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Integral solutions of the equation $\xi^2 + \eta^2 = \zeta^2$ in the quadratic realms of rationality

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Integral solutions of the equation $\xi^2 + \eta^2 = \zeta^2$ in the quadratic realms of rationality;

By Harris HANCOCK.

One of the simplest Diophantine équations is $x^2 + y^2 = z^2$.

The solution of this equation gives the so called Pythagorean numbers $x = 2pqt$, $y = (p^2 - q^2)t$, $z = (p^2 + q^2)t$, vhere p, q, t are rational intégers such that $p > q > 0$, $t > 0$ with the further condition that p and q are relatively prime and both must not be odd.

For exemple,

$$p = 2,\quad q = 1,\quad x = 4,\quad y = 3,\quad z = 5,\quad 5^2 = 4^2 + 3^2,$$
$$p = 5,\quad q = 4,\quad x = 40,\quad y = 9,\quad z = 41,\quad 41^2 = 40^2 + 9^2, \text{ etc.}$$

A treatment of such problems in the realm of natural integrals is found in the second volume of Dickson's admirable *History of the Theory of Numbers*. As every problem, whose history is given by Dickson, admits a generalized treatment in the algebraic realms of the second, third and higher degrees, it is seen that a vast field of further investigation is open to Mathematicians. And by developing this algebraic number-theory new light may be thrown upon the theory of algebraic equations.

By definition a quadratic algebraic integer is the root of the equation $t^2 + At + B = 0$, where A and B are rational integers and the coefficient of t^2 is unity.

Let

$$\xi = a_1 + \sqrt{a_1^2 + a_2},\qquad \eta = b_1 + \sqrt{b_1^2 + b_2},\qquad \zeta = c_1 + \sqrt{c_1^2 + c_2}$$

be three quadratic algebraic integers, being respectively the roots of

$$x^2 - 2a_1x - a_2 = 0, \qquad y^2 - 2b_1y - b_2 = 0, \qquad z^2 - 2c_1z - c_2 = 0,$$

where a_1, a_2, b_1, b_2, c_1, c_2 are arbitrary rational integers.

If then $\xi^2 + \eta^2 = \zeta^2$, it follows that

(I) $$2a_1\xi + a_2 + 2b_1\eta + b_2 = 2c_1\zeta + c_2,$$

or

(I^a) $$2a_1^2 + a_2 + 2a_1\sqrt{a_1^2 + a_2} + 2b_1^2 + b_2 + 2b_1\sqrt{b_1^2 + b_2}$$
$$= 2c_1^2 + c_2 + 2c_1\sqrt{c_1^2 + c_2}.$$

Our problem is to solve this latter equation in integral values of a_1, a_2, b_1, b_2, c_1, c_2.

First let $a_1 = 0$. It follows that

(II) $$a_2 + 2b_1^2 + b_2 = 2c_1^2 + c_2,$$

(III) $$b_1\sqrt{b_1^2 + b_2} = c_1\sqrt{c_1^2 + c_2}.$$

Writing (III) in the form

(IV) $$b_1^4 - c_1^4 = c_1^2c_2 - b_1^2b_2,$$

it is seen that b_1, c_1, may be taken at pleasure and c_2, b_2 so chosen as to satisfy (IV). Then from (II) a_2 is determined.

For example let $b_1 = 4$, $c_1 = 3$, then is $b_2 = -700$, $c_2 = -1225$ and $a_2 = -539$. And that is, if η is a root of $y^2 - 8y + 700 = 0$ and ζ a root ov $z^2 - 6z + 1225$, then $\eta^2 = \zeta^2 + 539$. See *Diophantus* II, 11; *Dickson* II, p. 402.

