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Transformations of surfaces applicable to a quadric

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Transformations of surfaces applicable to a quadric;

By LUTHER PFAHLER EISENHART (Princeton).

If a conjugate system of curves, or *net*, N on a surface S and a congruence G of straight lines are so related that the developables of G meet S in N, the net and congruence are said to be *conjugate*. Two nets conjugate to the same congruence are said to be in the relation of a *transformation* T, if the nets are not parallel. In a previous paper (1) the author developed a general theory of transformations T. When two surfaces S and $\overline{S}$ are applicable, there is a unique net on S which remains a net as S is deformed into $\overline{S}$; we call it the *permanent net* on S for the deformation. Let N and $\overline{N}$ denote these nets. Peterson (2) showed that if a net N′ parallel to N is known, a net $\overline{N}'$ parallel to $\overline{N}$ can be found by quadratures such that N′ and $\overline{N}'$ are applicable. In a former paper (3) the author showed that when two such parallel nets N′ and $\overline{N}'$ are known, two new applicable nets N_1 and $\overline{N}_1$ can be found by a quadrature such that N_1 and $\overline{N}_1$ are T transforms of N and $\overline{N}$ respectively. Subsequently (4) the author applied these results to the case where N is a permanent net in a

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(1) *Transactions of the Amer. Math. Soc.*, vol. XVIII, 1917, p. 97-124. — This paper will be referred to as M_1.

(2) *Ueber Curven und Flächen* (Moskau and Leipzig), 1868, p. 106.

(3) *Transactions of the Amer. Math. Soc.*, vol. XIX, 1918, p. 167-185. — This paper will be referred to as M_2.

(4) *Transactions of the Amer. Math. Soc.*, vol. XX, 1919, p. 323-338. — This paper will be referred to as M_3.

deformation of a quadric, and by making use of the theory of orthogonal nets in higher spaces established the following theorems :

THEOREM I. — *If* $\overline{N}$ *is a net applicable to a net* N *on a quadric* Q, *there exist an infinity of sets of* ∞^2 T *transforms* $\overline{N}_1$ *of* $\overline{N}$ *which are applicable to nets on* Q ; *these transforms are conjugate to* ∞^2 *congruences* G ; *their determination requires the solution of a completely integrable system of eight equations.*

THEOREM II. — *If* $\overline{N}$ *is a net applicable to a net* N *on a central quadric* Q, *not of revolution, there can be found by the solution of a Riccati equation and quadratures three families each of* ∞^2 T *transforms* $\overline{N}_1$ *applicable to nets on* Q ; *the transforms of each family are conjugate to* ∞^1 *congruence* G, *there being* ∞^1 *transforms conjugate to each congruence* G ; *the lines of the congruences* G *through a point of* $\overline{N}$ *form a quadric cone ; the tangent planes at points of a line of* G *to the nets* $\overline{N}_1$ *conjugate to it envelop a quadric cone and the points on* Q *corresponding to these points of the nets* $\overline{N}_1$ *on a line of* G *lie on a conic.*

THEOREM III. — *If* $\overline{N}_1$ *and* $\overline{N}_2$ *are transforms of a net* $\overline{N}$ *applicable to a net* N *on a quadric* Q *by means of transformations* T_{k_1} *and* T_{k_2} $(k_1 \neq_2 k_2)$, *there can be found without quadratures a net* $\overline{N}_{12}$ *applicable to a net* N_{12} *on* Q *which is in the relations of transformations* T'_{k_2} *and* T'_{k_1} *with* N_1 *and* N_2 *respectively.*

It was pointed out that when Q is a central quadric, the transformations T_k of $\overline{N}$ are the transformations discovered by Guichard ([1]) in an entirely different manner. However, this method did not reveal the relations between the nets on Q.

In the first part of the present paper we establish Theorems I and II by a method different from that used in the former paper. Thus in § 4 we determine the transformations T_k of a permanent net N on a

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([1]) *Mémoire sur la déformation des quadriques* (*Mémoires de l'Académie des Sciences*, vol. XXXIV, 1909).

quadric into permanents nets N_1 on the same quadric, and in § 5 show that when such a transformation is known there follows directly a transformation T_k of $\overline{N}$ applicable to N into $\overline{N}_1$ applicable to N_1. These results are developed for all types of quadrics.

In another paper read before the Strasbourg Congress [1] it was shown that certain pairs of solutions of the point equation of an R net N determine a new net $\hat{N}$ such that N and $\hat{N}$ lie on the focal sheets of a W congruence; in this case we called $\hat{N}$ a W *transform of* N. In § 5 it is shown that a net $\overline{N}$ applicable to a net N on a quadric is an R net, and in §§ 6, 7 that certain pairs of functions determining two T_k transforms of N determine also a W transform of N into a net applicable to a net on the given quadric. These transformations are in fact the transformations B_k, discovered by Bianchi [2]. Furthermore it is established in § 8 that the transformations T_k and B_k are permutable, that is, if $\overline{N}_1$ and $\hat{N}$ are respectively T_k and B_k transforms N, there can be found a net $\hat{N}_1$ which is a T_k transform of $\hat{N}$ and a B_k transform of N_1.

In § 9 it is shown that the transformations T_k are permutable also with the transformations H of Bianchi [3].

1. *Transformations* T *of applicable nets.* — If N is a net, the cartesian coordinates, x, y, z, of the net satisfy an equation of the form

$$\frac{\partial^2 \theta}{\partial u\, \partial v} = \frac{\partial \log a}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log b}{\partial u}\frac{\partial \theta}{\partial v}. \tag{1}$$

A net N' is said to be *parallel* to N when its tangents are parallel to the corresponding tangents to N. If x', y', z' are the coordinates

(1) *Conjugate nets* R *and their transformations* (*Annals of Mathematics*, ser. 2, vol. XXII, 1921); also an abstract in the proceedings of the Congress. — This paper will be referred to as M_4.

(2) *Lezioni di geometria differenziale*, vol. III. — This book will be referred to as *B*.

(3) *B.*, p. 214.

of N′, then

$$(2)\qquad \begin{cases} \dfrac{\partial x'}{\partial u} = h\dfrac{\partial x}{\partial u}, & \dfrac{\partial y'}{\partial u} = h\dfrac{\partial y}{\partial u}, & \dfrac{\partial z'}{\partial u} = h\dfrac{\partial z}{\partial u}, \\ \dfrac{\partial x'}{\partial v} = l\dfrac{\partial x}{\partial v}, & \dfrac{\partial y'}{\partial v} = l\dfrac{\partial y}{\partial v}, & \dfrac{\partial z'}{\partial v} = l\dfrac{\partial z}{\partial v}, \end{cases}$$

where h and l are a pair of solutions of

$$(3)\qquad \frac{\partial h}{\partial v} = (l-h)\frac{d\log a}{dv}, \qquad \frac{\partial l}{\partial u} = (h-l)\frac{\partial \log b}{\partial u}.$$

Conversely, every pair of solutions of (3) determines a parallel net N′.

We call equation (1) the *point equation* of N. The coordinates of N′ satisfy a similar equation. Moreover, to each solution θ of (1) there *corresponds* a solution θ' of the point equation of N′. It is determined by

$$(4)\qquad \frac{\partial \theta'}{\partial u} = h\frac{\partial \theta}{\partial u}, \qquad \frac{\partial \theta'}{\partial v} = l\frac{\partial \theta}{\partial v}.$$

If θ and θ' are any pair of *corresponding solutions* of the point equations of N and N′, the functions x_1, y_1, z_1, defined by equations of the form

$$(5)\qquad x_1 = x - \frac{\theta}{\theta'}x'$$

are the cartesian coordinates of a nat N_1 which is a T transform of N. Conversely, the most general transformation T is defined in this way [1]. By means of the above formulas we establish the following relations :

$$(6)\qquad \begin{cases} x_1 - \dfrac{\theta}{\theta'\dfrac{\partial}{\partial u}\left(\dfrac{\theta}{\theta'}\right)}\dfrac{\partial x_1}{\partial u} = x - \dfrac{\theta}{\dfrac{\partial\theta}{\partial u}}\dfrac{\partial x}{\partial u}, \\ x_1 - \dfrac{\theta}{\theta'\dfrac{\partial}{\partial v}\left(\dfrac{\theta}{\theta'}\right)}\dfrac{\partial x_1}{\partial v} = x - \dfrac{\theta}{\dfrac{\partial\theta}{\partial v}}\dfrac{\partial x}{\partial v}. \end{cases}$$

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[1] M_1, p. 109.

Consequently the tangents to the curves $v = \text{const.}$ or $u = \text{const.}$ at corresponding points of N and N_1 meet in the points, P_1 or P_2, whose coordinates are of the above forms. If we take another transformation T given by (5) with θ replaced by another solution θ_1 of (1), we find that the tangent plane to the transform meets the corresponding tangent planes to N and N_1 in the point whose coordinates ξ, η, ζ, are of the form

$$(6') \qquad \xi = x - \frac{\left(\theta_1 \frac{\partial\theta}{\partial v} - \theta \frac{\partial\theta_1}{\partial v}\right)\frac{\partial x}{\partial u} - \left(\theta_1 \frac{\partial\theta}{\partial u} - \theta\frac{\partial\theta_1}{\partial u}\right)\frac{\partial x}{\partial v}}{\frac{\partial\theta}{\partial v}\frac{\partial\theta_1}{\partial u} - \frac{\partial\theta}{\partial u}\frac{\partial\theta_1}{\partial v}}.$$

From the form of this expression it follows that this point lies also on the tangent plane to the transform of N whose coordinates are of the form obtained by replacing θ in (5) by $\theta + c\theta_1$, where c is any constant, and θ' by a solution of the corresponding equations (4). Hence the corresponding tangent planes of the transforms obtained by varying c envelop a cone. Moreover, the point of coordinates ξ, η, ζ generates a *derived net* of N [1].

