JOURNAL
DE
MATHÉMATIQUES
PURES ET APPLIQUÉES
FONDÉ EN 1836 ET PUBLIÉ JUSQU'EN 1874

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Journal de mathématiques pures et appliquées 9e série, tome 7 (1928), p. 409-416.

<http://www.numdam.org/item?id=JMPA_1928_9_7__409_0>

Conjugate Functions in Three Dimensions;

By E. R. HEDRICK and Louis INGOLD.

1. *Introduction.* — It is well known that certain facts relating to the theory of functions of a complex variable do not admit of generalizations to three dimensions. It also seems to be very generally believed that there are certain properties of such functions which do hold in three dimensions. However, in the literature at least, it is only isolated theorems that are referred to and regarded as generalizations of corresponding theorems of the theory of functions [1].

It has, therefore, seemed to the authors desirable to give a more or less systematic account of the possible analogies between the two cases. In doing this it is, of course, inevitable that much old material will be covered. The new point of view, however, seems to make this worth while.

The initial papers, « Analytic Functions in three Dimensions » [2], and « Beltrami's Equations in three Dimensions » [3], have already appeared.

In the present paper the set of functions which satisfy the generalized Beltrami Equations, or as a special case, the generalized Cauchy-Riemann equations, are considered and their analogy with the theory of conjugate functions is pointed out.

(1) The important Thesis by Nicolesco, *Fonctions complexes dans le plan et dans l'espace.* Paris, 1928, has appeared since this paper was written.

(2) *Transactions of the American Math. Soc.*, vol. 27, 1925, p. 551-555.

(3) *Ibid.*, vol. 27, 1925, p. 556-562.

2. *Consequences of Beltrami's Equations.* — Let u, v, and w, be any three functions satisfying the equations

Two generalizations of Laplace's equation are given and the connection of all of the results with line and surface integrals is briefly indicated.

$$(1)\quad \begin{cases} u_x=\lambda\begin{vmatrix} E_{11} & v_x & w_x \\ E_{12} & v_y & w_y \\ E_{13} & v_z & w_z \end{vmatrix}, & u_y=\lambda\begin{vmatrix} E_{21} & v_x & w_x \\ E_{22} & v_y & w_y \\ E_{23} & v_z & w_z \end{vmatrix}, & u_z=\lambda\begin{vmatrix} E_{31} & v_x & w_x \\ E_{32} & v_y & w_y \\ E_{33} & v_z & w_z \end{vmatrix}, \\ v_x=\lambda\begin{vmatrix} u_x & E_{11} & w_x \\ u_y & E_{12} & w_y \\ u_z & E_{13} & w_z \end{vmatrix}, & v_y=\lambda\begin{vmatrix} u_x & E_{21} & w_x \\ u_y & E_{22} & w_y \\ u_z & E_{23} & w_z \end{vmatrix}, & v_z=\begin{vmatrix} u_x & E_{31} & w_x \\ u_y & E_{32} & w_y \\ u_z & E_{33} & w_z \end{vmatrix}, \\ w_x=\lambda\begin{vmatrix} u_x & v_x & E_{11} \\ u_y & v_y & E_{12} \\ u_z & v_z & E_{13} \end{vmatrix}, & w_y=\lambda\begin{vmatrix} u_x & v_x & E_{21} \\ u_y & v_y & E_{22} \\ u_z & v_z & E_{23} \end{vmatrix}, & w_z=\begin{vmatrix} u_x & v_x & E_{31} \\ u_y & v_y & E_{32} \\ u_z & v_z & E_{33} \end{vmatrix}, \end{cases}$$

where λ and the E_{ij} are given functions of x, y and z, and where $E_{ij}=E_{ji}$.

