JOURNAL

DE

MATHÉMATIQUES

PURES ET APPLIQUÉES

FONDÉ EN 1836 ET PUBLIÉ JUSQU'EN 1874

PAR JOSEPH LIOUVILLE

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Instability and transitivity

Journal de mathématiques pures et appliquées 9e série, tome 14, nº 1-4 (1935), p. 49-71.

<http://www.numdam.org/item?id=JMPA_1935_9_14_1-4_49_0>

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Instability and transitivity;

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The question as to the general existence of topological or metric transitivity has been much emphasized of late. One can refer to the historical summary by Birkhoff and Koopman, Birkhoff [II]. References are here given to the contributions of various writers including the recent important papers by Birkhoff, Hopf, Koopman, v. Neumann, and P. A. Smith. Birkhoff's Ergodic Theorem and v. Neumann's Mean Ergodic Theorem stand out at basic conclusions. The importance of these two theorems is however conditioned by the validity of the hypothesis that metric transitivity exists in general. If the hypothesis of metric transitivity fails in general, the theory will appear relatively incomplete and complex, at least until further illuminating contributions are made.

In non-analytic problems one can show by simple geodesic problems that metric transitivity fails in many cases. It would seem however that problems which include the analytic case offer a fairer test. Metric transitivity fails in all cases among geodesics on closed surfaces of revolution. Metric transitivity holds for a simple type of spiral-like motion on the torus, as is easy to prove. Birkhoff conjectured that geodesic motion on closed surfaces of constant negative curvature would offer an example of metric transitivity, and a number of mathematicians have been seeking to verify this opinion. Hedlund [I] has recently announced a proof of the desired theorem.

The theorem of Hedlund while most interesting is by no means a justification of the hypothesis of metric transitivity. The problem of whether geodesics on closed surfaces of non-constant negative curvature form a metrically transitive system has been reduced by the author to the problem of determining whether a certain topological transformation is *absolutely continuous* or not. This result can be obtained by combining Hedlund's theorem with the theorems of the author in the paper Morse [II] cited below.

But even if it turns out that geodesics on closed surfaces of negative curvature form a metrically transitive system, the hypothesis of metric transitivity will not then be justified in general. For a casual study of geodesics on surfaces of mixed curvature, partly positive and partly negative, shows that surfaces of negative curvature stand apart as an extreme case of non-general simplicity.

Let M^* be a non-singular r-dimensional manifold without boundary. On M^* suppose we have a system T of trajectories which includes on and only one trajectory through each point of M^*. We regard the time t as the parameter along these trajectories and suppose each trajectory can be continued over the time interval $-\infty < t < \infty$. So continued a trajectory will be said to be complete. The system T will be saïd to be *topologically transitive* if there exists a complete trajectory whose closure (the trajectory and its limit points) is the manifold M^*. Topological transitivity has been referred to as regional or geometric transitivity as well as topological transitivity. Metric transitivity implies topological transitivity as follows from the theorems of v. Neumann and Birkhoff. In examples for which topological transitivity holds there is thus a possibility that metric transitivity holds.

It is one of the purposes of the present paper to show that among the geodesics on closed surfaces R^* *of genus* $p > 1$ *uniform instability implies topological transitivity.*

Uniform instability, as it will be defined, is a property of the individual geodesics holding uniformly for all geodesics. Birkhoff has conjectured that the hypothesis of instability relative to the closed geodesics of R^* implies topological transitivity. The developments of this paper do not tend to support this conjecture.

The surfaces admitted are Riemannian manifolds which are closed in a topological sense. Those whose geodesics possess uniform instability include all topologically closed surfaces of negative curvature, as well as surfaces which possess regions of positive curvature. As previously stated our principal theorem connects instability with transitivity. Questions as to the distribution and number of transitive geodesics are answered, and a theorem on the deferring of transitivity is established.

From the point of view of the calculus of variations the hypothesis of uniform instability may be roughly regarded as the hypothesis that the first conjugate point of each finite point on a given surface *lies beyond the point at infinity*.

A proof of the existence of topological transitivity has been indicated by Birkhoff for a special surface of negative curvature of genus 2. See Birkhoff [III], p. 248. The methods of the author are based on the paper Morse [II] and are different from those of Birkhoff.

1. THE COVERING SURFACE R. — We suppose that we have a Fuchsian group G of non-singular, fractional, linear transformations of the plane $z = u + iv$ carrying the interior S of the unit circle

$$u^2 + v^2 = 1$$

into S. We suppose all members of the group are of hyperbolic type with fixed points on the unit circle, and that the group has a finite number of generators. We regard H as a hyperbolic plane of non-Euclidean geometry with the circles orthogonal to the unit circle representing the H-straight lines (H is written for hyperbolic). The H-length of a curve of S may be given by the integral

$$\int \frac{d\sigma}{1 - u^2 - v^2}, \tag{1.1}$$

where $d\sigma$ is the differential of arc length in the (u, v)-plane.