If a net admits an applicable net, either net is said to be a *permanent* net. If $\overline{N}$ is a net applicable to N, its coordinates, $\overline{x}$, $\overline{y}$, $\overline{z}$, satisfy (1). Moreover, the net $\overline{N}'$ whose coordinates are given by quadratures of the form

$$(7) \qquad \frac{\partial \overline{x}'}{\partial u} = h\frac{\partial \overline{x}}{\partial u}, \qquad \frac{\partial \overline{x}'}{\partial v} = l\frac{\partial \overline{x}}{\partial v},$$

is parallel to $\overline{N}$ and is applicable to N'. Furthermore, the net whose coordinates $\overline{x}_1$, $\overline{y}_1$, $\overline{z}_1$ are defined by equatione of the form

$$(8) \qquad \overline{x}_1 = \overline{x} - \frac{\theta}{\theta'}\overline{x}'_1$$

is a T transform of N [2].

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[1] M_4.

[2] M_2, p. 170.

The common point equation of N' and $\overline{\text{N}}'$ admits the solution

$$(9) \qquad \theta' = k(\Sigma x'^2 - \Sigma \bar{x}'^2),$$

where k is a constant, the symbol Σ indicating the sum of three terms obtained from the three corresponding coordinates. Whe have shown ([1]) that for this value of θ' and the corresponding function θ, given by (4), the nets N_1 and $\overline{N}_1$ are applicable.

2. *Transformations* T *of nets on a quadric.* — Consider a net N on the general quadric, Q, whose equation is

$$(10) \quad ex^2 + fy^2 + gz^2 + 2axy + 2byz + 2czx + 2rx + 2sy + 2tz + w = 0.$$

Since the coordinates are solutions of an equation of the form (1), we have on differentiating (10) with respect to u and v

$$(11) \quad e\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + f\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + g\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + a\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) + b\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} + \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) + c\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) = 0.$$

Any net $N'(x')$ parallel to N is given by equations of the form (2). Consequently we have an equation of the form (11) in which x, y, z, are replaced by x', y', z'. From this it follows that the function

$$(12) \qquad \theta' = ex'^2 + fy'^2 + gz'^2 + 2ax'y' + 2by'z' + 2cz'x'$$

is a solution of the point of equation of N' ([2]). It is readily found that θ' and θ, given by

$$(13) \qquad \theta = 2[exx' + fyy' + gzz' + a(x'y + xy') + b(y'z + yz') + c(z'x + zx') + rx' + sy' + tz']$$

satisfy equations (4).

When these values are substituted in (5), it is found that the T transform $N_1(x_1)$ lies on Q. It can be shown that any congruence conjugate to a net N can be obtained by drawing through points of N

([1]) M_2, p. 170.

([2]) The function $\theta' \neq 0$, since N' cannot lie on a cone.

lines whose direction-parameters are the coordinates of some net parallel to N. Hence we have the theorem of Ribaucour :

Any congruence conjugate to a net on a quadric meets the quadric again in a net to which it is conjugate.

Let $N'(x')$ and $N''(x'')$ be two nets parallel to a net N, and $N_1(x_1)$ and $N_2(x_2)$ the T transform of N determined by the pairs of corresponding functions θ_1, θ'_1 and θ_2, θ''_2, where θ_1 and θ_2 are two solutions of (1). In place of (4) we have

$$(14) \qquad \frac{\partial \theta'_1}{\partial u} = h_1 \frac{\partial \theta_1}{\partial u}, \quad \frac{\partial \theta'_1}{\partial v} = l_1 \frac{\partial \theta_1}{\partial v}; \qquad \frac{\partial \theta''_2}{\partial u} = h_2 \frac{\partial \theta_2}{\partial u}, \quad \frac{\partial \theta''_2}{\partial v} = l_2 \frac{\partial \theta_2}{\partial v}.$$

In addition there are functions θ''_1 and θ'_2, determined to within additive constants by the quadratures

$$(15) \qquad \frac{\partial \theta''_1}{\partial u} = h_2 \frac{\partial \theta_1}{\partial u}, \quad \frac{\partial \theta''_1}{\partial v} = l_2 \frac{\partial \theta_1}{\partial v}; \qquad \frac{\partial \theta'_2}{\partial u} = h_1 \frac{\partial \theta_2}{\partial u}, \quad \frac{\partial \theta'_2}{\partial v} = l_1 \frac{\partial \theta_2}{\partial v},$$

since θ_1 and θ_2 are solution of (1). We have shown [1] that a net N'''_1 of coordinates x'''_1, y'''_1, z'''_1 is given by equations of the form

$$(16) \qquad x'''_1 = x'' - \frac{\theta''_1}{\theta'_1} x'$$

and that N'''_1 is parallel to N_1; also that the functions

$$(17) \qquad \theta_{12} = \theta_2 - \frac{\theta_1}{\theta'_1} \theta'_2, \qquad \theta'''_{12} = \theta''_2 - \frac{\theta''_1}{\theta'_1} \theta'_2$$

are corresponding solutions of the point equations of N_1 and N'''_1. Moreover, we have shown also that the functions x_{12}, y_{12}, z_{12}, of the form

$$(18) \qquad x_{12} = x_1 - \frac{\theta_{12}}{\theta'''_{12}} x'''_1$$

are the coordinates of a net N_{12} which is a T transform of N_1 and also N_2. Since θ''_1 and θ''_2 are determined only to within arbitrary additive constants, there are accordingly ∞^2 such nets N_{12}.

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(1) M_1, p. 111.

We apply these results true for any net to the particular case when N is on the quadric Q, and also N_1 and N_2, that is where θ_1 and θ_2 are of the form (13). In order that θ'''_{12} and θ_{12} be of the form (12) and (13) with x', y', z'; x, y, z replaced by x'''_1, y'''_1, z'''_1; x_1, y_1, z_1 respectively, we must have

$$(19)\qquad \theta''_1 + \theta'_2 = 2[ex'x'' + fy'y'' + gz'z'' + a(x'y'' + x''y') + b(y'z'' + y''z') + c(z'x'' + z''x')].$$

By differentiation it is found that left-hand member of this equation is constant, and consequently the additive constants in θ''_1 and θ'_2 can be chosen in ∞^1 ways so that (19) shall hold. Hence :

If N_1 *and* N_2 *are* T *transforms of* N *and all three nets lie on* Q (9), *there are* ∞^1 *other nets* N_{12} *on* Q *which are* T *transforms of* N_1 *and* N_2; *they can be found by a quadrature.*

5. *Permanent nets on a quadric.* — Servant (¹) has shown that if N is a permanent net on a quadric, Q, the parameters of N can be chosen, so that

$$(20)\qquad D + D'' = -\frac{1}{\sigma}$$

where D and D″ are the second fundamental coefficients of N, and $\sigma^4 = -K$; K being the total curvature of Q; also that the applicable net $\overline{N}$ is isothermal conjugate, and that its second fundamental coefficients, $\overline{D}$ and $\overline{D}''$, are such that

$$(21)\qquad \overline{D} + \overline{D}'' = 0.$$

Since N and $\overline{N}$ are applicable, we have

$$(22)\qquad \sigma^4 = \frac{DD''}{H^2} = \frac{\overline{D}\,\overline{D}''}{H^2}, \qquad H^2 = EG - F^2,$$

where the linear elements of Q and the applicable surface $\overline{S}$, referred

(¹) *Bulletin de la Société mathématique de France*, vol. XXX, 1902, p. 10.

to N and $\overline{\mathrm{N}}$, is written

$$(23)\qquad ds^2 = \mathrm{E}\,du^2 + 2\mathrm{F}\,du\,dv + \mathrm{G}\,dv^2.$$

If we define two functions a and b by means of the equations

$$(24)\qquad \mathrm{D} = -\sigma a^2, \qquad \mathrm{D}'' = \sigma b^2,$$

we have from (20)

$$(25)\qquad a^2 - b^2 = \frac{1}{\sigma^2}.$$

In consequence of (21) and (22), we may take

$$(26)\qquad -\overline{\mathrm{D}} = \overline{\mathrm{D}}'' = \sigma a b, \qquad a^2 b^2 = \mathrm{H}^2\sigma^2.$$

The Codazzi equations for N and $\overline{\mathrm{N}}$ are (¹)

$$(27)\qquad \left\{\begin{aligned} &\frac{\partial \mathrm{D}}{\partial v} = \mathrm{D}\begin{Bmatrix}12\\1\end{Bmatrix} - \mathrm{D}''\begin{Bmatrix}11\\2\end{Bmatrix}, && \frac{\partial \mathrm{D}''}{\partial u} = -\begin{Bmatrix}22\\1\end{Bmatrix}\mathrm{D} + \begin{Bmatrix}12\\2\end{Bmatrix}\mathrm{D}'';\\ &\frac{\partial \overline{\mathrm{D}}}{\partial v} = \overline{\mathrm{D}}\left(\begin{Bmatrix}12\\1\end{Bmatrix} + \begin{Bmatrix}11\\2\end{Bmatrix}\right), && \frac{\partial \overline{\mathrm{D}}}{\partial u} = \overline{\mathrm{D}}\left(\begin{Bmatrix}12\\2\end{Bmatrix} + \begin{Bmatrix}22\\1\end{Bmatrix}\right),\end{aligned}\right.$$

the Christoffel symbols $\begin{Bmatrix}rs\\t\end{Bmatrix}$ being formed with respect to (23).

