \in N^n} \frac{|a^{(\alpha+\delta)}_{(\beta+\gamma)}(x,\xi)| \rho^{|\delta+\gamma|}}{\langle\xi\rangle_h^{m-|\alpha+\delta|} |\delta+\gamma|!^d} < \infty. \tag{2.2}$$

Define for $\rho \in R$

$$e^{\rho<D>_h^{1/d}} u(x) = (2\pi)^{-n} \int e^{ix\xi + \rho\langle\xi\rangle_h^{1/d}} \hat{u}(\xi) d\xi,$$

for $u \in H^m_{\rho,d}$. Then $e^{\rho<D>_h^{1/d}}$ maps from $H^m_{\rho_1,d}$ to $H^m_{\rho_1-\rho,d}$ continuously. Moreover for $a \in \gamma^d_{\rho_0} S^{m_1}$ we can see that $a(x,D)$ maps from $H^m_{\rho,d}$ to $H^{m+m_1}_{\rho,d}$ continuously if $|\rho| \leq (24n^\kappa)^{-1}\rho_0^\kappa, \kappa = 1/d$ and consequently

$$a(\rho, x, D) = e^{-\rho\langle D\rangle_h^\kappa} a(x,D) e^{\rho\langle D\rangle_h^\kappa} \tag{2.3}$$

maps H^m to H^{m+m_1} continuously, where $H^m = \{u \in \mathcal{S}'(R^n); \langle\xi\rangle_h^m \hat{u}(\xi) \in L^2(R^n)\}$ is the usual Sobolev space. Moreover we can prove the following Lemma.

Lemma 2.1 *Let $d > 1, \rho \in R, \kappa = 1/d$ and $a \in \gamma^d_{\rho_0} S^m$. Then the symbol of $a(\rho, x, D)$ given by (2.3) belongs to S^m and satisfies for any integer $N \geq 0$,*

$$a(\rho, x, \xi) = \sum_{|\beta|<N} \beta!^{-1} a_{(\beta)}(x,\xi)\omega_\beta(\xi) + \rho r_N(a)(\rho, x, \xi), \tag{2.4}$$

where $\omega_\beta(\xi) \in S^{-|\beta|(1-\kappa)}$ are given by

$$\omega_\beta(\xi) = e^{-\rho\langle\xi\rangle_h^\kappa} \partial_\xi^\beta e^{\rho\langle\xi\rangle_h^\kappa} \tag{2.5}$$

and the remainder term $r_N(a)(\rho, x, \xi)$ belongs to $S^{m-N(1-1/d)}$ and moreover there are C_N independent of such that for $h \geq 1$

$$|r_N(a)|_l^{(m-N(1-1/d))} \leq C_N |a|_{\rho_0, d, N+M_1(l)}, \tag{2.6}$$

where

$$|\rho| \leq 24^{-1} n^{-\kappa} \rho_0^\kappa, \quad M(l); = [\ell(1-\kappa)^{-1}] + 2\ell + [n/2]. \tag{2.7}$$

The proof of this lemma is given in Lemma 1.2 of [3]. We can prove the following lemma by use of the above lemma.

Lemma 2.2 *(i) Let $a \in S^0$. Then there is a positive integer l_0 such that*

$$||a(x,D)u||_0 \leq C_0 |a|_{l_0}^{(0)} ||u||_0$$

for $u \in L_2$.
(ii) Let $a \in \gamma^d_{\rho_0} S^m$ and $\Lambda = \rho\langle\xi\rangle_h^\kappa$. Then if $|\rho| \leq (24^d n)^{-\kappa} \rho_0^\kappa$,

$$||e^\Lambda a(x,D)u||_s \leq C_0 |a|_{\gamma^d_{\rho_0} S^m, l_1} ||u||_{s+m}$$

for $u \in H^{s+m}$, where $l_1 = |s| + M(l_0)$.

We shall derive a priori estimates for a linear equation below

$$L[u] = \partial_t^2 u + a(t,x,D)u + b(t,x,D)u = f(t,x). \tag{2.8}$$

Assume that the principal part $a(t,x,\xi)$ and $b(t,x,\xi)$ satisfy

$$a(t,x,\xi) \geq 0, \quad (t,x,\xi) \in [0,T_0] \times R^{2n}. \tag{2.9}$$

$$a(t,x,\xi) \in C^2([0,T_0]; \gamma^d_{\rho_0} S^2). \tag{2.10}$$

$$b(t,x,\xi) \in C^0([0,T_0]; \gamma^d_{\rho_0} S^1). \tag{2.11}$$

Then the non-negativity of a implies

$$|a_t(t,x,\xi)| + |\nabla_x a(t,x,\xi)| + |\nabla_\xi a(t,x,\xi)| \leq C|a|_{C^2([0,T_0]\times R^{2n})} a(t,x,\xi), \tag{2.12}$$

for $(t,x,\xi) \in [0,T_0] \times R^{2n}$. Put

$$v(t,x) = e^{\Lambda(t,D)} u(t,x) \tag{2.13}$$

where $\Lambda(t,\xi) = \rho(t)\langle\xi\rangle_h^\kappa$. Then v satisfies

$$L_\Lambda[v] = (\partial_t - \Lambda_t)^2 v + a_\Lambda(t,x,D)v + b_\Lambda(t,x,D)v = e^\Lambda f(t,x), \tag{2.14}$$

$$v(0) = v_0, \quad (\partial_t - \Lambda_t)v(0) = v_1, \tag{2.15}$$

where for an operator a we denote by a_Λ the product $e^\Lambda a e^{-\Lambda}$. Introduce an energy of v below

$$e(t)^2 = \frac{1}{2}\{||(\partial_t - \Lambda_t)v(t)||_s^2 + \Re(a(t,x,D)v, v)_s + (\langle D\rangle_h^\kappa v, \langle D\rangle_h^\kappa v)_s\}, \tag{2.16}$$

where $\kappa = 1/d$, $||\cdot||_s$ and $(\cdot,\cdot)_s$ mean a norm and an inner product of Sobolev space H^s respectively.