Denoting the determinant $|E_{ij}|$ by H [1], we obtain

$$(2)\ (\textit{à suivre})\quad \begin{cases} \begin{vmatrix} v_y & w_y \\ v_z & w_z \end{vmatrix} = \dfrac{1}{\lambda H}\begin{vmatrix} u_x & E_{12} & E_{13} \\ u_y & E_{22} & E_{23} \\ u_z & E_{32} & E_{33} \end{vmatrix}. \\ \begin{vmatrix} v_z & w_z \\ v_x & w_x \end{vmatrix} = \dfrac{1}{\lambda H}\begin{vmatrix} E_{11} & u_x & E_{31} \\ E_{12} & u_y & E_{32} \\ E_{13} & u_z & E_{33} \end{vmatrix}. \\ \begin{vmatrix} v_x & w_x \\ v_y & w_y \end{vmatrix} = \dfrac{1}{\lambda H}\begin{vmatrix} E_{11} & E_{21} & u_x \\ E_{12} & E_{22} & u_y \\ E_{13} & E_{23} & u_z \end{vmatrix}. \\ \begin{vmatrix} w_y & u_y \\ w_z & u_z \end{vmatrix} = \dfrac{1}{\lambda H}\begin{vmatrix} v_x & E_{12} & E_{13} \\ v_y & E_{22} & E_{23} \\ v_z & E_{32} & E_{33} \end{vmatrix}. \\ \begin{vmatrix} w_z & u_z \\ w_x & u_x \end{vmatrix} = \dfrac{1}{\lambda H}\begin{vmatrix} E_{11} & v_x & E_{31} \\ E_{12} & v_y & E_{32} \\ E_{13} & v_z & E_{33} \end{vmatrix}. \end{cases}$$

(1) It will be assumed that λ and H are different from zero.

(2) (*suite*)

$$\left\{\begin{aligned}
&\begin{vmatrix} w_x & u_x \\ w_y & u_y \end{vmatrix} = \frac{1}{\lambda H}\begin{vmatrix} E_{11} & E_{21} & v_x \\ E_{12} & E_{22} & v_y \\ E_{13} & E_{23} & v_z \end{vmatrix},\\
&\begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix} = \frac{1}{\lambda H}\begin{vmatrix} w_x & E_{12} & E_{13} \\ w_y & E_{22} & E_{23} \\ w_z & E_{32} & E_{33} \end{vmatrix},\\
&\begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix} = \frac{1}{\lambda H}\begin{vmatrix} E_{11} & w_x & E_{31} \\ E_{12} & w_y & E_{32} \\ E_{13} & w_z & E_{33} \end{vmatrix},\\
&\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = \frac{1}{\lambda H}\begin{vmatrix} E_{11} & E_{21} & w_x \\ E_{12} & E_{22} & w_y \\ E_{13} & E_{23} & w_z \end{vmatrix},
\end{aligned}\right.$$

The connection between the jacobian J of the functions u, v, w, and the functions λ and H is easily found by writing J^2 in the form

$$J^2 = \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix} \cdot \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}.$$

Upon multiplying out and making use of the relations ([1]),

$$(3)\quad \left\{\begin{aligned} &\Sigma u_x^2 = u_x^2 + v_x^2 + w_x^2 = \lambda J E_{11}, \quad \Sigma u_y^2 = \lambda J E_{22}, \quad \Sigma u_z^2 = \lambda J E_{33},\\ &\Sigma u_x u_y = \lambda J E_{12}, \quad \Sigma u_y u_z = \lambda J E_{23}, \quad \Sigma u_z u_x = \lambda J E_{31}, \end{aligned}\right.$$

the result readily reduces to

$$J^2 = \lambda^3 J^3 H$$

or

$$(4)\qquad HJ = 1/\lambda^3.$$

With the aid of these formulas, it is now possible to prove the converse of certain results of our previous papers. In equations (2), multiply the first by u_x, the second by u_y, the third by u_z and add. After replacing J by $1/(\lambda^3 H)$ the result may be written

$$(5)\quad \frac{1}{\lambda^2 H} = \frac{1}{H}\left[u_x \begin{vmatrix} u_x & E_{12} & E_{13} \\ u_y & E_{22} & E_{23} \\ u_z & E_{32} & E_{33} \end{vmatrix} + u_y \begin{vmatrix} E_{11} & u_x & E_{31} \\ E_{12} & u_y & E_{32} \\ E_{13} & u_z & E_{33} \end{vmatrix} + u_z \begin{vmatrix} E_{11} & E_{21} & u_x \\ E_{12} & E_{22} & u_y \\ E_{13} & E_{23} & u_z \end{vmatrix} \right].$$

([1]) See the paper *Beltrami's Equations*, etc., *loc. cit.*, § 3, p. 559.