From these equations, in which D, D″ and $\overline{\mathrm{D}}$ are replaced by their expressions from (24) and (26), and the identities (²) :

$$(28)\qquad \frac{\partial \log \mathrm{H}}{\partial u} = \begin{Bmatrix}11\\1\end{Bmatrix} + \begin{Bmatrix}12\\2\end{Bmatrix}, \qquad \frac{\partial \log \mathrm{H}}{\partial v} = \begin{Bmatrix}22\\2\end{Bmatrix} + \begin{Bmatrix}12\\1\end{Bmatrix},$$

we obtain, in consequence of (25),

$$(29)\qquad \left\{\begin{aligned} &\begin{Bmatrix}11\\1\end{Bmatrix} = \frac{\partial u}{\partial}\log\frac{a}{\sigma}, && \begin{Bmatrix}12\\1\end{Bmatrix} = \frac{\partial}{\partial v}\log a, && \begin{Bmatrix}22\\1\end{Bmatrix} = \frac{b^2}{a^2}\frac{\partial}{\partial u}\log b\sigma,\\ &\begin{Bmatrix}11\\2\end{Bmatrix} = \frac{a^2}{b^2}\frac{\partial}{\partial v}\log a\sigma, && \begin{Bmatrix}12\\2\end{Bmatrix} = \frac{\partial}{\partial u}\log b, && \begin{Bmatrix}22\\2\end{Bmatrix} = \frac{\partial}{\partial v}\log\frac{b}{\sigma}.\end{aligned}\right.$$

(¹) *E.*, p. 155. — A reference of this sort is to the author's Differential Geometry.

(²) *E.*, p. 153.

If $\overline{x}$, $\overline{y}$, $\overline{z}$ denote the cartesian coordinates of $\overline{S}$, we have ([1])

$$(30)\quad \left\{ \begin{aligned} \frac{\partial^2 \overline{x}}{\partial u^2} &= \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} \frac{\partial \overline{x}}{\partial u} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \frac{\partial \overline{x}}{\partial v} + \overline{D} X, \\ \frac{\partial^2 \overline{x}}{\partial u \, \partial v} &= \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \frac{\partial \overline{x}}{\partial u} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \frac{\partial \overline{x}}{\partial v}, \\ \frac{\partial^2 \overline{x}}{\partial v^2} &= \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} \frac{\partial \overline{x}}{\partial u} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} \frac{\partial \overline{x}}{\partial v} + \overline{D}'' X, \end{aligned} \right.$$

and similar expressions in $\overline{y}$ and $\overline{z}$. In consequence of (21) and (29), we have that $\overline{x}$, $\overline{y}$, $\overline{z}$ are solutions of the equations

$$(31)\quad \left\{ \begin{aligned} \frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} &= 2 \frac{\partial \log a}{\partial u} \frac{\partial \theta}{\partial u} + 2 \frac{\partial \log b}{\partial v} \frac{\partial \theta}{\partial v}, \\ \frac{\partial^2 \theta}{\partial u \, \partial v} &= \frac{\partial \log a}{\partial v} \frac{\partial \theta}{\partial u} + \frac{\partial \log b}{\partial u} \frac{\partial \theta}{\partial v}. \end{aligned} \right.$$

We have seen ([2]) that when the coordinates of a net satisfy equations (31), it is an R net that is the tangente to the curves of either family of the net form a W congruence. Hence :

The permanent net on a deform of a quadric is an R *net* ([3]).

For the central quadric Q

$$(32)\quad e x^2 + f y^2 + g z^2 = 1$$

we have

$$(33)\quad X, Y, Z = \frac{e x, f y, g z}{\sqrt{\Sigma e^2 x^2}},$$

and the gaussian curvature is given by ([4])

$$(34)\quad K = -\frac{1}{c^4 (\Sigma e^2 x^2)^2}, \qquad \frac{1}{\sigma} = c \sqrt{\Sigma e^2 x^2},$$

where $c^4 = -\dfrac{1}{efg}$.

([1]) *E.*, p. 154.

([2]) M_4.

([3]) *Cf.* Tzitzeica, *Comptes rendus*, t. 152, 1911, p. 1077; also Bianchi, *Rendiconti dei Lincei*, 5e série, vol. XXII, 1913, p. 3.

([4]) *Cf.* C. Smith, *Solid Geometry*, 9th edition, p. 123.

If Q is referred to a net N whose point equation is (1), we have

$$(35)\quad \frac{\partial}{\partial v}\log\left[\Sigma e\left(\frac{\partial x}{\partial u}\right)^2\right]=\frac{\partial}{\partial v}\log a^2,\qquad \frac{\partial}{\partial u}\log\left[\Sigma e\left(\frac{\partial x}{\partial v}\right)^2\right]=\frac{\partial}{\partial u}\log b^2,$$

and also

$$(36)\qquad \Sigma e\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}=0.$$

Hence

$$(37)\qquad \Sigma e\left(\frac{\partial x}{\partial u}\right)^2=a^2\mathrm{U},\qquad \Sigma e\left(\frac{\partial x}{\partial v}\right)^2=b^2\mathrm{V},$$

where U and V are functions of u and v alone.

In consequence of (24), (29), and equations analogous to (30) we find

$$(38)\qquad \left\{\begin{aligned}\frac{\partial}{\partial u}\left[\Sigma e\left(\frac{\partial x}{\partial u}\right)^2\right]&=\frac{\partial}{\partial u}\log\frac{a^2}{\sigma^2}\cdot\Sigma e\left(\frac{\partial x}{\partial u}\right)^2-2\sigma a^2\Sigma e\mathrm{X}\frac{\partial x}{\partial u},\\ \frac{\partial}{\partial v}\left[\Sigma e\left(\frac{\partial x}{\partial v}\right)^2\right]&=\frac{\partial}{\partial v}\log\frac{b^2}{\sigma^2}\cdot\Sigma e\left(\frac{\partial x}{\partial v}\right)^2-2\sigma b^2\Sigma e\mathrm{X}\frac{\partial x}{\partial v}.\end{aligned}\right.$$

From (33) we have

$$\Sigma e\mathrm{X}\frac{\partial x}{\partial u}=\frac{\partial}{\partial u}\sqrt{\Sigma e^2x^2},\qquad \Sigma e\mathrm{X}\frac{\partial x}{\partial v}=\frac{\partial}{\partial v}\sqrt{\Sigma e^2x^2}.$$

When the expressions (37) are substituted in (38), the result is reducible by means of (34) to

$$\frac{1}{2}\frac{\partial\mathrm{U}}{\partial u}=\left(\mathrm{U}-\frac{1}{c}\right)\frac{\partial\log\sigma}{\partial u},\qquad \frac{1}{2}\frac{\partial\mathrm{V}}{\partial v}=\left(\mathrm{V}+\frac{1}{c}\right)\frac{\partial\log\sigma}{\partial v}.$$

Also on differentiating equations (36) with respect to u and v we get

$$(\mathrm{U}+\mathrm{V})\frac{\partial\log a}{\partial v}+\left(\mathrm{V}+\frac{1}{c}\right)\frac{\partial\log\sigma}{\partial v}=0,$$

$$(\mathrm{U}+\mathrm{V})\frac{\partial\log b}{\partial u}+\left(\mathrm{U}-\frac{1}{c}\right)\frac{\partial\log\sigma}{\partial u}=0.$$

From these two sets of equations it follows that $\mathrm{U}=-\mathrm{V}=\frac{1}{c}$, and consequently

$$(39) \qquad \Sigma e\left(\frac{\partial x}{\partial u}\right)^2 = \frac{a^2}{c}, \qquad \Sigma e\left(\frac{\partial x}{\partial v}\right)^2 = -\frac{b^2}{c}.$$

Hence from (25) and (34) we have the theorem :

For any permanent net on a central quadric (32) *the coordinates satisfy the condition*

$$(40) \qquad \Sigma e\left(\frac{\partial x}{\partial u}\right)^2 + \Sigma e\left(\frac{\partial x}{\partial v}\right)^2 = c\,\Sigma e^2 x^2,$$

where $c^1 = -\frac{1}{efg}$.

This equation may be written

$$(41) \qquad a^2 - b^2 = c^2\,\Sigma e^2 x^2.$$

For the paraboloid P

$$(42) \qquad ex^2 + fy^2 + 2z = 0$$

we have

$$(43) \qquad X, Y, Z = \frac{ex,\ fy,\ 1}{\sqrt{e^2x^2 + f^2y^2 + 1}},$$

and

$$(44) \qquad K = -\frac{1}{c^4(e^2x^2 + f^2y^2 + 1)^2}, \qquad \frac{1}{\sigma} = c\sqrt{e^2x^2 + f^2y^2 + 1},$$

where $c = -\frac{1}{4ef}$.

If P is referred to a net N whose point equation is (1), we have

$$e\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + f\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} = 0,$$

and by processes analogous to those used in the case of (32) we find

$$(45) \qquad e\left(\frac{\partial x}{\partial u}\right)^2 + f\left(\frac{\partial y}{\partial u}\right)^2 = \frac{a^2}{c}, \qquad e\left(\frac{\partial x}{\partial v}\right)^2 + f\left(\frac{\partial y}{\partial v}\right)^2 = -\frac{b^2}{c}.$$

Hence we have the theorem :

For any net on the paraboloid (42) *the coordinates satisfy*

$$(46)\qquad e\left(\frac{\partial x}{\partial u}\right)^2+f\left(\frac{\partial y}{\partial u}\right)^2+e\left(\frac{\partial x}{\partial v}\right)^2+f\left(\frac{\partial y}{\partial v}\right)^2=c(e^2x^2+f^2y^2+1)$$

where $c=\frac{1}{4ef}$.