Proposition 2.1 *Assume (2.9),(2.10) and (2.11) are valid and* $v \in \bigcap_{j=0}^2 C^2([0,T_0]; H^{s-j})$. *For any* $s \in R$ *there are a positive constant* C_s *,*h_s *and a positive function* $\rho(t) \in C^2([0,T_0]$ *such that*

$$e_s(t) \leq e^{c_s t}\{e_s(0) + \int_0^t ||L_\Lambda u(\tau)||_s d\tau\} \tag{2.17}$$

$$\int_0^t ||(\partial_t - \Lambda_t)v(\tau)||_\kappa^2 + ||v(\tau)||_{s+2\kappa}^2 d\tau \leq e^{c_s t}\{e(0)^2 + \int_0^t ||L_\Lambda v(\tau)||_s^2 d\tau\} \tag{2.18}$$

for $t \in [0,T_0]$ *and* $h \geq h_s$.

Proof. Differentiating $e_s(t)^2$ in t, we have

$$\begin{aligned}
2e_s(t)e_s'(t) &= \Re((\partial_t - \Lambda_t)^2 v, (\partial_t - \Lambda_t)v)_s + (\Lambda_t(\partial_t - \Lambda_t)v, (\partial_t - \Lambda_t)v)_s \\
&\quad + \frac{1}{2}\Re(a_t(t)v, v)_s + ((a(t) + a^*(t))(\partial_t - \Lambda_t)v, v)_s \\
&\quad + ((a(t) + a^*(t))\Lambda_t v, v)_s + (\partial_t - \Lambda_t)v, v)_{s+\kappa} + (\Lambda_t v, v)_{s+\kappa} \\
&= \Re(-(a_\Lambda(t) + b_\Lambda)v + L_\Lambda[v] + \Re a(t)v, (\partial_t - \Lambda_t)v)_s + (\Lambda_t(\partial_t - \Lambda_t)v, (\partial_t - \Lambda_t)v)_s \\
&\quad + \frac{1}{2}\Re(a_t(t)v, v)_s + ((a(t) + a^*(t))(\partial_t - \Lambda_t)v, v)_s
\end{aligned}$$

$$+((\Lambda_t(a(t)+a^*(t))\Lambda_t v, v)_s + (\partial_t - \Lambda_t)v, v)_{s+\kappa} + (\Lambda_t v, v)_{s+\kappa}. \tag{2.19}$$

Noting that from Lemma 1.1 we have

$$\sigma(-a_\Lambda(t) + \Re a(t))(x,\xi) = -\rho(t)a_1(t,x,\xi) - r(t,x,\xi), \tag{2.20}$$

where $r \in S^{2\kappa}$ and

$$a_1(t,x,\xi) = \sum_{|\alpha|=1} a_{(\alpha)}(t,x,\xi)\omega_\alpha(\xi), \tag{2.21}$$

we can estimate

$$\begin{aligned}
&|\Re(-(a_\Lambda(t)+b_\Lambda)v + L_\Lambda[v] + \Re a(t)v, (\partial_t - \Lambda_t)v)_s| \\
&\le 2|||\Lambda_t|^{-\frac{1}{2}}(-\rho(t)a_1(t,x,D) - r(t,x,D) + b_\Lambda)v + L_\Lambda v||_s^2 \\
&\quad + \frac{1}{8}|||\Lambda_t|^{\frac{1}{2}}(\partial_t - \Lambda_t)v||_s^2.
\end{aligned} \tag{2.22}$$

Since $r \in S^{2\kappa}$, and $\Lambda_t = \rho_t(t)\langle\xi\rangle_h^\kappa$, we can see

$$|||\Lambda_t|^{-\frac{1}{2}}(-r(t,x,D)v||_s \le \frac{C_s}{|\rho_t|}||\Lambda_t|^{\frac{1}{2}}v||_{s+\kappa} \tag{2.23}$$

Since $b_\Lambda \in S^1$, we get

$$|||\Lambda_t|^{-\frac{1}{2}}b_\Lambda v||_s \le \frac{C_s h^{1-2\kappa}}{|\rho_a t|}|||\Lambda_t|^{\frac{1}{2}}v||_{s+\kappa}. \tag{2.24}$$

Moreover we see

$$||(\rho(t)|\Lambda_t|^{-\frac{1}{2}}a_1(t,x,D)v||_s^2 = ((\rho(t)^2 a_1(t,x,D)^*|\Lambda_t|^{-1}a_1(t,x,D)v, v)_s, \tag{2.25}$$

and

$$\sigma(\rho(t)^2 a_1(t,x,D)^*|\Lambda_t|^{-1}a_1(t,x,D))(x,\xi) = \frac{\rho(t)^2\langle\xi\rangle_h^{-\kappa}}{|\rho_t(t)|}(|a_1(t,\xi)|^2 + r(t,x,\xi)) \tag{2.26}$$

where $r \in C^0([0,T_0];S^1)$. From (2,12) and (2.21) we have

$$\frac{\rho(t)^2\langle\xi\rangle_h^{-\kappa}}{|\rho_t(t)|}|a_1(t,\xi)|^2 \le C\frac{\rho(t)^2}{|\rho_t(t)|}a(t,x,\xi)\langle\xi\rangle_h^\kappa = C\frac{\rho(t)^2}{|\rho_t(t)|^2}|\Lambda_t|a(t,x,\xi). \tag{2.27}$$

If we choose $\rho(t)$ such that

$$C\frac{\rho(t)^2}{\rho_t(t)^2} \le \frac{1}{2}, \tag{2.28}$$

we can estimate from (2.26) and (2.27) by use of Fefferman-Phon inequality

$$\begin{aligned}
&(\rho(t)^2 a_1(t,x,D)^*|\Lambda_t|^{-1}a_1(t,x,D)v, v)_s \\
&\le \frac{1}{2}\Re(|\Lambda_t|a(t,x,D)v, v)_s + C_s\Re(|\Lambda_t|v, v)_s.
\end{aligned} \tag{2.29}$$