We suppose that the fundamental domain for G consists of a convex region S_0 on S bounded by a sequence, $p > 1$,

$$a_1 b_1 c_1 d_1, \quad a_2 b_2 c_2 d_2, \quad \ldots, \quad a_p b_p c_p d_p, \tag{1.2}$$

of segments of H-straight lines. The closure of S_0 contains no points on the unit circle. The successive H-lines in (1.2) shall form angles on S_0 equal to $\frac{\pi}{2p}$. The side a_k will be termed *conjugate* to c_k and the side b_k conjugate to d_k. There will be a transformation of the group carrying each side of S_0 into its conjugate side, carrying S_0 into an adjacent region. Points which are images of each other under G are termed *congruent.*

We make the usual convention that just one side of each pair of conjugate sides of S_0 shall be considered as belonging to S_0 and just one of S_0's vertices. The application of the transformations of G to S_0 will yield a set of regions covering S in a one-to-one manner, as is well known.

Instead of assigning S the H-metric defined by (1.1) we can assign S an R-metric (R is written for Riemannian) defined by a positive definite form

$$ds^2 = E(u, v)\,du^2 + 2F(u, v)\,du\,dv + G(u, v)\,dv^2 \tag{1.3}$$

with coefficients of class C^3 for (u, v) on S. The form (1.3) will define a Riemannian manifold R. We suppose that the form (1.3) is invariant under G. We then regard congruent points as identical. With this understood S_0 taken with the form (1.3) defines a *topologically closed surface* R^*. The genus of R^* will be the integer p. The manifold R is the *covering surface* belonging to R^*.

The form

$$ds^2 = \frac{du^2 + dv^2}{(1 - u^2 - v^2)^2},$$

derived from (1.1) is a special instance of (1.3) and defines a topologically closed surface of constant negative curvature.

2. The phase-space M. — Let angles θ in the (u, v)-plane be measured in the usual way from the positive u axis. A point (u, v) on S together with an angle θ will define an element E at (u, v) on S. Elements (u, v, θ) whose angles θ differ by an integral multiple of π will be regarded as identical. The set of all such elements E will define the *phase-space* M corresponding to S or R. The elements tangent to a regular curve g on R will make up a curve on M which we regard as the representative of g on M.

To any two points on R we assign an R-distance equal to the minimum length of R-geodesics joining the two points on R. If E′ and E″ are any two elements on M with directions represented by θ' and θ'', the R-angle between E′ and E″ will be taken as the minimum of

$$|\theta'' - \theta' + n\pi|,$$

for all integers n positive, negative, or zero. Two elements E′ and E″ on M will be said to possess an R-distance which is the sum of the R-angle and the R-distance between their initial points.

The set of elements with initial points on S_0 will be said to define the phase-space M^* corresponding to R^*. The *distance* between two elements E′ and E″ on M^* will be taken as the minimum of the R-distance between all pairs of elements respectively congruent to E′ and E″ on R.

3. Previous theorems. — The present paper is based on an earlier paper Morse [II]. We recall a few definitions and theorems of the earlier paper.

The hyperbolic plane H consists of the interior S of the unit circle. Let γ be a simple open arc lying on S. Let $\bar{\gamma}$ be the closure of γ. Suppose $\bar{\gamma}$ is a simple arc with end points P and Q. If P or Q lies on the unit circle, P or Q respectively will be termed *ideal end points* of γ.

Let A and B be two point sets on R. Suppose there exists a number λ such that A is a subset of the points at most an R-distance λ from B, and that B is a subset of points at most an R-distance λ from A. The greatest lower bound of such numbers λ will be termed the type-distance between A and B. Two simple open arcs on R will be said to be of the *same type* if they possess a finite type-distance.

An unending geodesic g on R will be said to be of *class* A if every finite segment of g affords an absolute minimum to the R-length relative to all rectifiable curves which join its end points on R. We state the following theorem.

Theorem 3.1. — *Every geodesic of class* A *on* R *is of the type of some* H-*straight line. Conversely there is at least one geodesic of class* A *of the type of each* H-*straight line on* S.

Moreover there exists a universal constant K *dependent only on* R *such that the type-distance between geodesics of class* A *of the same type or between geodesics of class* A *and* H-*straight lines of the same type never exceeds* K.

The surfaces admitted in Morse [II] are somewhat less general than the surfaces admitted here. Nevertheless the reader can readily see that the proofs in the earlier paper hold here practically unchanged. See Theorem 1 and Lemma 8, Morse [II].

If a curve h on S is invariant under a transformation T of the group G, h will be termed *periodic* mod T. The curve h will then define a closed curve on R^*.

Corresponding to each H-straight line h there will either exist a unique geodesic g of class A of the type of h, or else two non-intersecting geodesics g' and g'' of class A of the type of h between which lie all other geodesics of class A of the type of h. The geodesics g' and g'' will be called the *boundary geodesics* of the type of h. We apply this term even in the case where $g' = g''$.

We state the following theorem. Morse [II], Theorem 11.

THEOREM 3.2. — *The boundary geodesics of the type of a periodic* H-*straight line are themselves periodic and are invariant under the same transformations of* G *as* h.