This equation may be written

$$(47)\qquad a^2-b^2=c^2(e^2x^2+f^2y^2+1).$$

Conversely, we can establish the theorem :

If the coordinates of a net on the central quadric (32) *satisfy* (40), *or on the paraboloid* (42) *satisfy* (46), *the net is permanent net.*

4. *Transformations* T *of permanent nets on a quadric.* — Let N be a permanent net on the quadric Q (32). From (12) and (13) it follows that if in the equations (5) we take

$$(48)\qquad \theta=2(exx'+fyy'+gzz')\equiv 2\sum exx',\qquad \theta'=\sum ex'^2,$$

the T transform N_1 of N lies on Q. In order that N_1 be a permanent net, it is necessary and sufficient that

$$(49)\qquad \sum e\left(\frac{\partial x_1}{\partial u}\right)^2+\sum e\left(\frac{\partial x_1}{\partial v}\right)^2=c^2\sum e^2x_1^2.$$

From (5) we have by differentiation

$$(50)\qquad \frac{\partial x_1}{\partial u}=\frac{\tau}{\theta'^2}\left(x'\frac{\partial\theta}{\partial u}-\theta'\frac{\partial x}{\partial u}\right),\qquad \frac{\partial x_1}{\partial v}=\frac{\sigma}{\theta'^2}\left(x'\frac{\partial\theta}{\partial v}-\theta'\frac{\partial x}{\partial v}\right)$$

where

$$(51)\qquad \tau=h\theta-\theta',\qquad \sigma=l\theta-\theta'.$$

Substituting these expressions in (49), we can reduce the resulting equation by means of (39) to

$$(52)\qquad \tau^2a^2-\sigma^2b^2=c^2\theta'^2\sum e^2x_1^2.$$

In consequence of (3) we have from (51)

$$(53)\qquad \begin{cases} \dfrac{\partial}{\partial v}\left(\dfrac{\tau}{\theta'}\right) = \dfrac{\sigma-\tau}{\theta'}\,\dfrac{\partial}{\partial v}\log a - \dfrac{h\sigma}{\theta'^2}\,\dfrac{\partial\theta}{\partial v}, \\ \dfrac{\partial}{\partial u}\left(\dfrac{\sigma}{\theta'}\right) = \dfrac{\tau-\sigma}{\theta'}\,\dfrac{\partial}{\partial u}\log b - \dfrac{l\tau}{\theta'^2}\,\dfrac{\partial\theta}{\partial u}. \end{cases}$$

Differentiating (52) and making use of (53), the resulting equations are reducible to

$$\frac{\partial\varphi}{\partial u} + \frac{1}{\theta}\frac{\partial\theta}{\partial u}\left[-\frac{b^2}{\theta'^2}\,l\theta(\tau-\sigma) + c^2\Sigma e^2 x_1(x_1-x) + c^2\frac{\tau}{\theta'}\Sigma e^2 x(x_1-x)\right] = 0,$$

$$\frac{\partial\varphi}{\partial v} + \frac{1}{\theta}\frac{\partial\theta}{\partial v}\left[\quad\frac{a^2}{\theta'^2}\,h\theta(\sigma-\tau) + c^2\Sigma e^2 x_1(x_1-x) + c^2\frac{\sigma}{\theta'}\Sigma e^2 x(x_1-x)\right] = 0,$$

where

$$(54)\qquad \varphi = \frac{a^2\tau - b^2\sigma}{\theta'} + c^2\Sigma e^2 x x_1.$$

By means of (5), (41), (51), (52) and (54) these equations can be given the form

$$(55)\qquad \frac{\partial\varphi}{\partial u} + \frac{\varphi}{\theta}\frac{\partial\theta}{\partial u}\left(\frac{h\theta}{\theta'} - 2\right) = 0, \qquad \frac{\partial\varphi}{\partial v} + \frac{\varphi}{\theta}\frac{\partial\varphi}{\partial v}\left(\frac{l\theta}{\theta'} - 2\right) = 0$$

which can be integrated in the form

$$(56)\qquad \varphi\theta' = -\frac{kc^2}{2}\theta^2,$$

where k is any constant. When this value for φ is substituted in (54), this equation and (52) are equivalent, in consequence of (5) and (51), to

$$(57)\qquad \begin{cases} h^2a^2 - l^2b^2 = c^2\Sigma(e^2 - ke)x'^2, \\ h\ a^2 - l\ b^2 = c^2\Sigma(e^2 - ke)xx'. \end{cases}$$

Differentiating the second of these equations, we obtain

$$(58)\qquad \begin{cases} \dfrac{\partial h}{\partial u} + \dfrac{\partial \log a}{\partial u}h - \dfrac{b^2}{a^2}\,\dfrac{\partial\log b}{\partial u}\,l - \dfrac{c^2}{a^2}\Sigma(e^2 - ke)x'\dfrac{\partial x}{\partial u} = 0, \\ \dfrac{\partial l}{\partial v} + \dfrac{\partial \log b}{\partial v}l - \dfrac{a^2}{b^2}\,\dfrac{\partial\log a}{\partial v}\,h + \dfrac{c^2}{b^2}\Sigma(e^2 - ke)x'\dfrac{\partial x}{\partial v} = 0, \end{cases}$$

in consequence of (2).

It is readily found that equations (2), (3) and (58) form a completely integrable system, in consequence of (41). Moreover, for every set of solutions of this system equations (57) are satisfied to within additive constants, as is found by differentiation. Hence each set of solutions satisfying (57) determines a T transform which is a permamanent net. From (2), (3) and (5) it is seen that if x', y', z', h and l are multiplied by the same constant, the transform N_1 is unaltered. Hence when k is any constant different from e, f and g, there are ∞^2 sets of solutions satisfying (57) and giving distinct transforms.

When $k=e$, there are ∞^1 sets of solution y', z', h and l of (2), (3) and (58) satisfying (57). Then x' is given by the quadratures (2) and involves an additive constant, say m. In this case each set of solution y', z', h and l determines ∞^1 transformations, such that the corresponding points of the ∞^1 transforms lie on a conic, the section of quadric by a plane parallel to the lines from the origin to the points $(x+m, y, z)$ as m varies. Similar results hold when $k=f$ or $k=g$. Hence :

A permanent net on a central quadric $ex^2+fy^2+gz^2=1$ *admits* ∞^2 *transformations* T_k *into permanent nets on the quadric for each value of the constant* k; *when* k *is equal to* e, f *or* g, *the transforms* N_1 *may be grouped into* ∞^1 *families of* ∞^2 *transforms each such that corresponding points of the nets of a family lie on a conic.*

When N is a permanent net on the paraboloid P (42), we have as the equations analogous to (48), (52) and (55) the following :

$$\theta=2(exx'+fyy'+z'),\qquad \theta'=ex'^2+fy'^2, \tag{59}$$

$$\tau^2a^2-\sigma^2b^2=c^2\theta'^2(e^2x_1^2+f^2y_1^2+1), \tag{60}$$

$$\varphi=\frac{\tau}{\theta'}a^2-\frac{\sigma}{\theta'}b^2+c^2(e^2xx_1+f^2yy_1+1); \tag{61}$$

and (55) also. Now (60) and (61) are equivalent to

$$\left\{\begin{aligned} h^2a^2-l^2b^2&=c^2[(e^2-ke)x'^2+(f^2-kf)y'^2],\\ h\,a^2-l\,b^2&=c^2[(e^2-ke)xx'+(f^2-kf)yy'-kz'].\end{aligned}\right. \tag{62}$$

In this case we have the completely integrable system consisting of (2), (3) and

$$(63)\quad \begin{cases} \dfrac{\partial h}{\partial u}+\dfrac{\partial \log a}{\partial u}h-\dfrac{b^2}{a^2}\dfrac{\partial \log b}{\partial u}l-\dfrac{c^2}{a^2}\left[(e^2-ke)x'\dfrac{\partial x}{\partial u}+(f^2-kf)y'\dfrac{\partial y}{\partial u}\right]=0,\\ \dfrac{\partial l}{\partial v}+\dfrac{\partial \log b}{\partial v}l-\dfrac{a^2}{b^2}\dfrac{\partial \log a}{\partial v}h+\dfrac{c^2}{a^2}\left[(e^2-ke)x'\dfrac{\partial x}{\partial v}+(f^2-kf)y'\dfrac{\partial y}{\partial v}\right]=0. \end{cases}$$

From these results follows the theorem :

A permanent net on a paraboloid $ex^2+fy^2+2z=0$ *admits* ∞^2 *transformations* T_k *into permanent nets on the paraboloid for each value of the constant* k; *when* k *is equal to* e *or* f, *the transforms* N_1 *may be grouped into* ∞^1 *families of* ∞^1 *transforms such that corresponding points of the nets of a family lie on a conic.*

5. *Transformations* T *of surfaces applicable to a quadric.* — It is our purpose to show that each transformation T_k of a permanent net N on a quadric Q into a permanent net N_1 on Q leads directly to a transformation T_k of the net $\overline{N}$ applicable to N into the net $\overline{N}_1$ applicable to N_1. In fact, we shall show that it is possible to find without quadratures a net $\overline{N}'$ parallel to $\overline{N}$ such that θ' given by (48) can be put in the form (9), and then the desired transform defined by (8).