Next observe that from (I^a) follow the two equations

(1) $$2a_1^2 + a_2 + 2b_1^2 + b_2 = 2c_1^2 + c_2,$$

(2) $$a_1\sqrt{a_1^2 + a_2} + b_1\sqrt{b_1^2 + b_2} = c_1\sqrt{c_1^2 + c_2},$$

That these equations admit solution is seen at once if we put

$$a_1 = b_1 = c_1; \qquad c_2 = 2a_1^2 + a_2 + b_2. \qquad a_2 = b_2 = -\frac{a_1^2}{2}.$$

For it is evident that if ξ, η, ζ are respectively the roots of

$$x^2-2a_1x+\frac{a_1^2}{2}=0, \qquad y^2-2a_1y+\frac{a_1^2}{2}=0, \qquad z^2-2a_1z-a_1^2=0,$$

that $\xi^2+\eta^2=\zeta^2$.

Note that a_1 must be an even integer in order that ξ and η be algebraic integers.

For example,

$$x^2-4x+2=0, \qquad y^2-4y+2=0, \qquad z^2-4z-4=0,$$

have roots

$$\xi=\eta=2+\sqrt{2}, \qquad \zeta=2+2\sqrt{2}$$

such that

$$\xi^2+\eta^2=\zeta^2.$$

A solution of (1) and (2) is had as follows:

Put

(3) $$\sqrt{b_1^2+b_2}=m\sqrt{a_1^2+a_2},$$

where m is a rational integer to be determined.

We have at once

(4) $$c_1=a_1+mb_1 \quad \text{and} \quad \sqrt{a_1^2+a_2}=\sqrt{c_1^2+c_2}.$$

In the latter expression put for c_1 its value so that

$$\sqrt{a_1^2+a_2}=\sqrt{(a_1+mb_1)^2+c_2}=\sqrt{a_1^2+2a_1mb_1+m^2b_1^2+c_2}$$

and regarding a_2 fixed, make c_2 satisfy the relation

(5) $$a_2=2a_1mb_1+m^2b_1^2+c_2.$$

This expression may be simplified by observing from (1) that

(6) $$a_2+2b_1^2+b_2=4a_1mb_1+2m^2b_1^2+c_2.$$

From (5) and (6) it is seen that

(7) $$2b_1^2+b_2=a_2-c_2,$$

and also that

(8) $$m^2b_1^2+2a_1b_1m=2b_1^2+b_2.$$

The latter equation in m may be satisfied in an infinite number of ways. For let $2a_1 = 3mb_1$, thus making a perfect square on the left hand side and put respectively $b_2 = 2b_1^2$, $= 14b_1^2$, $= 34b_1^2$, $= 62b_1^2$, etc.

It is seen that m may take the values 1, 2, 3, 4, ...

For example let $m = 2$, so that $a_1 = 3b_1$; $b_2 = 14b_1^2$. From (3) it follows that $b_1^2 + b_2 = m^2(a_1^2 + a_2)$, or $-21b_1^2 = 4a_2$, an expression which offers an infinite number of values of a_2 and b_1. Write $b_1 = 2$, $a_2 = -21$, $a_1 = 6$, $b_2 = 56$. From (4) and (5) it is seen that $c_1 = 10$ and $c_2 = -85$.

It is thus shown that the roots of

$$x^2 - 12x + 21 = 0, \qquad y^2 - 4y - 56 = 0, \qquad z^2 - 20z + 85 = 0,$$

are

$$\xi = 6 + \sqrt{15}, \qquad \eta = 2 + 2\sqrt{15}, \qquad \zeta = 10 + \sqrt{15}$$

and that

$$\xi^2 + \eta^2 = \zeta^2.$$

A second method. — Write as above

$$x^2 - 2a_1 x + a_2 = 0, \qquad y^2 - 2b_1 y - b_2 = 0, \qquad z^2 - 2c_1 z - c_2 = 0$$

with roots, the algebraic integers,

$$\xi = a_1 + \sqrt{a_1^2 + a_2}, \qquad \eta = b_1 + \sqrt{b_1^2 + b_2}, \qquad \zeta = c_1 + \sqrt{c_1^2 + c_2}.$$