A geodesic on R which starts from a point P on R and is continued indefinitely in one sense from P will be called a *geodesic ray*. An H-straight line which emanates from P and is continued indefinitely in one sense will be called an H-*ray*. Two geodesics rays or a geodesic ray and an H-ray which emanate from a common point P on R and possess a finite type-distance will be said to be of the *same* type.

A geodesic ray g will be said to be of *class* A if every finite segment of g affords an absolute minimum to the R-length relative to rectifiable curves on R which join its end points.

With this understood we state the following theorem.

THEOREM 3.3. — *Corresponding to each* H-*ray issuing from a point* P *on* R *there exists a geodesic ray of class* A *of the same type. Conversely* *every geodesic ray of class* A *issuing from* P *is of the type of some* H-*ray.*

There exists a universal constant K *dependent only on* R *such that the type distance between two geodesic rays of class* A *of the same type or a geodesic ray of class* A *and an* H-*ray of the same type never exceeds* K.

Let k be an H-ray issuing from a fixed point P. There will either exist a unique geodesic ray of class A of the type of k, or there will exist two geodesic rays g' and g'' of class A of the type of k between which lie all other geodesic rays of class A of the type of k. The geodesics g' and g'' will be termed *boundary geodesic rays* of the type of k. We apply this term even in the case where $g'=g''$. The geodesic rays g' and g'' are either identical or else have at most the point P in common.

We state the following theorem.

THEOREM 3.4 — *Let* b *be a periodic* H-*straight line and* B *one of its ideal end points. Let* g *be a geodesic ray of class* A *whose initial point* P *does not lie between or on the boundary geodesics of the type of* b *and whose ideal end point coincides with* B. *The geodesic ray* g *will be asymptotic to one of the boundary geodesics of the type of* b.

That g is asymptotic to a periodic geodesic k of the type of b is stated in Lemma 10, Morse [II]. That k must be a boundary geodesic of the type of b follows from the affirmation in Lemma 11, Morse [II] that g cannot cross any periodic geodesic of class A of the type of b.

A geodesic g will be said to be a *limit geodesic* of a set of geodesics L not containing g if every element on g is a limit element of elements on geodesics of the set L.

In Morse [II] on p. 32 we stated the following lemma.

LEMMA A. — *There exists a transformation of the group* G *which has fixed points arbitrarily near the end points of any preassigned arc of the unit circle.*

The following theorem is a consequence of this lemma. Cf. Theorem 16, Morse [II].

THEOREM 3.5. — *The set of all boundary geodesics of periodic type includes all of the boundary geodesics among its limit geodesics.*

This theorem has an immediate corollary.

COROLLARY. — *The set of all geodesic rays of class* A *through a fixed point* P *asymptotic to periodic boundary geodesics includes all of the boundary geodesics among its limit geodesics.*

4. THE HYPOTHESIS OF UNICITY. — *We shall say that* R *satisfies the hypothesis of unicity if there is but one boundary geodesic of class* A *of the type of each* H-*straight line.*

The hypothesis of unicity always holds if the surface is a surface of negative curvature. For on surfaces of negative curvature it is impossible to have two unending geodesics of the same type. See Hadamard [I]. More generally the hypothesis of unicity will hold if the geodesics on R possess uniform instability, as we shall presently show.

We shall prove the following theorem.

THEOREM 4.1. — *If the hypothesis of unicity holds, there is but one geodesic ray of class* A *of a given type issuing from a given point* P *of* R.

Suppose the theorem is false and that g' and g'' are two geodesic rays of class A of the same type issuing from a point P.

Exactly as in the proof of Theorem 6, Morse [II], so here it follows that g' and g'' cannot be asymptotic. Let s be the arc length on g' measured from P. Let a be any positive constant. Since the geodesics g' and g'' are not asymptotic, the minimum R-distance from the point s on g' to g'' must have a positive lower bound k for $s > a$.

Let

$$s_1, \quad s_2, \quad \dots$$

be a sequence of points s on g' such that s_n becomes positively infinite with n and let

$$E_1, \quad E_2, \quad \dots$$

be the corresponding elements on g'. Let

$$e_1, \quad e_2, \quad \dots$$

be a set of elements e_n on g'' whose initial points on g'' are respectively at most the R-distance K from the corresponding points s_n on g'. The constant K is the universal constant of Theorem 3.3.

Let E_n be carried by a transformation T_n of the group G into an element E'_n with initial point on S_0. Under T_n, e_n will be carried into an element e'_n. The pairs

$$(E'_n, e'_n),$$

will have at least one cluster pair (E, e) on R since their initial points lie on the domain consisting of the points on R at most an R-distance K from the points of S_0.

Let γ' and γ'' be the unending geodesics on R defined by the elements E and e respectively. The type-distance between γ' and γ'' exists and cannot be less than k. The geodesics γ' and γ'' are thus of the same type but not identical.

From this contradiction we infer the truth of the theorem.