On the other hand

$$\sigma(\Re a_t(t,x,D)(x,\xi) = a_t(t,x,\xi) + r(t,x,\xi) \tag{2.30}$$

where $r \in C^0([0,T_0];S^1)$ and a_t satisfies from (2.12)

$$|a_t(t,x,\xi)|^2 \leq Ca(t,x,\xi)\langle\xi\rangle_h^2. \tag{2.31}$$

Hence we get

$$\frac{1}{2}\Re(a_t(t,x,D)v,v)_s \leq |||\Lambda_t|^{-\frac{1}{2}}\langle D\rangle_h^{-\kappa}\Re a_t v||_s^2 + |||\Lambda_t|^{\frac{1}{2}}\langle D\rangle_h^{\kappa} v||_s^2 \tag{2.32}$$

$$= (\Re a_t|\Lambda_t|^{-1}\langle D\rangle_h^{-2\kappa}\Re a_t v, v)_s + (|\Lambda_t|\langle D\rangle_h^{\kappa} v, \langle D\rangle_h^{\kappa} v)_s.$$

Besides from (2.31)

$$\sigma(\Re a_t|\Lambda_t|^{-1}\langle D\rangle_h^{-2\kappa}\Re a_t)(x,\xi) \leq \frac{1}{\langle\xi\rangle_h^{3\kappa}|\rho_t|}(|a_t|^2 + C\langle\xi\rangle_h^2)$$

$$\leq \frac{C\langle\xi\rangle_h^{2-4\kappa}|\Lambda_t(t,\xi)|}{\rho_t^2}(a(t,x,\xi)+1) \leq \frac{C|\Lambda_t(t,\xi)|}{h^{4\kappa-2}|\rho_t|}(a(t,x,\xi)+1). \tag{2.33}$$

Therefore using again Fehherman-Phon inequality we get from (2.31)-2.33

$$\frac{1}{2}(\Re(a_t(t,x,D)v,v)_s \leq \frac{1}{2}\Re(\Lambda_t av, v)_s + C_s(\langle D\rangle_h^{\kappa} v, v)_s, \tag{2.34}$$

if we take $h > 0, \rho(t)$ such that

$$\frac{Ch^{2-4\kappa}}{|\rho_t|} \leq \frac{1}{2}. \tag{2.35}$$

Thus we obtain (2.19),(2.22),(2.24),(2,29) and (2.34)

$$2e(t)_s\frac{d}{dt}e_s(t) \leq C_s e_s(t)^2 + \frac{1}{4}(\Lambda_t(\partial_t - \Lambda_t)v, \partial_t - \Lambda_t)v)_s \tag{2.36}$$

$$+\frac{1}{4}(\Lambda_t v, v)_{s+\kappa} + \Re(L_\Lambda[v], (\partial_t - \Lambda_t)v)_s,$$

for $t \in [0,T_0]$, if we choose $\rho(t) \in C^2([0,T_0]$ and $h > 0$ satisfying from (2.28), (2.35)

$$\rho_t(t) < 0, \quad |\rho_t(t)| \geq C_s(h^{2-4\kappa} + \rho(t)), \tag{2.37}$$

$$0 < \rho(t) \leq (24n^\kappa)^{-1}\rho_0^\kappa, \tag{2.38}$$

for $t \in [0,T_0]$. We can see easily that (2.36) implies (2.17) and (2.18). Q.E.D.

Using the estimates of Proposition 2.1 we can get the existence of solutions of the Cauchy problem for the equation (2.14)

Proposition 2.2 *Assume (2.9), (2.10) and (2.11) are valid. Then for any $v_0 \in H^{s+1}$, $v_1 \in H^s$ and $g \in C^0([0,T_0];H^s)$ there exists a unique solution $v \in \cap_{j=0}^2 C^j([0, T_0]; H^{s+1-j})$ of the Cauchy problem (2.14)-(1,15).*

Proof. We can prove this proposition by a standard method of regularization. Since $L_\varepsilon = L_\Lambda + \varepsilon\Delta$ is strictly hyperbolic for $\varepsilon > 0$, we can find a solution $v_\varepsilon \in \cap_{j=0}^2 C^j([0,T_0]; H^{s+1-j}$ of the Cauchy problem to $L_\varepsilon v = g$ with the initial data (2.15). Moreover we can derive the a priori extimates similar to (2.17),(2.18) with the constant C_s independent of $\varepsilon > 0$ and consequently we get the limit function v of v_ε tending to ε to zero. The a priori estimate (2.17) assures the uniqness of the solutions.
Q.E.D.

The following proposition is a direct result of Proposition 2.1.

Proposition 2.3 *Assume that same conditions as ones in Proposition 2.1 are valid. Then for any $u_0 \in H^{s+1}_{\rho_0,d}, u_1 \in H^s_{\rho_0,d}$ and $f \in C^0([0,T_0]; H^s_{\rho_0,d})$ there exists a unique solution u of the Cauchy problem (2.8)-(1.4) satisfying $e^\Lambda u \in \cap_{j=0}^2 C^j([0,T_0]; H^{s+1-j})$ ($\Lambda(0) = \rho_0\langle D\rangle_h^\kappa$), If $u(0) = u_t(0) = 0$, there is $C_s > 0$ such that*

$$||\partial_t e^\Lambda u||_s + ||e^\Lambda u||_{s+\kappa} \le e^{C_s t}\int_0^t ||e^\Lambda Lu||_s d\tau, \tag{2.39}$$

$$\int_0^t (||\partial_t e^\Lambda u(\tau)||^2_{s+\kappa} + ||e^\Lambda u(\tau)||^2_{s+2\kappa})d\tau \le e^{C_s t}\int_0^t ||e^\Lambda L[u](\tau)]||^2_s d\tau, \tag{2.40}$$

where $\Lambda = \rho(t)\langle D\rangle_h^\kappa$.