Put

$$\eta = m\xi + n,$$

so that

$$\eta^2 = m^2\xi^2 + 2mn\xi + n^2.$$

It follows that

$$2b_1\eta + b_2 = m^2(2a_1\xi + a_2) + 2mn\xi + n^2,$$

or

$$2b_1 m\xi + 2b_1 n + b_2 = 2a_1 m^2\xi + a_2 m^2 + 2mn\xi + n^2.$$

Il is clear from this last expression that

$$(1) \qquad b_1 = ma_1 + n,$$

$$(2) \qquad 2b_1 n + b_2 = m^2 a_2 + n^2.$$

If n is eliminated from these equation we have

$$(i) \qquad b_1^2 + b_2 = m^2(a_1^2 + a_2).$$

From equation (I), it follows that

$$2a_1\xi + a_2 + 2b_1 m\xi + 2b_1 n + b_2 = 2c_1\zeta + c_2.$$

If the rational and irrational part of this expression are equated, it is seen that

(3) $$a_1 + mb_1 = c_1,$$

(4) $$2a_1^2 + a_2 + 2b_1 ma_1 + 2b_1 n + b_2 = 2c_1^2 + c_2.$$

Eliminate n from (4) by means of (1), we have equation (*ii*) above

(*ii*) $$a_2 + 2b_1^2 + b_2 = 4a_1 mb_1 + 2m^2b_1^2 + c_2.$$

Noting that the relation

$$\sqrt{a_1^2 + a_2} = \sqrt{c_1^2 + c_2} = \sqrt{(a_1 + mb_1)^2 + c_2}$$

must be satisfied, it is seen that

(5) $$a_2 = 2a_1 mb_1 + m^2b_1^2 + c_2.$$

From (*ii*) and (5) the quantities $a_2 - c_2$ and m may be respectively eliminated, and the following equations take their places

(4^a) $$2b_1^2 + b_2 = m^2b_1^2 + 2a_1b_1m,$$

(5^a) $$2b_1^2 + b_2 = a_2 - c_2.$$

If as in the previous solution, we put in (4^a) the values $2a_1 = 3b_1m$, $b_2 = 14b_1^2$, we have $16b_1^2 = 4m^2b_1^2$, or $m = 2$, $2a_1 = 6b_1$. From (1) $n = -5b_1$ and from (2) $-21b_1^2 = 4a_2$. Put $b_1 = 2$, $a_2 = -21$, $a_1 = 6$, $b_2 = 56$. From (3) and (5^a) it is seen that $c_1 = 10$ and $c_2 = -85$. Observe, however, that equations (1), (2) et (4) may by eliminating n be replaced by

$$b_1^2 + b_2 = m^2(a_1^2 + a_2)$$

and

$$a_2 + 2b_1^2 + b_2 = 4a_1mb_1 + 2m^2b_1^2 + c_2;$$

and it is seen that this solution is only a different form of the preceding one.

A more general solution. — Writing as in the preceding method

$$\eta = m\zeta + n,$$

we have

$$(1) \qquad b_1 = mc_1 + n$$

and

$$(2) \qquad 2b_1 n + b_2 = m^2 c_2 + n^2.$$

Again note if n is eliminated from (1) and (2), that

$$b_1^2 + b_2 = m^2(c_1^2 + c_2).$$

Further write $\xi = p\zeta + q$ and it follows that

$$(3) \qquad a_1 = pc_1 + q$$

and

$$(4) \qquad 2a_1 q + a_2 = p^2 c_2 + q^2.$$

If q is eliminated from these latter equations, we have

$$a_1^2 + a_2 = p^2(c_1^2 + c_2).$$

The above values substituted in (1) cause that equation to become

$$2a_1(p\zeta + q) + a_2 + 2b_1(m\zeta + n) + b_2 = 2c_1\zeta + c_2.$$

In this expression equate the real parts and also the imaginary parts and it is seen that

$$(5) \qquad c_1 = pa_1 + mb_1,$$

$$2pa_1c_1 + 2mb_1c_1 + 2a_1q + 2b_1n + a_2 + b_2 = c_2 + 2c_1^2.$$

Due to (5) the last equation becomes

$$(6) \qquad 2a_1q + 2b_1n + a_2 + b_2 = c_2.$$

From (1), (3) and (5) it is seen that

$$(7) \qquad c_1 = \frac{mn + pq}{1 - m^2 - p^2};$$

and from (2), (4) and (6)

$$(8) \qquad c_2 = \frac{n^2 + q^2}{1 - m^2 - p^2}.$$

A solution of (7) and (8) is had, if we put

$$n = k(1 - m^2 - p^2) \quad \text{and} \quad q = l(1 - m^2 - p^2),$$

where k and l are arbitrary integers.