When the hypothesis of unicity holds the correspondence between elements at P which respectively define H-rays and geodesic rays of the same type is readily seen to be continuous as well as one-to-one.

In such a case each geodesic ray through P is of class A.

Since P is an arbitrary point on R we can say more generally that when the hypothesis of unicity holds each unending geodesic is of class A.

5. Topological transitivity. — The set of all elements on R^* defines the phase-space M^*. The elements on a geodesic g define a curve on M^*, the representative of g on M^*. If a geodesic or geodesic ray is represented by a curve on M^* whose closure is M^*, the geodesic or geodesic ray is termed *transitive*.

We shall prove the following theorem.

Theorem 5.1. — *Let* P *be an arbitrary point on* R *and* λ *an arbitrary open segment of the unit circle. If the hypothesis of unicity holds, there is at least one transitive geodesic ray issuing from* P *with an ideal end point on* λ.

Let

$$(5.1) \qquad \gamma_1, \quad \gamma_2, \quad \gamma_3, \quad \ldots$$

be a set of periodic geodesics on R which includes at least one geodesic congruent to each periodic geodesic on R. From (5.1) we form the sequence

$$\gamma_1, \quad \gamma_1\gamma_2, \quad \gamma_1\gamma_2\gamma_3, \quad \gamma_1\gamma_2\gamma_3\gamma_4, \quad \ldots, \tag{5.2}$$

in which each geodesic γ_i occurs infinitely many times. We denote the geodesics of the sequence (5.2) more simply by

$$g_1, \quad g_2, \quad g_3, \quad \ldots. \tag{5.3}$$

Let

$$e_1, \quad e_2, \quad e_3, \quad \ldots, \tag{5.4}$$

be a sequence of positive numbers which tend to zéro as the subscript becomes infinite.

We shall choose a sequence

$$h_1, \quad h_2, \quad h_3, \quad \ldots, \tag{5.5}$$

of geodesic rays emanating from P and terminating on λ.

The choice of h_1. — Let θ denote the arc length on λ measured on λ from one end point of λ. Let a point on λ be denoted by the corresponding value of θ. Let g'_1 be an unending geodesic congruent to g_1 with at least one end point θ_1 on λ. That such a geodesic exists follows from Lemma A in § **3**. For we have merely to choose a transformation T of G with at least one fixed point on λ and apply T or its inverse a sufficient number of times to g_1 to obtain the required geodesic g'_1.

It follows from Theorem 3.4 and the hypothesis of unicity that there is a geodesic ray which issues from P, terminates at the point θ_1 on λ, and is asymptotic to g'_1. There will accordingly exist a positive constant η_1, so small that each geodesic ray which issues from P and terminates on the segment of λ for which

$$\theta_1 - \eta_1 < \theta < \theta_1 + \eta_1, \tag{5.6}$$

will possess at least one element within an R-distance e_1 of some element on g'_1. We choose η_1 so small that the closure of the segment defined by (5.6) is interior to λ, and then choose h_1 as any geodesic ray which issues from P and terminates on the segment (5.6) of λ.

The choice of h_n. — Proceeding inductively we suppose that the entities

$$\theta_{n-1}, \quad \eta_{n-1}, \quad h_{n-1},$$

have been defined. We let g'_n be a periodic geodesic congruent to g_n with at least one end point θ_n on the segment of λ for which

$$(5.7) \qquad \theta_{n-1} - \eta_{n-1} < \theta < \theta_{n-1} + \eta_{n-1}.$$

There will then exist a positive constant η_n so small that the closure of the segment

$$(5.8) \qquad \theta_n - \eta_n < \theta < \theta_n + \eta_n,$$

of λ is a subsegment of (5.7) and is such that each geodesic ray which issues from P and terminates at a point θ on (5.8) will possess at least one element within an R-distance e_n of some element on g'_n. We choose h_n as any such geodesic ray.

Let E_n be the element on h_n whose initial point lies at P. Let E be a cluster element of the elements E_n, and g a geodesic ray issuing from P with the direction of E. The geodesic ray g will terminate on λ. I say moreover that g will be transitive.

For g will possess at least one element within an R-distance e_n of some element on g'_n. Let γ_m be an arbitrary geodesic of the set (5.1). By virtue of the choice of the sequence (5.2) the set of all elements on g or on geodesic rays congruent to g will include at least one element arbitrarily near some element on γ_m, and hence will include an element arbitrarily near *each* element on γ_m.

But according to Theorem 3.5 each geodesic on R is a limit geodesic of the set of all periodic geodesics on R. Thus each geodesic on R is a limit geodesic of the set of geodesics congruent to g. Hence g is a transitive ray, and the theorem is proved.

6. The transitivity function $\varphi_g(e)$. — We shall prove the following lemma.

Lemma. — *Let g be a transitive geodesic ray issuing from a point* P. *Corresponding to each positive number α there exists a number* $L_g(\alpha) \geqq 0$ *such that the segment of g consisting of points at most an* R-*distance* $L_g(\alpha)$

from P *defines a curve in the phase-space* M^* *at most an* R-*distance* α *from each element of* M^*.