We introduce a special symbol class of pseudodifferential operators in order to investigate nonlinear equations. Denote by $L^2_{\rho,d}S^m$ the set of symbols $a(x,\xi)$ satisfying

$$|a|_{L^2_{\rho,d}S^m,l} = \sup_{|\alpha+\beta|\le l} \frac{||a^{(\alpha)}_{(\beta)}(\cdot,\xi)||_{L^2_{\rho,d}}}{\langle\xi\rangle_h^{m-|\alpha|}} <\in fty. \tag{2.41}$$

Then we have similarly to Lemma 2.1,

Lemma 2.3 *Let $a \in L^2_{\rho,d}S^m$. Then $a(\rho,x,D) = e^{\rho\langle D\rangle_h^\kappa} a(x,D)e^{-\rho\langle D\rangle_h^\kappa}$ satisfies*

$$a(\rho,x,\xi) = \sum_{|\gamma|<N} \gamma!^{-1}a_{(\gamma)}(x,\xi)\omega_\gamma(\xi) + r_N(\rho,x,\xi) \tag{2.42}$$

where $\omega_\gamma = e^{-\rho\langle\xi\rangle_h^\kappa}\partial_\xi^\gamma e^{\rho\langle\xi\rangle_h^\kappa}$ and r_N satisfies

$$|r_N|_l^{(m-N(1-\kappa))} \le C_{l,N}|a|_{L^2_{\rho,d}S^m,N+M(l)} \tag{2.43}$$

and $M(l) = l(1-\kappa)^{-1} + 2l + [\frac{n}{2}]$.

Moreover we can prove the following lemma by use of the above lemma.

Lemma 2.4 *(i)For $a \in L^2_{\rho,d}S^m, u \in H^s_{\rho,d}$, and $s \ge 0$*

$$||a(x,D)u||_{H^s_{\rho,d}} \le C_l|a|_{L^2_{\rho,d}S^m,s+M(l_0)}||u||_{H^s_{\rho,d}}, \tag{2.44}$$

where $M(l_0) = l_0(1-\kappa)^{-1} + 2l_0 + [\frac{n}{2}]$ and l_0 is given by (i) of Lemma 2.2.
(ii) There is a positive integer $l = 2M(l_0)$ such that for $u,v \in H^l_{\rho,d}$ the product uv belongs to $H^l_{\rho,d}$ and satisfies

$$||uv||_{H^l_{\rho,d}} \le l(||u||_{H^l_{\rho,d}} + ||v||_{H^l_{\rho,d}})(||u||_{H^{l-1}_{\rho,d}} + ||v||_{H^{l-1}_{\rho,d}}). \tag{2.45}$$

The proof of this lemma can be found in [3].

For the operator a and b in (2.8) we assume (2.9), (2.10), (2.11) and the following conditions are valid

$$a(t,x,\xi) = a^1(t,x,\xi) + a^2(t,x,\xi)$$

$$a^i(t,x,\xi) = \sum_{j,k=1}^{n} a^i_{jk}(t,x)\xi_j\xi_k, i = 1,2 \tag{2.46}$$

where $a^1_{jk} \in C^2([0,T_0];\gamma^d_{\rho_0}(R^n))$, $a^2_{jk} \in L^2_\Lambda([0,T];H^l)$ (means $\langle D\rangle^l_h e^\Lambda a^2_{jk}$ belongs to $L^2((0,T_0)\times R^n))$, and

$$b(t,x,\xi) = b^1(t,x,\xi) + b^2(t,x,\xi),$$

$$b^i(t,x,\xi) = \sum_{j=1}^{n} b^i_j(t,x)\xi_j, i = 1,2 \tag{2.47}$$

where $b^1_j \in C^0([0,T_0];\gamma^d_{\rho_0}(R^n))$, $b^2_j \in L^2_\Lambda([0,T_0];H^l)$ and $\Lambda = \rho(t)\langle D\rangle^\kappa_h$ and l are given in Proposition 2.1 and in (ii) of Lemma 2.4 respectively. Then we can prove the following result by use of Lemma 2.2 and 2.4.

Proposition 2.4 *Assume* (2.9), (2.10), (2.11) *and the above conditions* (2.46), (2.47) *are valid. Then if we take* $l = 2M(l_0)$ *we have*

$$||e^\Lambda \partial_t u(t)||_l + ||e^\Lambda u||_{l+2\kappa} \le e^{C_1(t)} \int_0^t ||e^\Lambda L[u](\tau)||_l d\tau, \tag{2.48}$$

$$\int_0^t (||e^\Lambda \partial_t u(\tau)||^2_{l+\kappa} + ||e^\Lambda u(\tau)||^2_{l+2\kappa})d\tau \le C_2(t) \int_0^t ||e^\Lambda L[u](\tau)||^2_l d\tau, \tag{2.49}$$

for $e^\Lambda u \in \cap^2_{j=0} C^j([0,T_0];H^{l+2-j})$ *with* $u(0,x) = u_t(0,x) = 0$, *where*

$$C_i(t) = C_l(a^1_{jk}, b^1_j) \int_0^t ||a^2_{jk}(\tau,\cdot)||^i_l + ||b^2_j(\tau,\cdot)||^i_l d\tau, i = 1,2. \tag{2.50}$$

This implies the following proposition.