It follows that

$$c_1 = mk + pl,$$
$$c_2 = (k^2 + l^2)\mathrm{H}, \quad \text{where} \quad \mathrm{H} = 1 - m^2 - p^2.$$

From (1) and (3) we have

$$b_1 = k + p(ml - pk),$$
$$a_1 = l + m(pk - ml);$$

and from (2) and (4)

$$b_2 = [(ml - pk)^2 - k^2]\mathrm{H},$$
$$a_2 = [(pk - ml)^2 - l^2]\mathrm{H}.$$

Further note that

$$c_1^2 + c_2 = l^2 + k^2 - (kp - lm)^2,$$
$$a_1^2 + a_2 = p^2(c_1^2 + c_2), \quad b_1^2 + b_2 = m^2(c_1^2 + c_2).$$

For example, let $l = 2$, $k = 3$, $p = 5$, $m = 7$.

It is seen that

$$c_1^2 + c_2 = 12, \quad a_1^2 + a_2 = 5^2.12, \quad b_1^2 + b_2 = 7^2.12,$$
$$c_1 = 31, \quad a_1 = 9, \quad b_1 = -2,$$
$$\zeta = 3 + 2\sqrt{3}, \quad \xi = 9 + 10\sqrt{3}, \quad \eta = -2 + 14\sqrt{3},$$

where $\xi^2 + \eta^2 = \zeta^2$.

Again put

$$m = 3, \quad l = 1, \quad p = 2, \quad k = 5,$$

and we have

$$\xi = 22, + 2i\sqrt{23}, \quad \eta = 9 + 3i\sqrt{23}, \quad \zeta = 17 + i\sqrt{23},$$
$$\xi^2 + \eta^2 = \zeta^2.$$

Finally make $c_1^2 + c_2$ a perfect square, and we have the Pythagorean numbers. For example, let

$$m = 5, \quad l = 2, \quad p = 3, \quad k = 4,$$

and we have

$$c_1^2 + c_2 = 16, \qquad c_1 = 26, \qquad \zeta = 26 + 4 = 30,$$
$$a_1^2 + a_2 = 3^2 . 16, \qquad a_1 = 12, \qquad \xi = 12 + 12 = 24,$$
$$b_1^2 + b_2 = 5^2 . 16, \qquad b_1 = -2, \qquad \eta = -2 + 20 = 18,$$
$$\xi^2 + \eta^2 = 900 = \zeta^2.$$

The quadratic realms of rationality.

We are now in a position to express the above results in a more definite form, which is done through proof of the following theorem:

In every quadratic realm there is an infinite number of solutions of the equation

$$\xi^2 + \eta^2 = \zeta^2$$

through algebraic integers.

It is well known that any quadratic integer may be expressed in the form $a.1 + b.\omega$, where a and b are any two rational integers and where $1, \omega$ form the *basis* of the realm, say $R(\sqrt{t})$, t any integer. If $t \equiv 2 \pmod 4$ or $t \equiv 3 \pmod 4$, then is $\omega = \sqrt{t}$; if, however, $t \equiv 1 \pmod 4$, then is $\omega = \frac{1 + \sqrt{t}}{2}$. The case where t contains a squared factor may be reduced to one or the other of the above cases, so that it is unnecessary to consider the case $m \equiv 0 \pmod 4$. As the present paper is concerned particularly with the *existence* of such solutions as have been indicated, it is seen that the solutions of the second case necessarily imply those of the first case; for if