Let q_n be a segment of g with initial point at P and with an R-length n. Let q_n^* be the representative of q_n in the phase-space M^*.

Suppose the lemma fails to hold for a positive number α. Corresponding to each positive integer m there must then exist an element E_m on M^* at an R-distance from q_m^* greater than α. Let E be a cluster element of the elements E_m. The distance of E from the representative of g on M^* will be at least α, contrary to the fact that g is transitive. The lemma is accordingly true.

Corresponding to a transitive geodesic ray g and a positive number α there will exist a greatest lower bound $\varphi_g(\alpha)$ of the numbers $L_g(\alpha)$ of the lemma.

We term $\varphi_g(\alpha)$ *the transitivity function defined by* g.

The transitivity function is monotonically decreasing. For large values of α it is zero. For all positive values of α less than a positive constant α_g it is positive. It become infinite as α tends to zero. Cf. Birkhoff's Ergodic Function, Birkhoff [II].

We shall prove the following theorem on the *deferring* of transitivity.

THEOREM 6.1. — *Let* $\psi(\alpha)$ *be an arbitrary positive function which becomes positively infinite as* α *tends to zero. If the hypothesis of unicity holds, there exists a transitive geodesic ray* g *issuing from an arbitrary point* P *of* R *such that the corresponding transitivity function* $\varphi_g(\alpha)$ *satisfies the relation*

$$\varphi_g(\alpha) > \psi(\alpha),$$

for an infinite sequence of positive values of α *which tend to zero as a limit.*

We shall prove this theorem by suitably altering the proof of Theorem 5.1.

We begin by choosing the sets (5.1) to (5.4) as in § 5. We also choose θ_1 and g_1' as in § 5.

Let a_1 be a geodesic ray issuing from P with its ideal end point at the point θ_1 on λ. The geodesic ray a_1 will be asymptotic to g'_1. In the phase-space M^* the set A_1 of elements determined by a_1 will have at most the elements determined by g_1 as limit elements. There will accordingly exist an element E_1 on M^* and a positive constant $\alpha_1 < e_1$ such that

$$D(E_1, A_1) > \alpha_1, \tag{6.1}$$

where $D(E_1, A_1)$ denotes the R-distance on M^* between the element E_1 and the set A_1. We choose a positive constant L_1 such that

$$L_1 > \psi(\alpha_1), \tag{6.2}$$

and let H_1 denote the set of elements on M^* determined by an R-length L_1 on a geodesic ray with initial end point at P and ideal end point θ on an interval of the form (5.6). We subject the constant η_1 appearing in (5.6) to the restrictions imposed in § 5, requiring further that η_1 be so small that on M^*

$$D(E_1, H_1) > \alpha_1.$$

This is possible by virtue of (6.1). With η_1 so restricted the geodesic ray h_1 is chosen as any transitive geodesic ray which issues from P and terminates on the segment of λ defined by (5.6). We observe that

$$\varphi_{h_1}(\alpha_1) > L_1,$$

and hence that

$$\varphi_{h_1}(\alpha_1) > \psi(\alpha_1). \tag{6.3}$$

We proceed inductively, supposing that

$$\theta_{n-1}, \quad \eta_{n-1}, \quad h_{n-1}, \quad \alpha_{n-1}, \quad L_{n-1}$$

have already been defined. Let g'_n be a periodic geodesic congruent to g_n with at least one ideal end point at the point θ_n on the segment (5.7) of λ. Let a_n be a geodesic ray issuing from P with its ideal end point at the point θ_n on λ. We continue exactly as in the preceding paragraph, replacing all symbols with subscript 1 by the same symbol with subscript n. For the geodesic ray h_n thereby defined we have

$$\varphi_{h_n}(\alpha_n) > \psi(\alpha_n). \tag{6.4}$$

As in § 5, so here it follows that the geodesic rays h_n possess a limit geodesic ray g issuing from P, and that g is transitive. The geodesic g is an admissible choice for each of the geodesics h_n so that

$$\varphi_g(\alpha_n) > \psi(\alpha_n).$$

The sequence α_n tends to zero as a limit as n becomes infinite, and the theorem is satisfied as stated.

We continue with the following theorem.

THEOREM 6.2. — *If the hypothesis of unicity holds, the transitive geodesic rays issuing from a given point* P *of* R *possess directions at* P *which are everywhere dense and non-denumerable.*

That the directions of transitive geodesic rays issuing from P are everywhere dense follows from the fact stated in Theorem 3.1 that the ideal end points of transitive geodesic rays can be chosen on arbitrary segments of the unit circle.