Proposition 2.5 *Assume that same conditions as ones in Proposition 2.4 are valid. Then for any* $u_0 \in H^l_{\rho_0,d}$, $u_1 \in H^l_{\rho_0,d}$ *and* $f \in C^0([0,T_0];H^l_{\rho_0,d})$ *there exists a unique solution* u *of the Cauchy problem (2.8) with* $u(0) = u_t(0) = 0$ *satisfying (2.48) and (2.49).*

3. Local existence theorem of nonlinear equations

In this section we shall prove the local existence theorem of the Cauchy problem for nonlinear equation as follows

$$\partial_t^2 u(t,x) + \sum_{j,k=1}^{n} a_{jk}(t,x,u,\nabla_x u)D_jD_k u$$

$$+\sum_{j=1}^{n} b_j(t,x,u,\nabla_x u)D_j u + b_0(t,x,u,\nabla_x u)u = f(t,x), \quad t\in(0,T), x\in R^n. \tag{3.1}$$

$$u(0,x) = u_t(0,x) = 0, \quad x \in R^n. \tag{3.2}$$

Let $d \geq 1, s \geq 1$ and Ω a domain of R^m. We denote by $\gamma^{d,s}(R^n \times \Omega)$ the set of functions $f(x,y)$ satisfying that for any compact set $K \subset \Omega$ there are $C_K > 0, \rho_K > 0$ such that

$$|D_x^\alpha D_y^\beta f(x,y)| \leq C_K \rho_K^{-(|\alpha|+|\beta|)} |\alpha|!^d |\beta|!^s \tag{3.3}$$

for $x \in R^n, y \in K, \alpha \in N^n, \beta \in N^m$. Let B be a neighborhood of 0 in R^{n+1}. We assume that the coefficients a_{jk} of (3.1) belong to $C^2([0,T_0];\gamma^{d,1}(R^n \times B))$ and b_j to $C^0([0,T_0];\gamma^{d,1}(R^n \times B))$.

We can prove the following lemma by use of (ii) of Lemma 2.4

Lemma 3.1 *Let $f \in C^0([0,T_0];\gamma^{d,1}(R^n \times B))$ such that $f(t,x,0) = 0$, and $u = (u,\cdots,u_m) \in L^2_\Lambda([0,T_0];H^l)^m$ such that $e^\Lambda u \in C^0([0,T];H^{l-1})$ and $||e^\Lambda u||_{l-1} < C_l^{-1}\rho_0$,. Then $f(t,x,u(t,x))$ belongs to $L^2_\Lambda([0,T_0];H^l)$ and satisfies*

$$||f \circ u||_{L^2_\Lambda([0,T_0];H^l)} \leq \frac{C_l ||u||_{L^2_\Lambda([0,T_0];H^l)}}{\rho_0 - C_l||e^\Lambda u||_{l-1}}. \tag{3.4}$$

where $\Lambda = \rho(t)\langle\xi\rangle_h^\kappa$ and $l = M(l_0)$

For $M > 0, T > 0$ and $h > 0$(a parameter in $\langle\xi\rangle_h$) we define

$$X(M,T,h) = \{u \in L^2_\Lambda([0,T];H^{l+1}) | e^\Lambda u \in \cap_{j=0}^1 C^j([0,T];H^l) s.t.$$

$$||u||_{L^2_\Lambda([0,T];H^{l+1})} + \sum_{j=0}^1 \sup_{0\leq t\leq T} ||e^\Lambda u(t)||_l \leq M\}, \tag{3.5}$$

where $\Lambda = \rho(t)\langle D\rangle_h^\kappa$. For $v \in X(T,M,h)$ we conider the following linear equation

$$\partial_t^2 u(t,x) + \sum_{j,k=1}^n a_{jk}(t,x,v,\nabla_x v) D_j D_k u$$

$$+\sum_{j=1}^n b_j(t,x,v,\nabla_x v) D_j u + b_0(t,x,v)u = f(t,x), \quad t \in (0,T), x \in R^n. \tag{3.6}$$

Then it follows from Proposition 2.5 that we have a unique solution u of the Cauchy problem (3.6)-(3.2). Denote $u = \Phi(v)$ which satisfies from (2.48),(2.49)

$$||\partial_t e^\Lambda \Phi(v)(t)||_l + ||\Phi(v)(t)||_{l+\kappa} \leq C(M)\int_0^t ||e^\Lambda f(t)||_l \leq C(M)T \leq M,$$

if we choose T such that $C(M)T \leq M$ and

$$||\partial_t \Phi(v)||_{L^2_\Lambda([0,T];H^{l+\kappa})} + ||\Phi(v)(t)||_{L^2_\Lambda([0,T];H^{l+2\kappa})} \leq C(M)||f||_{L^2_\Lambda([0,T];H^l)} \leq M,$$

if we choose $T > 0$ small enough, because $e^\Lambda f$ is in $C^0([0,T];H^l)$. Therefore Φ is a mapping from $X(T,M)$ to itself. Moreover to $w = \Phi(v) - \Phi(v')$ applying (2.49), we can get

$$||w||_{L^2_\Lambda([0,T];H^l)} \leq \frac{C(M)}{h^{2\kappa-1}}||v - v'l||_{L^2_\Lambda([0,T];H^{l-1})} \leq \frac{1}{2}||v - v'||_{L^2_\Lambda([0,T];H^l)}$$

if we choose $h > 0$ such that $\frac{C(M)}{h^{2\kappa-1}} \leq \frac{1}{2}$. Here we note that $C(M)$ does not depend on the parameter h. Thus we have proved

Proposition 3.1 *Assume that a_{jk} of (3.1) belong to $C^2([0,T_0];\gamma^{d,1}(R^n\times B))$ and $b_j\in C^0([0,T_0];\gamma^{d,1}(R^n\times B))$. There are $T>0, M>0$ and $h>0$ such that Φ is a mapping from $X(T,M,h)$ to itself and moreover satisfies*

$$||\Phi(v)-\Phi(v')||_{L^2_\Lambda([0,T];H^l)} \le \frac{1}{2}||v-v'||_{L^2_\Lambda([0,T];H^l)} \tag{3.7}$$

for $v,v'\in X(M,T.h)$

This proposition implies immediately

Theorem 3.1 *Assume that a_{jk} of (3.1) belong to $C^2([0,T_0];\gamma^{d,1}(R^n\times B))$ and b_j to $C^0([0,T_0];\gamma^{d,1}(R^n\times B))$. Then for any $f\in C^0([0,T_0];H^l_{\rho 0})$ there is $T\in(0,T_0]$ such that we have a unique solution u of the Cauchy problem (3.1)-(3.2) satisfying that $e^\Lambda u\in\cap_{j=0}^2 C^j([0,T];H^{l-j})$ and (2.48)-(2.49).*