$$\xi^2 + \eta^2 = \zeta^2,$$

then necessarily

$$(2\xi)^2 + (2\eta)^2 = (2\zeta)^2.$$

We may therefore consider here only the first case: If

$$\xi = m_1 + m\sqrt{t},$$

where m_1 and m are any rational integers and if ξ satisfies the equation

$$x^2 - 2 a_1 x - a_2 = 0,$$

where a_1, a_2 are arbitrary rational integers, we have

$$m_1^2+2mm_1\sqrt{t}+m^2t=2a_1m_1+2a_1m\sqrt{t}+a_2.$$

It follows that

$$m_1=a_1 \qquad \text{and} \qquad m^2t=a_1^2+a_2.$$

Similarly if η and ζ are roots of the two equations

$$y^2-2b_1y-b_2=0 \qquad \text{and} \qquad z^2-2c_1y-c_2=0,$$

we have

$$\xi=a_1+mt^{\frac{1}{2}},$$
$$\eta=b_1+nt^{\frac{1}{2}},$$

where n like m is an arbitrary integer,

$$\zeta=c_1+pt^{\frac{1}{2}},$$

p being likewise any rational integer; with the conditions

$$m^2t=a_1^2+a_2, \qquad n^2t=b_1^2+b_2, \qquad p^2t=c_1^2+c_2.$$

Writing their values in equation (I) namely

$$2a_1\xi+a_2+2b_1\eta+b_2=2c_1\zeta+c_2,$$

it is seen that

$$2a_1^2+a_2+2a_1m\sqrt{t}+2b_1^2+b_2+2b_1n\sqrt{t}=2c_1^2+c_2+2c_1p\sqrt{t}.$$

Equating the rational and irrational terms in this expression, we have the following equations to solve

(1) $$2a_1^2+a_2+2b_1^2+b_2=2c_1^2+c_2,$$
(2) $$a_1m+b_1n=c_1p,$$
(3) $$m^2t=a_1^2+a_2,$$
(4) $$n^2t=b_1^2+b_2,$$
(5) $$p^2t=c_1^2+c_2.$$

From (1), (3), (4) and (5) it follows that

$$2(m^2+n^2-p^2)t=a_2+b_2-c_2;$$

or, what is the same thing,

(6) $$(m^2+n^2-p^2)t=c_1^2-a_1^2-b_1^2,$$

Eliminate respectively a_1, b_1, c_1, between (1) and (2) after (1) has been put in the form

$$a_1^2+b_1^2+(m^2+n^2)t=c_1^2+p^2t,$$

and we have

(7) $$p^2(m^2+n^2-p^2)t=(a_1m+b_1n)^2-p^2(a_1^2+b_1^2).$$

(8) $$n^2(m^2+n^2-p^2)t\equiv n^2(c_1^2-a_1^2)-(c_1p-a_1m)^2.$$

(9) $$m^2(m^2+n^2-p^2)t=m^2(c_1^2-b_1^2)-(c_1p-b_1n)^2.$$

Note that if we put

$$m^2+n^2=p^2,$$

in these equations, we have

$$\frac{a_1}{m}=\frac{b_1}{n}=\frac{c_1}{p}.$$

If then Pythagorean numbers are chosen for m, n, p, the same are had for a_1, b_1, c_1, and from (6) the number t is indeterminate.

Observe that equations (2) are satisfied if we put

$$m=ka_1,\qquad n=kb_1,\qquad p=kc_1\qquad (k \text{ any rational integer})$$

and choose fur a_1, b_1, c_1 any Pythagorean numbers π_1, π_2, π_3, $\pi_1^2+\pi_2^2=\pi_3^2$. For example, in any realm of nationality $R(\sqrt{t})$, it is evident that the algebraic integers

$$\xi=\pi_1+k\pi_1\sqrt{t},\qquad \eta=\pi_2+k\pi_2\sqrt{t},\qquad \zeta=\pi_3+k\pi_3\sqrt{t}$$

satisfy the equation

$$\xi^2+\eta^2=\zeta^2.$$

The theorem has resolved itself in the solution of the equation

(6) $$(m^2+n^2-p^2)t=c_1^2-a_1^2-b_1^2,$$

subject to the condition

(2) $$a_1m+b_1n=c_1p.$$

One method of procedure is the following:

The integer t being fixed, give to m, n, p fixed values and put

$$p^2 - m^2 - n^2 = k,$$

where k is constant.