To complete the proof of the theorem let us suppose that the set of transitive geodesics issuing from P are denumerable and can accordingly be given by a sequence

$$b_1, \quad b_2, \quad \ldots. \tag{6.5}$$

Let

$$\varphi_1, \quad \varphi_2, \quad \ldots$$

be the transitivity functions corresponding to the respective geodesic rays of the set (6.5). Let

$$\alpha_1 > \alpha_2 > \alpha_3 > \ldots$$

be a sequence of positive numbers which tend to zero as the subscript becomes infinite. Let $\psi(\alpha)$ be a function which is given by the sum

$$\psi(\alpha) = \varphi_1(\alpha) + \ldots + \varphi_n(\alpha),$$

for α on the interval

$$\alpha_n \geqq \alpha > \alpha_{n+1} \qquad (n = 1, 2, \ldots),$$

and which is 1 for values of $\alpha > \alpha_1$.

According to the preceding theorem there will exist a transitive geodesic ray g issuing from P for which the corresponding transitivity function will satisfy the relation

$$\varphi_g(\alpha) > \psi(\alpha),$$

for an infinite sequence of positive values of α tending to zero as the enumerating subscript becomes infinite. Hence for a fixed m,

$$\varphi_g(\alpha) > \varphi_m(\alpha)$$

for some value of α, so that g cannot be identical with b_m. Thus g cannot appear in the sequence (6.5), contrary to our hypothesis.

We conclude that the theorem is true.

Recall that the set of elements on R determines the phase-space M. The preceding theorem concerning the geodesic rays through a fixed point P are special cases of a more general class of theorems. The set of elements with a common initial point P determines a curve μ on M. This curve has the property that the geodesics determined by its elements are not identical with a single geodesic. In general any arc in the phase-space M whose elements define more than one geodesic on R will be termed *general*.

We state the following theorem.

THEOREM 6.3. — *If the hypothesis of unicity holds, the transitive geodesics defined by elements on a general arc μ in the phase-space* M *are everywhere dense on μ and non-denumerable.*

As the element E ranges over a general arc μ, the geodesic g defined by E will vary through a one-parameter family of geodesics. At least one of the ideal end points of these geodesics g will cover the whole of a segment of the unit circle at least once. For otherwise the geodesics g would reduce to a single geodesic.

The proof of the theorem can be obtained by making obvious modifications in the proofs of Theorems 5.1, 6.1 and 6.2, in particular by replacing the phrase « a geodesic issuing from P » by the phrase « a geodesic defined by an element on μ ». The geodesic rays h_n in the present proof will not be uniquely determined by their ideal end points on the unit circle, nor can these ideal end points in general be chosen arbitrarily subject to the earlier restrictions. Each such ideal end point must here be chosen among the totality of ideal end points which satisfy the earlier restrictions and are defined by elements on μ. Further details are unnecessary.

7. Uniform instability. — If there is no conjugate point of a point P on any geodesic ray issuing from P, the geodesic rays issuing from P will form a field covering R in a one-to-one manner P alone excepted. If there are no pairs of conjugate points on any geodesic on R, it follows from the Weierstrass field theory of the calculus of variations that each geodesic on R is of class A.

But the hypothesis of unicity is not necessarily valid even when each geodesic on R is of class A as one can readily show by examples. To insure the validity of the hypothesis of unicity we must introduce more stringent restrictions on the geodesics g.

Let g be an arbitrary geodesic. Let g be referred to normal geodesic coordinates (x, y). For such coordinates

$$ds^2 = C^2(x, y)\,dx^2 + dy^2, \tag{7.0}$$

where $y = 0$ along g and $C(x, 0) \equiv 1$, while the curves $x =$ constant are geodesics normal to g and y gives the R-distance along these geodesic normals measured in an arbitrary sense from g. The equation of normal variation from g takes the form

$$\frac{d^2 w}{dx^2} + K(x) w = 0, \tag{7.1}$$

where x is the arc length along g measured from an arbitrary point of g and $K(x)$ is the curvature of the surface at the point x on g. Solutions $w(x)$ of (7.1) will be said to be *based* on g. A solution $w(x)$ of (7.1) such that

$$w^2(0) + w'^2(0) = 1, \tag{7.2}$$

will be termed *normal* at $x = 0$.

We shall say that the geodesics g on R are *uniformly* unstable if there are no pairs of conjugate points on g and if there exists a function $M(x)$ with the following properties :

a. The function $M(x)$ *is positive and continuous for all values of* x *exceeding some positive constant* λ, *and becomes infinite with* x.

b. Any solution of (7.1) *based on g and normal at* $x = 0$ *satisfies the relation*

$$|w(x_1)| + |w(x_2)| > M(x), \tag{7.3}$$

for $x > \lambda$ *and* $-x_1$ *and* x_2 *greater than* x.

c. The function $M(x)$ *is independent of the choice of the point* P *on g from which* x *is measured and of the choice of g on* R.

Geodesics on surfaces of negative curvature are uniformly unstable as one can readily show. But one can clearly replace considerable areas on surfaces of negative curvature by areas of positive curvature and still retain the property of uniform instability for the geodesics thereby defined.

The field F. — Let g' and g'' be two distinct boundary geodesics of the same type. Under the assumption that there are no pairs of conjugate points on any geodesic on R we shall construct a field F of geodesics related to g' and g''.