The right hand side of this equation is the well known differential parameter $\Delta_1 u$, and a similar procedure shows that $\Delta_1 v$ and $\Delta_1 w$ have the same value, and also that $\Delta_1 uv = \Delta_1 vw = \Delta_1 wu = 0$, where

$$(6) \qquad \Delta_1(uv) = \frac{1}{H}\left[u_x \begin{vmatrix} v_x & E_{12} & E_{13} \\ v_y & E_{22} & E_{23} \\ v_z & E_{32} & E_{33} \end{vmatrix} + u_y \begin{vmatrix} E_{11} & v_x & E_{31} \\ E_{12} & v_y & E_{32} \\ E_{13} & v_z & E_{33} \end{vmatrix} + u_z \begin{vmatrix} E_{11} & E_{21} & v_x \\ E_{12} & E_{22} & v_y \\ E_{13} & E_{23} & v_z \end{vmatrix} \right].$$

These were the facts assumed in order to obtain Beltrami's equations, and the result shows that these conditions are both necessary and sufficient.

3. *Extensions of Laplace's Equation.* — In two dimensions, Laplace's equation is obtained by eliminating one of the pair of functions u, v, from the Cauchy-Riemann equations.

In three dimensions there are two possibilities. We may either eliminate one of the set u, v, w, to obtain a system of differential equations satisfied by the other two, or we may eliminate two of the three functions and obtain an equation which the remaining function must satisfy.

From the first three of equations (2) it is easily seen that

$$(7) \qquad \frac{\partial}{\partial x}\left[\frac{1}{\lambda H}\begin{vmatrix} u_x & E_{21} & E_{31} \\ u_y & E_{22} & E_{32} \\ u_z & E_{23} & E_{33} \end{vmatrix}\right] + \frac{\partial}{\partial y}\left[\frac{1}{\lambda H}\begin{vmatrix} E_{11} & u_x & E_{31} \\ E_{12} & u_y & E_{32} \\ E_{13} & u_z & E_{33} \end{vmatrix}\right] + \frac{\partial}{\partial y}\left[\frac{1}{\lambda H}\begin{vmatrix} E_{11} & E_{21} & u_x \\ E_{12} & E_{22} & u_y \\ E_{13} & E_{23} & u_z \end{vmatrix}\right] = 0.$$

and from the remaining equations of the set (2) it is clear that v and w satisfy the same equation.

Equation (7) is the generalization to curved spaces and to curvilinear coordinates of equation (17) of the earlier paper « *Analytic Functions in three Dimensions* » [1]. If $E_{ii} = 1$ and $E_{ij} = 0$ when $i \neq j$, the

[1] *Loc. cit.*, p. 555 and seq.

equation becomes

$$\frac{\partial}{\partial x}\left(\frac{1}{\lambda}u_x\right)+\frac{\partial}{\partial y}\left(\frac{1}{\lambda}u_y\right)+\frac{\partial}{\partial z}\left(\frac{1}{\lambda}u_z\right)=0.$$

A better analogy, in some respects, to Laplace's equation is obtained from equation (1) by equating the two values for each of the second derivatives w_{xy}, w_{yz}, w_{zx}. The result may be written

$$(8)\quad \left\{\begin{aligned}
\frac{\partial}{\partial y}\left[\lambda\begin{vmatrix} u_x & v_x & E_{11}\\ u_y & v_y & E_{12}\\ u_z & v_z & E_{13}\end{vmatrix}\right] &= \frac{\partial}{\partial x}\left[\lambda\begin{vmatrix} u_x & v_x & E_{21}\\ u_y & v_y & E_{22}\\ u_z & v_z & E_{23}\end{vmatrix}\right],\\
\frac{\partial}{\partial z}\left[\lambda\begin{vmatrix} u_x & v_x & E_{21}\\ u_y & v_y & E_{22}\\ u_z & v_z & E_{23}\end{vmatrix}\right] &= \frac{\partial}{\partial y}\left[\lambda\begin{vmatrix} u_x & v_x & E_{31}\\ u_y & v_y & E_{32}\\ u_z & v_z & E_{33}\end{vmatrix}\right],\\
\frac{\partial}{\partial x}\left[\lambda\begin{vmatrix} u_x & v_x & E_{31}\\ u_y & v_y & E_{32}\\ u_z & v_z & E_{33}\end{vmatrix}\right] &= \frac{\partial}{\partial z}\left[\lambda\begin{vmatrix} u_x & v_x & E_{11}\\ u_y & v_y & E_{12}\\ u_z & v_z & E_{13}\end{vmatrix}\right].
\end{aligned}\right.$$

Each of the pairs of functions v, w and u, w must satisfy the same system of equations.