From (6), it is seen that

$$a_1^2 + b_1^2 - c_1^2 = t.k.$$

If c_1 is eliminated from this latter equation and (2), we have

$$\text{(II)} \qquad (p^2 - m^2)a_1^2 - 2mna_1b_1 + (p^2 - n^2)b_1^2 = p^2tk,$$

which equation is a quadratic form in the two unknown quantities a_1, b_1, whose discriminant D is $-p^2k$.

The problem has now resolved itself into the solution of the quadratic form (II) with negative, zero or positive, discriminant.

1° If k is positive the discriminant is negative. In this case multiply (II) by $4(p^2 - m^2)$ and put

$$(i) \qquad 2(p^2 - m^2)a_1 - 2mnb_1 = s.$$

It is seen that (II) becomes

$$s^2 + 4p^2kb_1^2 = 4(p^2 - m^2)tkp^2,$$

or

$$(ii) \qquad s^2 = 4kp^2[(p^2 - m^2)t - b_1^2].$$

Hence through a finite number of trials, we may determine whither or not there is a value fur b_1 which makes the right hand side of (*ii*) a perfect square — a condition witch is evidently necessary for the solution of the problem.

For example write $p = 3$, $m = 2$, $n = 1$, so that $k = 4$. Take $t = 5$. From (*ii*) it is seen that, when

(*a*) $b_1 = 0$, $s = 4.3.5$;
(*b*) $b_1 = 3$, $s = 4.3.4$;
(*c*) $b_1 = 4$, $s = 4.3.3$;
(*d*) $b_1 = 5$, $s = 0$.

Hence from (i) and (2), we have

(a)	$a_1 = 6$,	$b_1 = 0$,	$c_1 = 4$;
(b)	$a_1 = 6$,	$b_1 = 3$,	$c_1 = 5$;
(c)	*no solution*;		
(d)	$a_1 = 2$,	$b_1 = 5$,	$c_1 = 3$.

It is also evident from (ii) that negative values may be given to b_1 and s so that, for example, $b_1 = -3$, $s = -4.3.4$.

In this case $a_1 = -b_1$, $b_1 = -3$, $c_1 = -4$.

Hence in the realm $R(\sqrt{5})$ if values 2, 1, 3 are taken respectively for m, n, p, it is seen that:

$$(a)\quad \begin{cases} \xi = \pm 6 + 2\sqrt{5}, \\ \eta = \quad 0 + \sqrt{5}, \\ \zeta = \pm 4 + 3\sqrt{5}; \end{cases}$$

$$(b)\quad \begin{cases} \zeta = \pm 6 + 2\sqrt{5}, \\ \eta = \pm 3 + \sqrt{5}, \\ \zeta = \pm 5 + 3\sqrt{5}; \end{cases}$$

$$(c)\quad \begin{cases} \xi = \pm 2 + 2\sqrt{5}, \\ \eta = \pm 5 + \sqrt{5}, \\ \zeta = \pm 3 + 3\sqrt{5}; \end{cases}$$

are solutions of the equation

$$\xi^2 + \eta^2 = \zeta^2.$$

It is evident that if -2, -1, -3 had been chosen respectively at values of m, n, p, the same values of a_1, b_1, c_1 as those above would have been derived.

If we put $m = 1$, $n = 3$, $p = 2$, it is seen that $k = -6$ and $D = +24$. Equation (II) becomes, if t is taken $= 3$,

$$5b_1^2 + 6b_1a_1 - 3a_1^2 = 72.$$

A solution is $a_1 = 3$, $b_1 = 3$, $c_1 = \dfrac{a_1m + b_1n}{p} = 6$. The corresponding values of ξ, η, ζ are

$$\xi = 3 + \sqrt{3}, \qquad \eta = 3 + 3\sqrt{3}, \qquad \zeta = 6 + 2\sqrt{3}.$$

If $t = 10$, the equation

$$5b_1^2 + 6b_1a_1 - 3a_1^2 = 240,$$

admits solution $a_1 = 2$, $b_1 = 6$, $c_1 = 10$.