Equating these expressions for θ', we have

$$(64)\qquad (e-k)x'^2+(f-k)y'^2+(g-k)z'^2+k\Sigma\overline{x}'^2=0.$$

Differentiating this expression and assuming that equations of the form (7) hold, we obtain

$$(65)\quad \begin{cases} (e-k)x'\dfrac{\partial x}{\partial u}+(f-k)y'\dfrac{\partial y}{\partial u}+(g-k)z'\dfrac{\partial z}{\partial u}+k\Sigma\overline{x}'\dfrac{\partial \overline{x}}{\partial u}=0,\\ (e-k)x'\dfrac{\partial x}{\partial v}+(f-k)y'\dfrac{\partial y}{\partial v}+(g-k)z'\dfrac{\partial z}{\partial v}+k\Sigma\overline{x}'\dfrac{\partial \overline{x}}{\partial v}=0. \end{cases}$$

If these equations are differentiated with respect to u and v and in the reduction use is made of (30) and analogous equations for $\overline{N}$, two of the resulting equations are satisfied identically in consequence of (65) and the other two are reducible to

$$h\Sigma e\left(\frac{\partial x}{\partial u}\right)^2 + \mathrm{D}.\Sigma(e-k)x'\mathrm{X} + k\overline{\mathrm{D}}\ \Sigma\overline{x}'\overline{\mathrm{X}} = 0,$$

$$l\,\Sigma e\left(\frac{\partial x}{\partial v}\right)^2 + \mathrm{D}''\Sigma(e-k)x'\mathrm{X} + k\overline{\mathrm{D}}''\Sigma\overline{x}'\overline{\mathrm{X}} = 0.$$

In consequence of (24), (26), (33), (39) and the second of (57) these two equations are equivalent to

(66) $$(h-l)\sigma ab + kc\Sigma\overline{x}'\mathrm{X} = 0.$$

Solving equations (65) and (66) for $\overline{x}'$, $\overline{y}'$, $\overline{z}'$, we have expressions of the form

(67) $$k\overline{x}' = \frac{\sigma ab}{c}(l-h)\overline{\mathrm{X}} + \frac{1}{\mathrm{H}^2}\left[\frac{\partial\overline{x}}{\partial u}\Sigma(e-k)x'\left(\mathrm{F}\frac{\partial x}{\partial v} - \mathrm{G}\frac{\partial x}{\partial u}\right) + \frac{\partial\overline{x}}{\partial v}\Sigma(e-k)x'\left(\mathrm{F}\frac{\partial x}{\partial u} - \mathrm{E}\frac{\partial x}{\partial v}\right)\right].$$

If we differentiate these expressions, we find that $\overline{x}'$, $\overline{y}'$, $\overline{z}'$ satisfy equations (7), by making use of equations of §§ **3**, **4** and of § **65**, *E.*, p. 152. Also from (67) we have

(68) $$\begin{aligned} k^2\Sigma\overline{x}'^2 &= \frac{\sigma^2a^2b^2}{c^2}(l-h)^2 \\ &\quad + \frac{1}{\mathrm{H}^2}\left\{\left[\Sigma(e-k)x'\frac{\partial x}{\partial v}\right]^2\mathrm{E} + \left[\Sigma(e-k)x'\frac{\partial x}{\partial u}\right]^2\mathrm{G} \right. \\ &\quad \left. - 2\mathrm{F}\left[\Sigma(e-k)x'\frac{\partial x}{\partial u}\right]\left[\Sigma(e-k)x'\frac{\partial x}{\partial v}\right]\right\} \\ &= \frac{\sigma^2a^2b^2}{c^2}(l-h)^2 + \Sigma(e-k)^2x'^2 - [\Sigma(e-k)x'\mathrm{X}]^2. \end{aligned}$$

Substituting this expression in (64), we find that it is satisfied in virtue of the first of (57).

From these results and the first theorem of § **4** follows Theorem I of the introduction when k is not equal to 0, e, f or g.

When $k=e$, the function x' is determined only to within an additive constant m, as seen in § **4**. There are only ∞^1 sets of solutions y', z', h and l, and consequently from (67) it follows that there are only ∞^1 congruences G of the ∞^2 transformations. As m varies we obtain ∞^1 transforms N_1 conjugate to the same congruence. They are defined by (8) where

$$\theta = 2\Sigma e.xx' + 2em\,x, \qquad \theta' = e(x'+m)^2 + fy'^2 + gz'^2.$$

From this expression for θ and the results of § 1 it follows that the tangent planes at corresponding points of these nets N_1 envelope a cone. If ξ_0, η_0, ζ_0 are the coordinates of the vertex, the equation of the tangent plane is

$$(\xi - \xi_0)X_1 + (\eta - \eta_0)Y_1 + (\zeta - \zeta_0)Z_1 = 0,$$

where ξ, η and ζ are current coordinates, and X_1, Y_1 and Z_1 are direction-parameters of the normal to N_1. When their expressions are calculated it is found that they involve m to the second degree, and consequently the cone is quadric.

When $k = e$, x' does not appear in (64) and (65). Solving the latter for y' and z', and substituting in (64), we obtain a homogeneous quadric equation in $\overline{x}'$, $\overline{y}'$, $\overline{z}'$. Hence the lines of the congruences G through a point of N form a quadric cone. Since similar results hold when k is equal to f or g, we have in consequence of the first theorem of § 4, Theorem II of the introduction.

When Q is the quadric of revolution $e(x^2+y^2)+gz^2=1$, the transformations of Theorem I hold, as in the case of the general quadric. There is only one set of transformations of the type described in Theorem II; they are T_g. The transformations T_e possess the following properties [1].

Let $\overline{N}$ *be a net applicable to a net* N *on a central quadric of revolution* Q; *the lines joining points of* N *to the foci of* Q *on the axis of revolution become lines of two normal congruences* G_1 *and* G_2 *conjugate to* $\overline{N}$, *when* N *is applied to* $\overline{N}$; *there can be found by two quadratures* ∞^2 *nets* $\overline{N}_1$ *conjugate to* G_1 *and* ∞^2 *nets* $\overline{N}_2$ *conjugate to* G_2 *which are applicable to* ∞^2 *nets* N_1 *and* ∞^2 *nets* N_2 *respectively on* Q; *the nets* $\overline{N}_1$, *or* $\overline{N}_2$, *can be grouped into* ∞^1

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(1) M_3, p. 337.

families of ∞^1 nets each such that their tangent planes at points on the same line of G_1, *or* G_2, *envelop a quadric cone and the corresponding points of the applicable nets* N_1, *or* N_2, *on* Q *lie on a conic.*

When the quadric is a sphere, real or imaginary, with the equation $a(x^2+y^2+z^2)=1$, there are no transformations of the type of Theorem II, but Theorem I holds. The nets $\overline{N}$ and $\overline{N}_1$ consist of the lines of curvature of surfaces of constant total curvature, and $\overline{N}$ and $\overline{N}_1$ are on the sheets of the enveloppe of a two parameter family of spheres ([1]).

When the quadric is a paraboloid with the equation (42), we have from (9) and (59) in place of (64) the equation

$$(e-k)x'^2+(f-k)y'^2-kz'^2+k\Sigma x'^2=0. \tag{69}$$

Proceeding as in the case of (64), we find equations obtained from (65), (67) and (68) by putting $g=0$. Hence we find that Theorem I holds for deforms of P, and that there are only two sets of transformations of the type of Theorem II, namely F_e and F_f.

When $e=f$, that is when P is a paraboloid of revolution, the transformations T_e possess the following properties ([2]).

If $\overline{N}$ *is a net applicable to a net* N *on a paraboloid of revolution* P, *the lines from points of* N *to the focus of* P *on its axis and the normals to the tangent plane to* P *at its vertex become lines of two normal congruences,* G_1 *and* G_2, *conjugate* $\overline{N}$ *when* N *is applied to* $\overline{N}$; *there can be found by two quadratures* ∞^2 *nets* $\overline{N}_1$ *conjugate to* G_1, *and* ∞^2 *nets* $\overline{N}_2$ *conjugate to* G_2 *which are applicable to* ∞^2 *nets* N_1 *and* ∞^2 *nets* N_2 *respectively on* P; *the nets* $\overline{N}_1$, *or* $\overline{N}_2$, *can be grouped into* ∞^1 *families of* ∞^1 *nets each such that their tangent planes at points on the same line of* G_1, *or* G_2, *envelop a quadric cone and the corresponding points of* N_1, *or* N_2, *on* P *lie on a conic.*

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(1) M_3, p. 337.
(2) M_3, p. 338.