4. *Conjugate Functions.* — A set of functions u, v, w, satisfying equation (1) will be called a set of conjugate functions.

If u and v are any two functions satisfying relations $\Delta_1 u = \Delta_1 v$, $\Delta_1 uv = 0$, and also the system of differential equations (8), where λ is defined by the equation $\Delta_1 u = 1/(\lambda^2 H)$ then there exists a third function w such that u, v, w, form a conjugate set. For, if we write

$$\Delta_1 u = \frac{1}{\lambda^2 H},\qquad a=\lambda\begin{vmatrix} u_x & v_x & E_{11}\\ u_y & v_y & E_{12}\\ u_z & v_z & E_{13}\end{vmatrix},\qquad b=\lambda\begin{vmatrix} u_x & v_x & E_{21}\\ u_y & v_y & E_{22}\\ u_z & v_z & E_{23}\end{vmatrix},$$

$$c=\lambda\begin{vmatrix} u_x & v_x & E_{31}\\ u_y & v_y & E_{32}\\ u_z & v_z & E_{33}\end{vmatrix},$$

then equations (8) are the necessary and sufficient conditions that a, b, and c, are the derivatives with respect to x, y, and z, of some function w. Thus w_x, w_y, w_z are known and w is known except for an arbitrary constant.

It can be shown that u, v, and w, as thus obtained, satisfy equations (1). They are therefore a set of conjugate functions. Before giving the proof of this statement we discuss in the next section certain important relations connecting the differential parameters which are used in the proof.

5. *Relations among the Invariants.* — We have already introduced the invariants $\Delta_1 u$, $\Delta_1 v$, and $\Delta_1 uv$; another important invariant occurs in the expression for the jacobian J when written in terms of the functions u and v. If we denote the determinants of the matrix

$$\begin{matrix} u_x & u_y & u_z \\ v_x & v_y & v_z \end{matrix}$$

by L, M, N, respectively, we have $J = w_x L + w_y M + w_z N$. When w_x, w_y, w_z, are replaced by their values,

$$w_x = \lambda E_{11} L + \lambda E_{12} M + \lambda E_{13} N. \qquad \ldots.$$

the expression for J reduces to

$$J = \lambda [E_{11} L^2 + E_{22} M^2 + E_{33} N^2 + 2 E_{12} LM + 2 E_{23} MN + 2 E_{31} NL].$$

The quantity in brackets divided by H is an invariant which we denote by ∇uv.

If we omit the divisor H in the expressions for $\Delta_1 u$, $\Delta_1 v$, and $\Delta_1 uv$, the resulting expressions may be written

$$(10) \quad \begin{cases} P = \mathcal{E}_{11} u_x^2 + \mathcal{E}_{22} u_y^2 + \mathcal{E}_{33} u_z^2 + 2\mathcal{E}_{12} u_x u_y + 2\mathcal{E}_{23} u_y u_z + 2\mathcal{E}_{31} u_z u_x, \\ Q = \mathcal{E}_{11} v_x^2 + \mathcal{E}_{22} v_y^2 + \mathcal{E}_{33} v_z^2 + 2\mathcal{E}_{12} v_x v_y + 2\mathcal{E}_{23} v_y v_z + 2\mathcal{E}_{31} v_z v_x, \\ R = \mathcal{E}_{11} u_x v_x + \mathcal{E}_{22} u_y v_y + \mathcal{E}_{33} u_z v_z + \mathcal{E}_{12} (u_x v_y + u_y v_x) \\ \qquad + \mathcal{E}_{23} (u_y v_z + u_z v_y) + \mathcal{E}_{31} (u_z v_x + u_x v_z), \end{cases}$$

where $\mathcal{E}_{ij}$ is the cofactor of E_{ij} in the determinant $|E_{rs}|$. If we subtract the square of the last expression from the product of the other two, the result reduces to

$$PQ - R^2 = H [E_{11} L^2 + \ldots] = HJ/\lambda;$$

but, by hypothesis, $R = 0$ and $P = Q = 1/\lambda^2$.