However if $t = 5$, the equation

$$5b_1^2 + 6b_1a_1 - 3a_1^2 = 120$$

does *not* admit solution, as is evident from elementary consideration.

Having *one* solution it is possible by means of the Pell's equation to derive an infinity of others.

In the first case, for example, where $D = 24$, the equation of Pell, namely $T^2 - 24U^2 = 1$, admits the solution $T = 5$, $U = 1$.

Note that the equation

$$5b_1^2 + 6b_1a_1 - 3a_1^2 = 72,$$

may be written in the form

$$(5b_1 + 3a_1)^2 - 24a_1^2 = 5.72.$$

Hence writing

$$5b_1 + 3a_1 + \sqrt{24}a_1 = (24 + 3\sqrt{24})(5 + \sqrt{24})^k,$$

we have the solutions corresponding to values of $k = 1, 2, 3, \ldots.$

If $k = 1$, it is seen that

$$a_1 = 39, \qquad b_1 = 15, \qquad c_1 = 42,$$

and

$$\xi = 39 + \sqrt{3}, \qquad \eta = 15 + 3\sqrt{3}, \qquad \zeta = 42 + 2\sqrt{3}.$$

In the two examples given above it is seen that a_1, b_1, c_1 have a common factor other than unity.

Take, however,

$$m = 3, \qquad n = 2, \qquad p = 1$$

and let

$$t = 22.$$

It is seen that $k = -12$, and the discriminant $D = +12$. Equation (II) becomes

$$8a_1^2 + 12a_1b_1 + 3b_1^2 = 12.22.$$

Note that 12 is a quadratic residue of 12.22, the congruence

$$n^2 \equiv 12 \qquad (\mathrm{mod}\, 12.22)$$

admitting the solution $n = 54$; and observe that

$$a_1 = 3, b_1 = 4$$

is a *proper* solution of the quadratic form, that is, one in which a_1 and b_1 are relativily prime.

The corresponding Pell's equation

$$T^2 - 12U^2 = 1$$

admits the solution

$$T = 7, \qquad U = 2.$$

Write

$$8a_1^2 + 12a_1 b_1 + 3b_1^2 = 12.22$$

in the form

$$(8a_1 + 6b_1)^2 - 12b_1^2 = 8.12.22,$$

and it is seen that the general solution is

$$8a_1 + 6b_1 + \sqrt{12}\, b_1 = (48 + 4\sqrt{12})(7 + 2\sqrt{12})^k \qquad (k = 0, 1, 2, 3, \ldots).$$

When

$k = 1$,	$a_1 = -39$,	$b_1 = 127$,	$c_1 = 131$,
$k = 2$,	$a_1 = -549$,	$b_1 = 1732$,	$c_1 = 1817$,
$k = 3$,	$a_1 = -7647$,	$b_1 = 24124$,	$c_1 = 25307$,
.....,	............,	...........,	....

The corresponding values of ξ, η, ζ are

$k = 0$,
$\xi = 3 + 3\sqrt{22}$,
$\eta = 4 + 2\sqrt{22}$,
$\zeta = 17 + \sqrt{22}$;

$k = 1$,
$\xi = -39 + 3\sqrt{22}$,
$\eta = 124 + 2\sqrt{22}$,
$\zeta = 131 + \sqrt{22}$;

$k = 2$,
$\xi = -549 + 3\sqrt{22}$,
$\eta = 1732 + 2\sqrt{22}$,
$\zeta = 1817 + \sqrt{22}$;
................,

$k = 3$,
$\xi = -7647 + 3\sqrt{22}$,
$\eta = 24124 + 2\sqrt{22}$,
$\zeta = 25307 + \sqrt{22}$;
....

Note that on the transition from a negative to a positive discriminant we must have

$$D = o = - p^2 k.$$

If $k = o$, whe have the Pythagorean numbers considered above. And if $p = o$, the quadratic form (II) becomes

$$a_1 m + b_1 n = o$$

and the further solution is without difficulty.