6. W *transforms of surfaces applicable to a central quadric.* — Let $\overline{N}(\overline{x})$ be a net applicable to a net $N(x)$ on the central quadric (32). We consider a T_k transformation of $\overline{N}(\overline{x})$ and $N(x)$ as treated in § 5. We have

(70) $$\theta = 2\Sigma e x x'.$$

From (33) and (34) we have

(71) $$\Sigma e x' X = c\sigma \Sigma e^2 x x'.$$

By differentiation of (70) we have

(72) $$\frac{\partial\theta}{\partial u} = 2\Sigma e x' \frac{\partial x}{\partial u}, \qquad \frac{\partial\theta}{\partial v} = 2\Sigma e x' \frac{\partial x}{\partial v},$$

and with the aid of (30), (39) and (57) we find

(73) $$\left\{\begin{aligned} \frac{\partial^2\theta}{\partial u^2} &= \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} \frac{\partial\theta}{\partial u} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \frac{\partial\theta}{\partial v} - ck\sigma^2 a^2 \theta + \frac{2\sigma^2 a^2 b^2}{c}(l-h), \\ \frac{\partial^2\theta}{\partial v^2} &= \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} \frac{\partial\theta}{\partial u} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} \frac{\partial\theta}{\partial v} + ck\sigma^2 b^2 \theta + \frac{2\sigma^2 a^2 b^2}{c}(h-l). \end{aligned}\right.$$

Because of (25) and (29) we have

(74) $$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = 2\frac{\partial \log a}{\partial u}\frac{\partial\theta}{\partial u} + 2\frac{\partial \log b}{\partial v}\frac{\partial\theta}{\partial v} - ck\theta.$$

From the results of a previous paper ([1]) it follows that if we take two transformations of $\overline{N}(\overline{x})$ of this type and write

(75) $$\theta_1 = 2\Sigma e x x', \qquad \theta_2 = 2\Sigma e x x''$$

the net $\overline{N}(\overline{\xi})$ defined by equations of the form

(76) $$\overline{\xi} = \overline{x} + p\frac{\partial \overline{x}}{\partial u} + q\frac{\partial \overline{x}}{\partial v},$$

where

(77) $$\left\{\begin{aligned} &p = \frac{1}{\Delta}\left(\theta_1 \frac{\partial\theta_2}{\partial v} - \theta_2 \frac{\partial\theta_1}{\partial v}\right), \qquad q = \frac{1}{\Delta}\left(\theta_2 \frac{\partial\theta_1}{\partial u} - \theta_1 \frac{\partial\theta_2}{\partial u}\right), \\ &\Delta = \frac{\partial\theta_2}{\partial u}\frac{\partial\theta_1}{\partial v} - \frac{\partial\theta_2}{\partial v}\frac{\partial\theta_1}{\partial u}, \end{aligned}\right.$$

([1]) M_4.

is a W transform of $\overline{N}(\overline{x})$, that is these two nets are on the focal surfaces of a congruence. It is our purpose to show that $\overline{N}(\overline{\xi})$ is applicable to a net on Q.

On substituting the expressions for the derivatives of θ_1 and θ_2 analogous to (72) and (73) in the derivatives of $\overline{\xi}$ we reduce the resulting expressions to

$$(78)\quad \left\{\begin{aligned} \frac{\partial\overline{\xi}}{\partial u} &= p\Big\{\overline{D}\,\overline{X} + \frac{2\sigma^2 a^2}{c\Delta}\Big[\left(\frac{\partial\overline{x}}{\partial u}\frac{\partial\theta_2}{\partial v} - \frac{\partial\overline{x}}{\partial v}\frac{\partial\theta_2}{\partial u}\right)\left(b^2(l_1-h_1) - c^2k\frac{\theta_1}{2}\right) \\ &\qquad - \left(\frac{\partial\overline{x}}{\partial u}\frac{\partial\theta_1}{\partial v} - \frac{\partial\overline{x}}{\partial v}\frac{\partial\theta_1}{\partial u}\right)\left(b^2(l_2-h_2) - c^2k\frac{\theta_2}{2}\right)\Big]\Big\}, \\ \frac{\partial\overline{\xi}}{\partial v} &= q\Big\{\overline{D}''\overline{X} - \frac{2\sigma^2\beta^2}{c\Delta}\Big[\left(\frac{\partial\overline{x}}{\partial u}\frac{\partial\theta_2}{\partial v} - \frac{\partial\overline{x}}{\partial v}\frac{\partial\theta_2}{\partial u}\right)\left(a^2(l_1-h_1) - c^2k\frac{\theta_1}{2}\right) \\ &\qquad - \left(\frac{\partial\overline{x}}{\partial u}\frac{\partial\theta_1}{\partial v} - \frac{\partial\overline{x}}{\partial v}\frac{\partial\theta_1}{\partial u}\right)\left(a^2(l_2-h_2) - c^2k\frac{\theta_2}{2}\right)\Big]\Big\}. \end{aligned}\right.$$

By means of the same functions θ_1 and θ_2 we obtain a derived net $N(\xi)$ of $N(x)$. Its equations are of the form

$$(79)\qquad \xi = x + p\frac{\partial x}{\partial u} + q\frac{\partial x}{\partial v},$$

where p and q are given by (77). On substituting the expressions for θ_1 and θ_2, as given by (75) in the expression (77) for Δ, we find, in consequence of (33) and (*cf.* 26)

$$(80)\qquad H\sigma = ab$$

that

$$(81)\quad \Delta = 4\Sigma ef(x''y' - x'y'')\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) = 4H\Sigma ef(x''y' - x'y'')Z$$
$$= 4cabefg\Sigma x(y''z' - y'z'').$$

Hence the expression for ξ is reducible to

$$(82)\qquad \xi = \frac{y'z'' - y''z'}{e\Sigma x(y'z'' - y''z')}.$$

From this we have

$$(83)\qquad \Sigma e\xi x = 1.$$

The equations analogous to (57) are

$$(84)\quad \begin{cases} h_1^2 a^2 - l_1^2 b^2 = c^2 \Sigma(e^2 - ke)x'^2, & h_2^2 a^2 - l_2^2 b^2 = c^2 \Sigma(e^2 - ke)x''^2, \\ h_1 a^2 - l_1 b^2 = c^2 \Sigma(e^2 - ke)xx', & h_2 a^2 - l^2 b^2 = c^2 \Sigma(e^2 - ke)xx''. \end{cases}$$

By differentiation it can be shown that the left-hand member of the following equation is constant for any two transformations T_k :

$$(85)\qquad h_1 h_2 a^2 - l_1 l_2 b^2 - c^2 \Sigma(e^2 - ke)x'x'' = 0.$$

We choose the nets N′ and N″ so that (85) is satisfied. By means of these relations we show that

$$\begin{aligned}(86)\qquad & \Sigma(f^2 - kf)(g^2 - kg)(y'z'' - y''z') \\ & = \Sigma(e^2 - ke)x'^2 \Sigma(e^2 - ke)x''^2 - (\Sigma(e^2 - ke)x'x'')^2 \\ & = -\frac{a^2 b^2}{c^4}(l_1 h_2 - l_2 h_1)^2 ;\end{aligned}$$

also that

$$(87)\quad (e^2 - ke)(f^2 - kf)(g^2 - kg)[\Sigma x(y'z'' - y''z')]^2 = \frac{ka^2 b^2}{c^4}(l_1 h_2 - l_2 k_1)^2.$$

In consequence of these identities we have

$$(88)\qquad \sum \frac{\xi^2}{\frac{1}{e} - \frac{1}{k}} = 1,$$

that is N (ξ) lies on a quadric Q_k confocal with Q.

The equations for N(ξ) analogous to (78) are obtained from (78) by removing the bars form $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{D}$, $\bar{D}''$ and $\bar{X}$. Substituting from (75) the expressions for the derivatives of θ_1 and θ_2 and making use of (24), (33), (34) and (80), we reduce the resulting equations to the form

$$\begin{aligned}\frac{\partial \xi}{\partial u} = \frac{\sigma^2 a^2 p}{\Delta}\Big\{ -c\Delta e x + 4abfg\Big[& (zy'' - z''y)\left(b^2(l_1 - h_1) - c^2 k \frac{\theta_1}{2}\right) \\ & - (zy' - z'y)\left(b^2(l_2 - h_2) - c^2 k \frac{\theta_2}{2}\right)\Big]\Big\},\end{aligned}$$

$$\begin{aligned}\frac{\partial \xi}{\partial u} = \frac{\sigma^2 b^2 q}{\Delta}\Big\{ \quad c\Delta e x - 4abfg\Big[& (zy'' - z''y)\left(a^2(l_1 - h_1) - c^2 k \frac{\theta_1}{2}\right) \\ & - (zy' - z'y)\left(a^2(l_2 - h_2) - c^2 k \frac{\theta_2}{2}\right)\Big]\Big\}.\end{aligned}$$

By means of (75) and (81) these are reducible to

$$(89)\quad \begin{cases} \dfrac{\partial\xi}{\partial u} = \dfrac{4\sigma^2 a^3 bpfg}{\Delta}\big\{ c^2(e^2-ke)x\Sigma x(y'z''-y''z') + c^2k(y'z''-y''z') \\ \qquad + b^2[(zy''-z''y)(l_1-h_1)-(zy'-z'y)(l_2-h_2)]\big\}, \\ \dfrac{\partial\xi}{\partial v} = -\dfrac{4\sigma^2 ab^3 qfg}{\Delta}\big\{ c^2(e^2-ke)x\Sigma x(y'z''-y''z') + c^2k(y'z''-y''z') \\ \qquad + a^2[(zy''-z''y)(l_1-h_1)-(zy'-z'y)(l_2-h_2)]\big\}. \end{cases}$$

From (84) we have

$$\begin{aligned}(zy''-z''y)(h_1a^2-l_1b^2) &- (zy'-z'y)(h_2a^2-l_2b^2) \\ &= c^2\big\{(e^2-ke)x\Sigma x(y'z''-y''z') + (z'y''-z''y')\Sigma(e^2-ke)x^2\big\} \\ &= c^2\left\{(e^2-ke)x\Sigma x(y'z''-y''z') + (z'y''-z''y')\left(\frac{1}{c^2\sigma^2}-1\right)\right\}.\end{aligned}$$

Adding the left-hand member of this expression to the expressions in parentheses in (89) and subtracting the right-hand member, we get in consequence *of* (25)

$$(90)\quad \begin{cases} \dfrac{\partial\xi}{\partial u} = \dfrac{4a^3bpfg}{\Delta}[h_1(zy''-z''y) - h_2(zy'-z'y) + (z''y'-z'y'')], \\ \dfrac{\partial\xi}{\partial v} = -\dfrac{4ab^3qfg}{\Delta}[l_1(zy''-z''y) - l_2(zy'-z'y) + (z''y'-z'y'')]. \end{cases}$$