Let A_1 and A_2 be two points on g'. Let h_1 and h_2 be two sensed geodesics which join the points A_1 and A_2 on g' to points B_1 and B_2 respectively on g''. Suppose moreover that h_1 and h_2 give paths from A_1 and A_2 to g'' as short as possible. The R-lengths of h_1 and h_2 will be at most the universal constant K of Theorem 3.1.

The points A_1, A_2, B_1, B_2 are the vertices of a geodesic quadrilateral which we shall cover with a special field of geodesics. To that end we regard points on h_1 and h_2 which divide h_1 and h_2 in the same ratio with respect to R-length as corresponding.

Let F *be the family of geodesics which join corresponding points of* h_1 *and* h_2. *We shall investigate the representation of this family.*

We turn to an arbitrary geodesic g of the family F. Let α represent the arc length on h_2 measured along h_2 from g. Let the neighborhood of g be referred to normal geodesic coordinates (x, y) as in (7.0). Let g^α be the geodesic of F which joins the point α on h_2 to the corresponding point on h_1. For α near zero the geodesics g^α can be represented in the form

$$y = \varphi(x, \alpha).$$

The functions $\varphi(x, \alpha)$ will be of class C^2 for α near o and x on an interval of the form

$$\underline{x}(\alpha) \leqq x \leqq \overline{x}(\alpha)$$

where the functions $\underline{x}(\alpha)$ and $\overline{x}(\alpha)$ give the values of x as functions of α at the respective intersections of g^α with h_1 and h_2.

A. *We shall show that*

$$\varphi_\alpha(x, 0) \neq 0 \tag{7.4}$$

along g.

Turning to the final end point of g we have

$$\varphi_\alpha[x_2(0), 0] \neq 0. \tag{7.5}$$

For if h_2 is represented in terms of the arc length α, for α near o, in the form

$$y = \overline{y}(\alpha), \qquad x = \overline{x}(\alpha),$$

we obtain the identity

$$\overline{y}(\alpha) \equiv \varphi[\overline{x}(\alpha), \alpha],$$

from which it follows that

$$\frac{d\overline{y}}{d\alpha} \equiv \varphi_x \frac{d\overline{x}}{d\alpha} + \varphi_\alpha. \tag{7.6}$$

If φ_α were zero at the intersection of g and h_2 it would follow from (7.6) that h_2 and g would be tangent and hence be continuations of identical geodesics. But this is impossible, since in passing from h_2 to h_1 g would then intersect g' or g'' twice. Hence (7.5) holds as stated.

We could prove in a similar manner that $\varphi_\alpha(x, 0)$ does not vanish at the intersection of g and h_1. Thus $\varphi_\alpha(x, 0)$ vanishes at neither end point of g.

We can now establish (7.4). For if $\varphi_\alpha(x, 0)$ vanished at an intermediate point of g the geodesics of F neighboring g would cross g near the zero of $\varphi_\alpha(x, 0)$ and hence cross g twice, which is impossible. Thus (A) holds as stated.

A second representation of the field F. — It follows from (7.4) that the trajectories orthogonal to the field F are well defined at each point of F. In fact the equation

$$y - \varphi(x, \alpha) = 0,$$

can be solved for α as a function $\alpha(x, y)$ of class C^2 neighboring each point on g. The differential equation of the trajectories orthogonal to the geodesics of F can then be locally represented in the form

$$C^2(x, y)\, dx + \varphi_x[x, \alpha(x, y)]\, dy = 0.$$

These orthogonal trajectories are accordingly without singularity.

We shall obtain a new representation of the field F. Let γ be a trajectory orthogonal to the geodesics of F issuing from a point p on g' midway between A_1 and A_2. Let $d(p, g'')$ be the R-distance from p to g''. We know that

$$(7.7) \qquad 0 < d(p, g'') \leqq K,$$

where K is the universal constant of Theorem 3.1. We shall *admit* only those geodesic quadrilaterals whose sides A_1A_2 on g' have an R-length greater than $4K$.

On an admissible quadrilateral γ can be continued on F over an R-length at least $d(p, g'')$ without passing off from F. For during such a continuation γ can at most reach g'', since the R-distance from p to g'' is $d(p, g'')$. Moreover during such a continuation γ cannot reach either h_1 or h_2. For if h_1, for example, were reached, A_1 and p could be joined by a combination of arcs of h_1 and γ of R-length at most $2K$ contrary to the choice of A_1A_2 on g'. Thus γ can be continued on F for an R-length at least $d(p, g'')$.

Let μ be the R-length along γ measured from p. Let μ_0 be a value of μ on the interval

$$(7.8) \qquad 0 \leqq \mu \leqq d(p, g''),$$

and let g^0 be the geodesic of the field F which passes through the point μ_0 on γ. Let the neighborhood of g^0 be referred to normal geodesic coordinates (x, y) with g^0 as the base. The geodesics of F neighboring g^0 can be represented in the form

$$(7.9) \qquad y = \psi(x, \mu),$$

where the function $\psi(x, \mu)$ is of class C^2 in its arguments. The function

$$w(x) = \psi_\mu(x, \mu_0) \tag{7.10}$$

will be a solution of the equation of variation

$$\frac{d^2 w}{dx^2} + K(x) w = 0 \tag{7.11}$$

based on g^0. Concerning $w(x)$ we shall prove the following statement.