4. Proof of Theorm 1.1

Denote

$$B(T,M)=\{a(t)\in C^2([0,T]);a(t)\ge 0,|a|_{C^2([0,T]}\le M\}, \tag{4.1}$$

where $|a|_{C^2([0,T]}=\sup_{k\le 2,t\in[0,T]}|\frac{d^k}{dt^k}a(t)|$. For $a(t)\in B(T,M)$ we consider

$$\partial_t^2 u+a(t)A[u]=f(t,x),\quad (t,x)\in(0.T)\times R^n \tag{4.2}$$

$$u(0)=u_0,\quad u_t(0)=u_1(x),\quad x\in R^n, \tag{4.3}$$

where A is given by (1.4). Then it follows from Theorem 3.1 that we can prove

Proposition 4.1 *Let $a(t)\in B(T,M)$. Assume that A satisfies (1.5)-(1.7) and there are a positive integer $\rho_0>0, 1<d<2$ such that the initial data $u_0,u_1\in H^{l+1}_{\rho_0,d}$ and $f\in C^0([0,T_0];H^l_{\rho_0,d})$, where $l=2M(l_0)$ given in Lemma 2.4. Then there are $T_0\ge T>0, M>0$ and ρ_1 such that for any $a(t)\in B_{T,M}$ the Cauchy problem (4),(2) has a unique solution $u\in C^2([0,T];H^l_{\rho_1,d})$ satisfying*

$$||\partial_t^2 u(t)||_{H^{l-2}_{\rho_1,d}}+||\partial_t u(t)||_{H^{l-1}_{\rho_1,d}}+||u(t)||_{H^l_{\rho_1,d}}\le e^{C(M)T}(C_0+C(M)T),\quad t\in[0,T], \tag{4.4}$$

where C_0 is independent of M,T.

Proof Put $w=u-u_0-tu_1$. Then if u satisfies (4.2)-(4.3), w does (3.1)-(3.2) modifying f in (3.2). The assumption (1.6) assures the hyperbolicity of the equation (3.1) and consequently (2.48) implies (4.4).
Q.E.D.

It follows from (4.4) that taking $T>0$ such that $C(M)T\le 1$,we get

$$||\partial_t^2 u(t)||_{H^{l-2}_{\rho_1.d}}+||\partial_t u(t)||_{H^{l-1}_{\rho_1.d}}+||u(t)||_{H^l_{\rho_1,d}}\le e(C_0+1),\quad t\in[0,T]. \tag{4.5}$$

For $a\in B(T,M)$ denote by u_a the solution of (4.2)-(4.3). Define

$$\Psi(a)(t)=(A[u_a],u_a)_{L^2}=\sum(a_{jk}(x,\nabla_x u_a)D_k u_a, Dju_a)_{L^2}. \tag{4.6}$$

By use of the above proposition we can prove similarly to Proposition 2.1 in [6] that Ψ is a contraction mapping in $B(M,T)$, if M,T are chosen suitably.

5. Proof of Theorem 1.2

To investigate the analyticity of solutions to the Cauchy problem (1,3)-(1.4), we introduce a convinient norm in $C^0([0,T];L^2_{\rho_0,d}(R^n))$ following the idea of Lax [7], Mizohata [8] and Alnihac and Métivier [1]. Let a positive integer $N \geq 2, \rho > 0$, and $0 < r \leq 1$. For $e^{\Lambda}u \in C^0([0,T];H^l)$, where $\Lambda = \Lambda(t,\rho) = (\rho(t)+\rho)\langle\xi\rangle_h^\kappa, 0 < \rho < \rho_0/2 =: \rho_2$ is given in Proposition 2.1, we denote

$$|u|^t_{\rho,r,N} = \sup_{0<s<t, 2\leq|\beta|\leq N} \frac{||e^{\Lambda(s,\rho)}D_x^\beta u||_l r^{|\beta|-2}}{\Gamma_2(|\beta|)}, \tag{5.1}$$

where $\Gamma_2(k) = \lambda_0 k! k^{-2}$ for $k \geq 1$ and $\Gamma_2(0) = \lambda_0$. Here we choose $\lambda_0 > 0$ such that

$$\sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \Gamma_2(|\alpha'|+p)\Gamma_2(|\alpha-\alpha'|) \leq \Gamma_2(|\alpha|+p), \tag{5.2}$$

for $p = 0,1,2,...$ and $\alpha \in N^n$. In brief we write $|u|_{r,N} = |u|^t_{\rho,r,N}$, if there is no confusion.

Lemma 5.1 *For $e^\Lambda v_i \in C^0([t_0,T];H^l), i = 1,...,n$, denote $v^\beta = v_1^{\beta_1}v_2^{\beta_2}\cdots v_n^{\beta_n}$. Then there is $C_0 > 0$ such that for $2 \leq |\beta| \leq N, t \in [0,T]$,*

$$(i) \qquad |v^\beta|_{r,N} \leq C_0^{|\beta|-1}(\sup_{t_0\leq s\leq t}||e^\Lambda v(s)||_{l+1} + r^2|v|_{r,N})^{|\beta|-1}|v|_{r,N},$$

and for $|\beta| > N, t \in [0,T]$

$$(ii) \qquad |v^\beta|_{r,N} \leq C_0^{|\beta|-1}(||e^\Lambda v(t)||_l + r^2)^{|\beta|-N}((||e^\Lambda v(s)||_{l+1} + r^2|v|_{r,N})^{N-1}|v|_{r,N})$$
$$+C_0^{|\beta|}||e^\Lambda v||^{|\beta|}_{l+1}.$$