For $N(\xi)$ the expressions analogous to $\Sigma e\left(\frac{\partial x}{\partial u}\right)^2$ and $\Sigma e\left(\frac{\partial x}{\partial v}\right)^2$ for N are $\sum\frac{ke}{k-e}\left(\frac{\partial\xi}{\partial u}\right)^2$ and $\sum\frac{ke}{k-e}\left(\frac{\partial\xi}{\partial v}\right)^2$. Making use of well-known theorems on determinants, we find ultimately that

$$(91)\quad \begin{cases} \sum\dfrac{ke}{k-e}\left(\dfrac{\partial\xi}{\partial u}\right)^2 = -ka^2p^2, \qquad \sum\dfrac{ke}{k-e}\dfrac{\partial\xi}{\partial u}\dfrac{\partial\xi}{\partial v} = 0, \\ \qquad\qquad \sum\dfrac{ke}{k-e}\left(\dfrac{\partial\xi}{\partial v}\right)^2 = kb^2q^2. \end{cases}$$

For $\overline{N}(\overline{x})$ we have $\overline{D} = -\overline{D}'' = -\sigma ab$ (*cf.* 26). Hence from (75) and the analogous equations for $N(\xi)$ we have

$$(91)\quad \begin{cases} \sum\left(\dfrac{\partial\overline{\xi}}{\partial u}\right)^2 - \sum\left(\dfrac{\partial\xi}{\partial u}\right)^2 = p^2(\overline{D}^2 - D^2) = -a^2p^2, \\ \sum\dfrac{\partial\overline{\xi}}{\partial u}\dfrac{\partial\overline{\xi}}{\partial v} - \sum\dfrac{\partial\xi}{\partial u}\dfrac{\partial\xi}{\partial v} = 0, \\ \sum\left(\dfrac{\partial\overline{\xi}}{\partial v}\right)^2 - \sum\left(\dfrac{\partial\xi}{\partial v}\right)^2 = q^2(\overline{D}''^2 - \overline{D}''^2) = b^2q^2. \end{cases}$$

From (91) and (92) we have

$$\Sigma\left(\frac{\partial\bar{\xi}}{\partial u}\right)^2=\Sigma\frac{k}{k-e}\left(\frac{\partial\xi}{\partial u}\right)^2,\qquad \Sigma\left(\frac{\partial\bar{\xi}}{\partial v}\right)^2=\Sigma\frac{k}{k-e}\left(\frac{\partial\xi}{\partial v}\right)^2,$$

$$\Sigma\frac{\partial\bar{\xi}}{\partial u}\frac{\partial\bar{\xi}}{\partial v}=\Sigma\frac{k}{k-e}\frac{\partial\xi}{\partial u}\frac{\partial\xi}{\partial v}.$$

Hence if we put

$$(93)\qquad \hat{x}=\sqrt{\frac{k}{k-e}}\xi,\qquad \hat{y}=\sqrt{\frac{k}{k-f}}\eta,\qquad \hat{z}=\sqrt{\frac{k}{k+g}}\zeta,$$

the net $\bar{N}(\bar{\xi})$ is applicable to the net $N(\hat{x})$, which in consequence of (88) lies on Q. The equations (93) define the *relation of Ivory* between a quadric and a confocal quadric; the point of coordinates $\hat{x}$, $\hat{y}$, $\hat{z}$ is the intersection with Q of the orthogonal trajectory of the family of confocal quadrics which passes through point of coordinates ξ, η, ζ of Q_k (¹).

The functions of a transformation W are x', y', z', x'', y'', z'', h_1, l_1, h_2 and l_2. They satisfy a completely integrable system of the form (2), (3) and (58). Moreover, the five conditions (84) and (85) must be satisfied. However, these equations are satisfied also by the functions

$$(94)\qquad \begin{cases} \alpha x'+\beta x'', & \alpha y'+\beta y'', \quad \alpha z'+\beta z'', \quad \alpha h_1+\beta h_2, \quad \alpha l_1+\beta l_2,\\ \gamma x'+\delta x'', & \gamma y'+\delta y'', \quad \gamma z'+\delta z'', \quad \gamma h_1+\delta h_2, \quad \gamma l_1+\delta l_2,\end{cases}$$

where α, β, γ and δ are constants. In this case, as follows froms (76), (77) and (79) we get the same nets $\bar{N}(\bar{\xi})$ and $N(\xi)$. Consequently for each value of k there are ∞^1 transformations of the kind sought.

If $N_1(x_1)$ and $N_2(x_2)$ denote the T transforms of N, we have

$$x_1=x-\frac{\theta_1}{\theta'_1}x',\qquad x_2=x-\frac{\theta_2}{\theta''_2}x''.$$

In consequence of (82) and (83) we have

$$\Sigma e\xi x_1=1,\qquad \Sigma e\xi x_2=1.$$

Hence the point of coordinates, ξ, η, ζ, is the pole of the plane of the corresponding points on N, N_1 and N_2.

(¹) *B.*, p. 59.

We may state the foregoing results as follows.

If N *is a permanent net on a central quadric* Q, *there are* ∞^1 *sets of transformations* T_k *of* N *into nets* N_1 *and* N_2 *so that the condition* (85) *is satisfied; the locus of the pole* M_k *of the plane* MM_1M_2 *with respect to* Q *is a net* N_k *on a quadric confocal to* Q; *as* N *rolls on its applicable net* $\overline{N}$, *the point* M_k *describes a net* $\overline{N}_k$, *such that* $\overline{N}$ *and* $\overline{N}_k$ *are on the focal surfaces of a* $\overline{W}$ *congruence, and* $\overline{N}_k$ *is applicable to the net on* Q *which is the Ivory transform of the net* $\overline{N}_k$.

These transformations are equivalent to those found by Bianchi by entirely different processes (¹).

7. W *Transforms of surfaces applicable to a paraboloid.* — In this section we establish for surfaces applicable to the paraboloid $ex^2 + fy^2 + 2z = 0$ transformations analogous to those found in the preceding section. Equations (73) and (74) hold in this case also. In place of (75) we have

$$\theta_1 = z(exx' + fyy' + z'), \qquad \theta_2 = z(exx'' + fyy'' + fyy'' + z'').$$

In place of (81) and (82) we have

$$\Delta = 4cabef(x''y' - x'y''),$$

and

$$\xi = \frac{1}{e}\,\frac{y''z' - y'z''}{x''y' - x'y''}, \qquad \eta = \frac{1}{f}\,\frac{z''x' - z'x''}{x''y' - x'y''},$$

$$\zeta = -z + \frac{y(x''z' - x'z'') + x(y'z'' - y''z')}{x''y' - x'y''}.$$

Now

$$ex\xi + fy\eta + \zeta + z = 0.$$

The first two of equations (84) with $g = 0$ hold and in place of the second we have

$$(95)\qquad \begin{cases} h_1 a^2 - l_1 b^2 = c^2[(e^2 - ke)xx' + (f^2 - kf)yy' - kz'], \\ h_2 a^2 - l_2 b^2 = c^2[(e^2 - ke)xx'' + (f^2 - kf)yy'' - kz''], \end{cases}$$

and (85) holds with $g = 0$.

(¹) *B.*

By making use of the expressions for

$$x(y'z''-y''z') \quad \text{and} \quad y(z'x''-z''x')$$

which are obtainable from (95), we find that ξ, η and ζ satisfy the condition

$$\frac{\xi^2}{\frac{1}{e}-\frac{1}{k}}+\frac{\eta^2}{\frac{1}{f}-\frac{1}{k}}+2\zeta+\frac{1}{k}=0,$$

that is the net N (ξ) lies on a quadric confocal with P.

In place of (89) we have

$$\begin{aligned}\frac{\partial\xi}{\partial u} &= p\frac{\sigma^2a^2}{\Delta}\{\Delta c(e-k)x+4abf\\ &\qquad\times[b^2y''(l_1-h_1)-b^2y'(l_2-h_2)+c^2k(y'z''-y''z')]\},\\ \frac{\partial\xi}{\partial v} &=-q\frac{\sigma^2b^2}{\Delta}\{\Delta c(e-k)x+4abf\\ &\qquad\times[a^2y''(l_1-h_1)-a^2y'(l_2-h_2)+c^2k(y'z''-y''z')]\}.\end{aligned}$$

From (95) we have

$$\begin{aligned}&y''(h_1a^2-l_1b^2)-y'(h_2a^2-l_2b^2)\\ &\quad=c^2[(e^2-ke)x(x'y''-y'x'')+k(y'z''-y''z').\end{aligned}$$

With the aid of this identity the above equations are reducible to

$$\frac{\partial\xi}{\partial u}=\frac{4pa^3bf}{\Delta}(y''h_1-y'h_2), \qquad \frac{\partial\xi}{\partial v}=-\frac{4qab^3f}{\Delta}(y''l_1-y'l_2).$$

From these and analogous equations we obtain

$$\begin{aligned}\frac{ek}{k-e}\left(\frac{\partial\xi}{\partial u}\right)^2+\frac{fk}{k-e}\left(\frac{\partial\eta}{\partial u}\right)^2&=-ka^2p^2,\\ \frac{ek}{k-e}\left(\frac{\partial\xi}{\partial v}\right)^2+\frac{fk}{k-e}\left(\frac{\partial\eta}{\partial v}\right)^2&=kb^2q^2,\\ \frac{ek}{k-e}\frac{\partial\xi}{\partial u}\frac{\partial\xi}{\partial v}+\frac{fk}{k-e}\frac{\partial\eta}{\partial u}\frac{\partial y}{\partial v}&=0.\end{aligned}$$

From these equations and (92) we have

$$\Sigma\left(\frac{\partial\xi}{\partial u}\right)^2=\Sigma\left(\frac{\partial\hat{x}}{\partial u}\right)^2, \qquad \Sigma\frac{\partial\xi}{\partial u}\frac{\partial\xi}{\partial v}=\Sigma\frac{\partial\hat{x}}{\partial u}\frac{\partial\hat{x}}{\partial v},$$

$$\Sigma\left(\frac{\partial\xi}{\partial v}\right)^2=\Sigma\left(\frac{\partial\hat{x}}{\partial v}\right)^2 \text{ (1)}.$$

(1) Here the symbol Σ refers to the three variables.

where

$$\hat{x} = \sqrt{\frac{k}{k-e}}\,\xi, \qquad \hat{y} = \sqrt{\frac{k}{k-f}}\,\eta, \qquad \hat{z} = \zeta + \frac{1}{2k}.$$

These are the equations of the transformation of Ivory for P.