It follows, therefore, that

$$J = \lambda PQ/H = \lambda H\, \Delta_1 u\, \Delta_1 v = 1/(\lambda^3 H).$$

Incidentally this proves that the jacobian J does not vanish.

6. *Proof of the Cauchy-Riemann Equations.* — We are now in position to prove the statement at the end of § 4. From the definition of w_x, w_y, w_z, we have

$$(11)\quad \begin{cases} w_x = \lambda E_{11}\begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix} + \lambda E_{12}\begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix} + \lambda E_{13}\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}, \\ w_y = \lambda E_{21}\begin{vmatrix} u_y & b_y \\ u_z & v_z \end{vmatrix} + \lambda E_{22}\begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix} + \lambda E_{23}\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}, \\ w_z = \lambda E_{31}\begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix} + \lambda E_{32}\begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix} + \lambda E_{33}\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}. \end{cases}$$

From these we obtain

$$(12)\quad \begin{cases} \varepsilon_{11} w_x + \varepsilon_{12} w_y + \varepsilon_{13} w_z = \lambda H \begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix}, \\ \varepsilon_{21} w_x + \varepsilon_{22} w_y + \varepsilon_{23} w_z = \lambda H \begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix}, \\ \varepsilon_{31} w_x + \varepsilon_{32} w_y + \varepsilon_{33} w_z = \lambda H \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}. \end{cases}$$

Multiplying these equations in turn by w_x, w_y, w_z, and adding, we find

$$\lambda HJ = H\, \Delta_1 w,$$

or, since $J = 1/(\lambda^3 H)$,

$$\Delta_1 w = 1/(\lambda^2 H) = \Delta_1 u = \Delta_1 v.$$

In a similar manner it is found that

$$\Delta_1 uw = \Delta_1 vw = 0.$$

The equations $\Delta_1 u = 1/(\lambda^2 H)$, $\Delta_1 uv = 0$, $\Delta_1 uw = 0$, when solved for u_x, u_y, u_z, will give the first row of equations (1). In the same way the second row of equations (1) is obtained from the equations

$\Delta_1 v = 1/(\lambda^2 H)$, $\Delta_1 vu = 0$, $\Delta_1 vw = 0$. This completes the proof that a pair of functions u and v subject to the conditions specified in § 4 determines, except for a constant, a third function w such that u, v, and w constitute a conjugate set.

7. *Integrals independent of the Path.* — The connexion between ordinary analytic functions of a complex variable, and integrals which are independent of the path of integration, is well known. Thus if v is one of the components of an analytic function of (x, y) the integral

$$\int \begin{vmatrix} dx & v_x \\ dy & v_y \end{vmatrix}$$

vanishes around any closed contour which does not enclose singular points.

Conversely, if this integral vanishes around any such closed contour, v is a component of an analytic function and the conjugate function u can be determined.

In a strictly analogous fashion the conditions that a pair of functions u, v in a curved space, satisfying the conditions $\Delta_1 u = \Delta_1 v$, $\Delta_1 uv = 0$, be a pair of functions of a conjugate set may be expressed by the vanishing of a certain integral around a closed contour.

In fact, equation (8), which we have just seen are the necessary and sufficient conditions on u and v, are equivalent to the necessary and sufficient conditions that the line integral

$$(13)\quad I = \int \lambda \left[\begin{vmatrix} u_x & v_x \\ u_y & v_y \\ u_z & v_z \end{vmatrix} dx + \begin{vmatrix} u_x & v_x & E_{21} \\ u_y & v_y & E_{22} \\ u_z & v_z & E_{23} \end{vmatrix} dy + \begin{vmatrix} u_x & v_x & E_{31} \\ u_y & v_y & E_{32} \\ u_z & v_z & E_{33} \end{vmatrix} dz \right]$$

should vanish around every closed contour.

The conditions that u and v be components of a function of a given class in ordinary space may be regarded as a special case of this condition. The quantities E_{ij} in this special case are the fundamental quantities of a space of zero curvature.