(iii) Let $a(x) \in \mathcal{A}(R^n)$ satisfying

$$|a|_{R^n,\rho_1,1} = \sup_{\beta\in N^n, x\in R^n} \frac{|D_x^\beta a(x)|\rho_0^{|\beta|}}{|\beta|!} < \infty.$$

Then there is $C > 0$ such that for $|\rho(t)+\rho| \leq (24n^\kappa)^{-1}\rho_1^\kappa$,

$$|av|_{r,N} \leq C|a|_{R^n,\rho_1,1}|v|_{r,N}.$$

(iv) Let G an open set in R^N and $A(x,v)$ is analytic in $R^n \times G$ and satisfies

$$|A|_{R^n\times G,\rho_1,1} = \sup_{\alpha,\beta\in N^n, x\in R^n, v\in G} \frac{|D_x^\alpha D_v^\beta A(x,v)|\rho_1^{|\alpha|+|\beta|}}{|\alpha|!|\beta|!} < \infty.$$

Then if $||e^\Lambda v(t)||_{l+1} < \rho_1/(nC_l)$, a composite function $A \circ v(t,x) = A(x,v(t,x))$ satisfies that there is $C > 0$ such that for $t \in [0,T]$

$$|A\circ v|_{r,N} \leq C|A|_{R^n\times G,\rho_0,1}(1 + \sum_{j=0}^{N-1} C^j(||e^\Lambda v(s)||_{l+1} + r^2|v|_{r,N})^j|v|_{r,N}).$$

The proof of this lemma is given in [5]. We remark that it follows from (iv) of Lemma 5.1 that if $A(x,y)$ is analytic in $(x,y) \in R^n \times G$, and $||e^{\Lambda}\nabla_x v(t)||_{l+1} < \rho_1/(nC_l)$, we have for $2 \le |\alpha| \le N$,

$$||e^{\Lambda}D_x^{\alpha}(A \circ \nabla_x v)||_l \le Cr^{-|\alpha|+2}\Gamma_2(|\alpha|+1) \tag{5.3}$$

$$\times\{1 + \sum_{j=0}^{N-1} C^j(\sup_{t_0 \le s \le t} ||v(s)||_{l+2} + r^2|v|_{r,N})^j|\nabla_x v|_{r,N}\},$$

where C is independent of N.

Now we shall turn to prove Theorem 1.2. Let u be satisfied with (1.3)-(1.4). Since $u \in C^0([0,T]; H^l_{\rho_0})$, for any $\epsilon > 0$ there is $\tau > 0$ such that for $\Lambda(t) = (\rho(t)+\rho)\langle\xi\rangle_h^{\kappa}$ satisfying (2.37)-(2.38),

$$||e^{\Lambda}(t)(u(t,\cdot) - u(i\tau,\cdot))||_{l+2} < \epsilon, \tag{5.4}$$

for $t \in [i\tau,(i+1)\tau], i = 0,1,...,[T/\tau]$ and $t \in [[T/\tau]\tau, T]$, where $[\cdot]$ stands for the Gauss notation. Denote $\tau_i = i\tau$ for $i = 0,1,\cdots,[T/\tau]$, and $\tau_{[T/\tau]+1} = T$. We shall prove that $u(\tau_i,x)$ is in $L^2_{\rho_i,1}$ for $i = 0,1,...,[T/\tau]+1$ by induction of i. First it is trivial that $u(0,x) = u_0(x)$ is in $L^2_{\rho_0,1}(R^n)$. Assume that $u(\tau_i,x)$ is in $L^2_{\rho_i,1}(R^n)$. Then we show that $u(\tau_{i+1},x)$ belongs to $L^2_{\rho_{i+1}}(R^n)$. Then we may assume that

$$||e^{\Lambda}D_x^{\alpha}u(\tau_i,\cdot)||_{l+2} \le Cr_i^{-|\alpha|}|\alpha|!. \tag{5.5}$$

For the simplicity of notation we consider the case of $i = 0$. Put $v(t,x) = u(t,x) - u_0(t,x)$,where $u_0(t,x) = u(0,x) + tu_t(0,x)$. Since u is a solution of (1.3), v satisfies

$$\partial_t^2 v - \sum_{j,k=1}^{n} \tilde{a}_{jk}(t,x,\nabla_x v)D_jD_k v = \tilde{f}(t,x,v), (t,x) \in (0,\tau) \times R^n, \tag{5.6}$$

$$v(0,x) = v_t(0,x) = 0, \quad x \in R^n, \tag{5.7}$$

where

$$\tilde{a}_{jk}(t,x,\nabla_x v) = a_{jk}(t,x,\nabla_x(v + u_0(t,x))) \tag{5.8}$$

$$+\sum_{m=1}^{n} \partial_{y_m}a_{jm}(x,\nabla_x(v+u_0(t,x))\partial_{x_m}(v(t,x+u_0(t,x))$$

and

$$\tilde{f}(t,x,\nabla_x v) = f(t,x) - \sum_{j,k=1}^{n} \tilde{a}_{jk}(t,x,\nabla_x v)D_jD_k u_0(t,x) \tag{5.9}$$

Then it follows from (3.8) that $\tilde{A}_j$ and $\tilde{f} - f$ satisfy the condition of (iv) in Lemma 3.1. Therefore $\tilde{A}_j$ and $\tilde{f} - f$ satisfy (5.3). Differentiating (5.6) with respect to x we get

$$L[D^{\alpha}v] = f_{\alpha}, \quad D^{\alpha}v(0,x) = D^{\alpha}v_t(0,x) = 0, \tag{5.10}$$

where

$$f_{\alpha} = D^{\alpha}\tilde{f} + \sum_{\alpha' < \alpha}\binom{\alpha}{\alpha'}\sum_{j,k=1}^{n} D^{\alpha-\alpha'}\tilde{a}_{jk}D^{\alpha'}D_jD_k v. \tag{5.11}$$