The other observations for transformations of a central quadric hold also for the case of the paraboloids, and consequently we have the analogous theorem :

If N *is a permanent net on a paraboloid* P, *there are* ∞^1 *pairs of transformations* T_k *of* N *into nets* N_1 *and* N_2 *so that the condition*

$$h_1 h_2 a^2 - l_1 l_2 b^2 = c^2[(c^2 - ke)x'x'' + (f^2 - kf)y'y'']$$

holds; the locus of the pole M_k *of the plane* MM_1M_2 *with respect to* P *is a net* N_k *on a paraboloid confocal to* P; *as* N *rolls on its applicable net* $\overline{N}$ *the point* M_k *describes a net* $\overline{N_k}$ *such that* $\overline{N}$ *and* $\overline{N_k}$ *are the focal nets of a* W *congruence, and* $\overline{N_k}$ *is applicable to the net on* P *which is the Ivory transform of the net* N_k.

These transformations are equivalent to Bianchi's transformation B_k of surfaces applicable to a paraboloid (¹).

8. *Permutability to transformations* T_k *and* B_k. — Let $\overline{N}(\bar{x})$ be the permanent net on a deform of the quadric Q(32) and $\hat{N}(\xi)$ its B_k transform by means of functions x', y', z'; x'', y'', z''; h_1, l_1, h_2, l_2. Let x''', y''', z''', h_3, l_3, be another set of solutions satisfying the conditions

(96) $$h_3^2 a^2 - l_3^2 b^2 = c^2 \Sigma (e^2 - k'e) x'''^2, \qquad h_3 a^2 - l_3 b^2 = c^2 \Sigma (e^2 - k'e) x x''',$$

so that the $T_{k'}$ transform N_3 of N defined by equations of the form

$$x_3 = x - \frac{\theta_3}{\theta_3'''} x'''$$

where

$$\theta_3 = 2 \Sigma e x x''', \qquad \theta_3''' = \Sigma e x'''^2$$

is on Q.

(¹) *Loc. cit.*

From the results (1) of general transformations T.

It follows that the equations of the forms

$$(97)\qquad (x_3)' = x' - \frac{\theta'_3}{\theta'''_3}x''', \qquad (x_3)'' = x'' - \frac{\theta''_3}{\theta'''_3}x'''$$

define nets parallel to N_3 applicable to $\overline{N}_3$; the corresponding functions h_{31}, l_{31} and h_{32}, l_{32} are given by

$$h_{31} = \frac{h_3\theta'_3 - h_1\theta'''_3}{h_3\theta_3 - \theta'''_3}, \qquad l_{31} = \frac{l_3\theta'_3 - l_1\theta'''_3}{l_3\theta_3 - \theta'''_3},$$

$$h_{32} = \frac{h_3\theta''_3 - h_2\theta'''_3}{h_3\theta_3 - \theta'''_3}, \qquad l_{32} = \frac{l_3\theta''_3 - l_2\theta'''_3}{l_3\theta_3 - \theta'''_3}.$$

The functions a_3 and b_3 appearing in the point equation of N_3 are of the form

$$a_3 = a\left(h_3\frac{\theta_3}{\theta'''_3} - 1\right), \qquad b_3 = b\left(l_3\frac{\theta_3}{\theta'''_3} - 1\right).$$

In order that these functions may satisfy equations analogous to (84) and (85), namely

$$\begin{aligned}
a_3^2h_{31}^2 - b_3^2l_{31}^2 &= c^2\Sigma(e^2 - ke)(x_3)'^2,\\
a_3^2h_{32}^2 - b_3^2l_{32}^2 &= c^2\Sigma(e^2 - ke)(x_3)''^2,\\
a_3^2h_{31} - b_3^2l_{31} &= c^2\Sigma(e^2 - ke)x_3(x_3)',\\
a_3^2h_{32} - b_3^2l_{32} &= c^2\Sigma(e^2 - ke)x_3(x_3)'',\\
a_3^2h_{31}h_{32} - b_3^2l_{31}l_{32} &= c^2\Sigma(e^2 - ke)(x_3)'(x_3)'',
\end{aligned}$$

it is necessary and sufficient that

$$(98)\qquad \begin{cases} a^2h_1h_3 - b^2l_1l_3 = c^2\Sigma(e^2 - ke)x'x''' + \dfrac{c^2}{2}(k - k')\theta'_3,\\[2ex] a^2h_2h_3 - b^2l_2l_3 = c^2\Sigma(e^2 - ke)x''x''' + \dfrac{c^2}{2}(k - k')\theta''_3.\end{cases}$$

Differentiating these equations with respect to u and v, and making use of (58), and analogous equations, we find that the resulting equations are satisfied identically. There are two cases to be considered, according as k and k' are equal or not.

If $k' \neq k$, the functions θ'_3 and θ''_3 are uniquely determined by (98),

(1) M_1, p. 111.

and consequently $(x_3)'$ and $(x_3)''$ are uniquely determined by (97). Then by means of the functions

$$(99)\qquad \theta_{31} = 2\sum e x_1'(x_3)', \qquad \theta_{32} = 2\sum e x_1 (x_3)''$$

we obtain a B_k transform $\widehat{N_3}$ of $\overline{N_3}$ (applicable to N_3) which is applicable to a net on Q.

It is readily found by differentiation that the left hand members of the equations

$$(100)\qquad \theta_3' + \theta_1''' - 2\sum e x' x''' = 0, \qquad \theta_3'' + \theta_2''' - 2\sum e x'' x''' = 0$$

are constants. If we take θ_1''' and θ_2''' as given by (100), we find that

$$\theta_{31} = \theta_1 - \frac{\theta_3}{\theta_3'''}\theta_1''', \qquad \theta_{32} = \theta_2 - \frac{\theta_3}{\theta_3'''}\theta_2'''.$$

These are the conditions that N and $\widehat{N_3}$ are in relation T [1].

When $k = k'$, the functions θ_3' and θ_3'' are not defined by (98), and consequently there are 2 families of ∞^1 nets parallel to N_3 defined by (97). Hence there are ∞^1 functions θ_{31}, and ∞^1 functions θ_{32}, which give nets $\widehat{N_3}$ in relation B_k with $\overline{N_3}$ and relation T_k with $\widehat{N}$. Equations (96) and (98) are satisfied by

$$h_3 = \alpha h_1 + \beta h_2, \qquad l_3 = \alpha l_1 + \beta l_2, \qquad x''' = \alpha x' + \beta x'', \qquad \ldots,$$

where α and β are constants, in consequence of (84) and (85).

Similar results hold for the transformations T_k and B_k of nets applicable to the paraboloid.

Hence :

The transformations B_k *and* T_k *of a surface applicable to a quadric are permutable.*

9. *Permutability of transformations* T_k *and* H. — By definition two surfaces S and $\hat{S}$ are *conjugate in deformation*, if the asymptotic lines correspond on S and $\hat{S}$ and to each deform of S there corresponds a deform of $\hat{S}$ with asymptotic lines in correspondence on

(1) M_4.

these deforms. Servant has shown that aside from surfaces applicable to a surface of revolution the only surfaces admitting surfaces conjugate in deformation are surfaces applicable to a quadric, and any such surface has this property. If S and $\hat{S}$ are two such surfaces, the quadrics Q and $\hat{Q}$ to which they are applicable are conjugate in deformation and are in fact projective transforms of one mother. Moreover, if a surface S applicable to Q is known, the corresponding surface applicable to $\hat{Q}$ is known intrinsically, that is its fundamental coefficients are known. Bianchi (¹) says that S and $\hat{S}$ are in the relation of *a transformation* H.

It is readily shown that a transformation H transforms the permanent net N on S into the permanent net $\hat{N}$ on $\hat{S}$. Let N_0 and $\widehat{N_0}$ denote the corresponding nets on Q and $\hat{Q}$. Let N_{01} denote a T_k transform of N_0, and N_1 the corresponding T_k transform of N, in accordance with the results of §§ 4, 5. Since Q is a projective transform of $\hat{Q}$, and any transformation T is transformed into a transformation T by a projectivity, it follows that the projective transform of N_0, is a permanent net $\widehat{N_{01}}$, which is a T_k transform of $\widehat{N_0}$. Consequently the net $\hat{N}_1$, applicable to $\widehat{N_{01}}$, is a T_k transform of $\hat{N}$. Hence we have the theorem :

If N *and* $\hat{N}$ *are nets in relation* H, *and* N_1 *is a* T_k *transform of* N, *there exists a net* $\hat{N}_1$ *which is a* T_k *transform of* $\hat{N}$ *and the* H *transform of* N_1.

Thus we have established the permutability of transformations T_k and H.

(¹) *B.*, p. 214.

ERRATA.

Page 48, *ajouter* a la fin du théorème la note suivante :
Cf. Calapso, *Annali di Matematica*, 3ᵉ série, vol. XIX, 1912, p. 61.