B. *The solution $w(x)$ of (7.11) given by (7.10) satisfies the initial condition $w(0) = 1$.*

For μ near μ_0, γ can be given in the form

$$y = y^*(\mu), \qquad x = x^*(\mu), \tag{7.12}$$

and we have the identity

$$y^*(\mu) \equiv \psi[x^*(\mu), \mu].$$

From this identity we obtain a second identity

$$\frac{dy^*}{d\mu} \equiv \psi_x \frac{dx^*}{d\mu} + \psi_\mu. \tag{7.13}$$

But

$$\psi_x(x, \mu_0) \equiv 0.$$

Moreover at the intersection of g^0 and γ, $\frac{dy^*}{d\mu} = 1$ as follows from the choice of γ and of our coordinate system (x, y). Statement (B) follows from (7.13).

We come to a basic theorem.

THEOREM 7.1. — *Uniform instability of the geodesics on* R *implies that the hypothesis of unicity holds on* R.

We suppose that the theorem is false and that g' and g'' are two different geodesics of the same type. We shall prove that $g' = g''$ thereby arriving at a contradiction.

We identify g' and g'' with the preceding geodesics g' and g'' and construct a geodesic quadrilateral $A_1 A_2 B_1 B_2$ and field F as before.

Neighboring the geodesic g^0 of F we make use of the second representation (7.9) of the field F. Let s_1 and s_2 be the arc lengths on h_1 and h_2 respectively, measured from g'. Let g_μ be the geodesic of the field F which intersects the curve γ at the point on γ with parameter μ. Let

$$s_1(\mu), \quad s_2(\mu)$$

be the respective values of the arc lengths on h_1 and h_2, measured from g^0, at the points on h_1 and h_2 at which the geodesic g_μ meets h_1 and h_2. In the normal geodesic coordinate system (x, y) based on g^0 suppose x is measured from the point of intersection of g^0 and γ in the sense that leads from h_1 to h_2. Let

$$x_1(\mu), \qquad y_1(\mu),$$

be the coordinates of the point of intersection of g_μ with h_1. For $\mu = \mu_0$ we have

$$\left(\frac{ds_1}{d\mu}\right)^2 = \left(\frac{dx_1}{d\mu}\right)^2 + \left(\frac{dy_1}{d\mu}\right)^2 \geqq \left(\frac{dy_1}{d\mu}\right)^2,$$

so that

$$\frac{ds_1}{d\mu} \geqq \psi_\mu[a_1, \mu_0] = w(a_1) > 0$$

where $a_1 = x_1(\mu_0)$.

Similarly let $x_2(\mu)$, $y_2(\mu)$ be the point of intersection of g_μ with h_2. We can show as above that for $\mu = \mu_0$

$$\frac{ds_2}{d\mu} \geqq w(a_2) > 0,$$

where $a_2 = x_2(\mu_0)$. We thus find that for $\mu = \mu_0$

$$\frac{d}{d\mu}[s_1(\mu) + s_2(\mu)] \geqq w(a_1) + w(a_2). \tag{7.14}$$

Let the R-distance between A_1 and A_2 on g' be 2σ. We have denoted the intersection of g' and γ by p. The segment $A_2 p$ of g' has the R-length σ. It can be joined by a broken geodesic arc consisting of a segment of h_2 of R-length at most K, a segment of g^0 of R-length a_2, and a segment of γ of R-length at most K. From the minimizing property of the segment $A_2 p$ of g' we see that

$$\sigma < K + a_2 + K,$$

A similar relation holds upon replacing a_2 by $-a_1$. Thus

$$a_2 > \sigma - 2K, \quad -a_1 > \sigma - 2K.$$

According to (B), $w(0)=1$. Hence $w(x)$ will be « normal » at $x=0$ if multiplied by a suitable positive constant at most 1. It follows from the hypothesis of uniform instability that

$$w(a_1) + w(a_2) > M(\sigma - 2K), \tag{7.15}$$

if σ is sufficiently large. If σ is sufficiently large, the right member of (7.15) will be arbitrarily large according to the nature of $M(x)$, and in particular will be greater than $\frac{2K}{a}$ where

$$a = d(p, g'').$$

For such a choice of σ, (7.15) gives the relation

$$a[w(a_1) + w(a_2)] > 2K. \tag{7.16}$$

By virtue of (7.14) and (7.16) the sum of the R-lengths of h_1 and h_2 will be greater than $2K$. From this contradiction we infer that $g' = g''$.

The proof of the theorem is complete.

Theorems 6.2 and 7.1 combine into the following theorem.

THEOREM 7.2. — *If the geodesics on* R *are uniformly unstable the transitive geodesic rays issuing from an arbitrary fixed point* P *on* R *have directions at* P *which are everywhere dense and non-denumerable.*

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