Therefore by use of (2.39) of Proposiyion 2,3 we obtain

$$||e^{\Lambda(t,\rho)}D^\alpha\nabla_x v(t)||_l \le C\int_0^t ||e^{\Lambda(s,\rho)}f_\alpha(s)||_{l+1-\kappa}ds. \tag{5.12}$$

$$\le C\int_0^t (\rho'-\rho)^{-1}||e^{\Lambda(s,\rho')}f_\alpha||_l ds,$$

for $\rho' \in (\rho,\rho_2)$. On the other hand from (5.2),(5.3) and (5.4) we have if in brief $||\cdot|| = ||e^{\Lambda(s,\rho')}\cdot||_l$,

$$||f_\alpha(s)|| \le ||D^\alpha\tilde{f}|| + \sum_{1\le|\alpha'|\le|\alpha|-1}\binom{\alpha}{\alpha'}\sum_{j,k=1}^{n}(||D^{\alpha-\alpha'}\tilde{a}_{jk}||||D^{\alpha'}D_jD_kv||) \tag{5.13}$$

$$\le |f|^s_{r,N}r^{-|\alpha|+2}\Gamma_2(|\alpha|) + C\sum_{1\le|\alpha'|\le|\alpha|-2}\binom{\alpha}{\alpha'}r^{-(|\alpha-\alpha'|+2)}\Gamma_2(|\alpha-\alpha'|)$$

$$\times\{1+\sum_{j=0}^{N-1}C^j(\epsilon+r^2|\nabla_x v|^s_{r,N})^j|\nabla_x v|^t_{r,N}\}|\nabla_x v|^s_{r,N}r^{-|\alpha'|+1}\Gamma_2(|\alpha'|+1)$$

$$\le \{|f|^s_{r,N}r^{-|\alpha|+2}\Gamma_2(|\alpha|)$$

$$+C\{1+\sum_{j=0}^{N-1}C^j(\epsilon+r^2|\nabla_x v|^s_{r,N})^j|\nabla_x v|^s_{r,N}\}(r^{-|\alpha|+3}|v|^s_{r,N}\Gamma_2(|\alpha|+1).$$

Here we choose $r = r(s,\rho') = r_0(\sigma(\rho_2-\rho')-s)$, where $0 < r_0 \le 1, \sigma = \frac{\tau}{\rho_2}$ and $0<\rho<\rho_2$. Denote

$$y(t) = \sup_{0\le s\le\min\{t,\sigma(\rho_2-\rho)\},0<\rho<\rho_2} r(s,\rho)|v|^s_{r(s,\rho),N}.$$

We have from (5.13)

$$||f_\alpha|| \le |f|^s_{r,N}r(s,\rho')^{-|\alpha|+2}\Gamma_2(|\alpha|)$$

$$+C\{r(s,\rho')+\sum_{j=0}^{N-1}C^j(\epsilon+y(s))^jy(s)\}y(s)r(s,\rho')^{-|\alpha|+1}\Gamma_2\}(|\alpha|+1).$$

Hence we have from (5.12) and (5.13)

$$||e^\Lambda D^\alpha\nabla_x v(t)||_l \le C\int_0^t(\rho'-\rho)^{-1}\{(|f|^s_{r,N}r(s)^{-|\alpha|+2}\Gamma_2(|\alpha|)$$

$$+C(r+\sum_{j=0}^{N-1}C^j(\epsilon+y(s))^jy(s))y(s)r(s)^{-|\alpha|+1}\Gamma_2(|\alpha|+1)\}ds.$$

Here we take $\rho'-\rho = r(s,\rho')r_0^{-1} = \sigma(\rho_2-\rho')-s$, that is $\rho' = \frac{\tau+\rho-s}{1+\sigma}$. Then noting that $y(s)\le y(t)$ for $s<t$ and $r(s,\rho')\le r_0\tau\le r_0$ we get from the above estimate

$$||e^{\Lambda(t,\rho)}D^\alpha\nabla_x v(t)||_l \le |f|^t_{r,N}\int_0^t r_0r(s,\rho')^{-|\alpha|+1}ds\Gamma_2(|\alpha|) \tag{5.14}$$

$$+C\int_0^t(r_0+\sum_{j=1}^{N}C^j(\epsilon+y(s))^jy(s)r_0r(s,\rho')^{-|\alpha|}ds\Gamma_2(|\alpha|+1).$$

Since $|f|^t_{r,N} \leq C, r(t) \leq r_0\tau \leq r_0$ and $r_0 \int_0^t r(s)^{-|\alpha|}ds \leq r(t)^{-|\alpha|+1}|\alpha|^{-1}$ for $|\alpha| \geq 2$, we get from (5.14)

$$y(t) \leq Cr_0\{1 + \sum_{j=0}^{N} C^j(\epsilon + y(t))^j y(t)\}),$$

for $t \in [0,\tau]$. We shall prove that $y(t) < \epsilon$ for $t \in [0,\tau]$, if we choose $r_0 > 0, \varepsilon > 0$ small enough. Assume that there is $t_1 \in [0,\tau]$ such that $y(t_1) = \epsilon$ and $y(t) < \epsilon$ for $t \in (0,t_1)$. Since $y(0) = 0$, we have $t_1 > 0$. It follows from (5.15) that if $2C\varepsilon < 1$

$$y(t) \leq Cr_0(1 + \frac{y(t)}{1 - 2C\epsilon})$$

for $t \in [0,t_1]$. Hence we have

$$y(t_1) \leq \frac{Cr_0}{1 - \frac{Cr_0}{1-2C\varepsilon}} < \varepsilon,$$

which contradicts to $y(t_1) = \varepsilon$ if we choose r_0 sufficient small, and consequently we have proved Theorem 3.

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