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On the infinitesimal deformations of tensor submanifolds

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On the infinitesimal deformations of tensor submanifolds;

By P. DIENES and E. T. DAVIES.

General conventions. — I. The summation symbol Σ is suppressed if it applies to terms with identical suffixes.

II. The first letters of Latin and Greek alphabets as suffixes vary from 1 to n, the middle letters $i, j, k, \ldots$, from 1 to $m (< n)$, and the end letters $p, q, \ldots$ from $m+1$ to n.

Part I. — Definitions, General Properties.

1. The deformation of a tensor manifold. — A general n-dimensional linearly connected tensor manifold or space A_n is determined by two independent sets of functions:

(i) the metric parameters $a_{\alpha\beta}$ which assign a measure ds to the distance between the neighbouring points $P(x)$ and $Q(x + dx)$ by the formula

$$(1) \qquad ds^2 = a_{\alpha\beta}\, dx^\alpha\, dx^\beta.$$

(ii) the connexion parameters $\Gamma^\alpha_{\beta\gamma}$ which define parallelism (equipollence) between vectors and tensors at neighbouring points by the formulae

$$(2) \quad v^\alpha(Q \| P) = v^\alpha(Q) + \Gamma^\alpha_{\beta\gamma} v^\beta\, dx^\gamma, \qquad v_\beta(Q \| P) = v_\beta(Q) - \Gamma^\alpha_{\beta\gamma} v_\alpha\, dx^\gamma.$$

The substitution of $v^\alpha(Q \| P)$ at P for $v^\alpha(Q)$ at Q is called the « parallel transport » of $v^\alpha(Q)$ from Q to P.

Special spaces are defined by special sets of metric and connexion parameters, or else by relations between these two sets. For example, a Riemann space V_n is specified by the two conditions

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}, \tag{3}$$

$$\nabla_{\gamma} a_{\alpha\beta} \equiv \partial_{\gamma} a_{\alpha\beta} - \Gamma^{\delta}_{\alpha\gamma} a_{\delta\beta} - \Gamma^{\delta}_{\beta\gamma} a_{\alpha\delta} = 0, \tag{4}$$

in which case the functions $\Gamma^{\alpha}_{\beta\gamma}$ reduce to the three-index symbols $\left\{ {\alpha \atop \beta\gamma} \right\}$ of Christoffel (¹).

A space in which (4) is satisfied will be called a *metric space* (²), (or a Riemann space with torsion), since in such a case length of a vector and angle between two vectors are unchanged by parallel transport. In the classification of spaces given by Schouten (1924, p. 75) such a space would be of the type III Aγ.

In order to define the deformation of A_n we remark that a change of variables

$$'x^a = f^a(x^1, \ldots . x^n) \equiv f^a(x^{\alpha}) \equiv f^a(x), \tag{5}$$

admits two different geometrical interpretations. It can be regarded either (i) as a mapping of the x-space upon the $'x$-space, i. e. as a straightforward *transformation* of the x-space into the $'x$-space, or (ii) as a mapping of the x-space upon itself, in which case the point of coordinates $'x^{\alpha}$ is regarded as the point of coordinates $'x$ in the x-space. In this second interpretation the same change of variables will be called a *displacement*, and to indicate the fact that the new points are also in the x-space we shall replace the Latin suffix a by a Greek one (³).

A change of variables of the *particular form*

$$'x^a = x^a + \epsilon\xi^a(x), \tag{6}$$

where ϵ is a small constant, is called an *infinitesimal transformation*, or an *infinitesimal displacement* of the x-space according to the interpretation chosen.

(¹) Eisenhart and Veblen (1922, p. 20).

(²) In the terminology of Cartan (1923) and Lagrange (1926) this would be an « espace à connexion euclidienne ».

(³) For the distinction between Latin and Greek suffixes see Dienes (1932).

For an infinitesimal transformation, we have

$$(7) \qquad 'dx^a = dx^a + \epsilon\, \partial_\alpha \xi^a . dx^\alpha = A^a_\alpha\, dx^\alpha \qquad \left(\partial_\alpha = \frac{\partial}{\partial x^\alpha}\right),$$

where

$$(8) \qquad A^a_\alpha = \delta^a_\alpha + \epsilon\, \partial_\alpha \xi^a.$$

The reciprocals A^β_b are defined as usual by

$$(9) \qquad A^\beta_b A^a_\beta = \delta^a_b \qquad \text{or} \qquad A^\beta_b A^b_\alpha = \delta^\beta_\alpha,$$

giving

$$(10) \qquad A^\beta_b = \delta^\beta_b - \epsilon\, \partial_b \xi^\beta \qquad \left(\partial_b = \frac{\partial}{\partial' x^b}\right).$$

Hence, to first order quantities in ϵ, vectors and tensors are transformed by the rules

$$(11) \qquad v^a = v^\alpha A^a_\alpha, \qquad v_b = v_\beta A^\beta_b,$$

$$(12) \qquad T^{a_1 \ldots a_p}_{b_1 \ldots b_q}('x) = T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}(x)\, A^{a_1 \ldots a_p}_{\alpha_1 \ldots \alpha_p} A^{\beta_1 \ldots \beta_q}_{b_1 \ldots b_q}$$
$$= T^{a_1 \ldots a_p}_{b_1 \ldots b_q}(x) + \epsilon \sum_{s=1}^{p} T^{a_1 \ldots \gamma \ldots a_p}_{b_1 \ldots b_q} \partial_\gamma \xi^{a_s} - \epsilon \sum_{s=1}^{q} T^{a_1 \ldots a_p}_{b_1 \ldots \gamma \ldots b_q} \partial_{b_s} \xi^\gamma.$$

Metric and connexion parameters are transformed by the usual formulae

$$(13) \qquad a_{ab} = a_{\alpha\beta} A^{\alpha\beta}_{ab},$$

$$(14) \qquad \Gamma^a_{bc} = \Gamma^\alpha_{\beta\gamma} A^a_\alpha A^{\beta\gamma}_{bc} + A^a_\alpha\, \partial_c A^\alpha_b.$$

The tensor space so constructed will be referred to as the transform $'A_n$ of A_n. Like every transform it represents, to first approximation, the *same geometry* A_n in new variables, since corresponding vectors have the same lengths, and since parallelism is preserved in the transformation.

In the second interpretation of the change of variables the point x is displaced *in the original space* A_n to $'x$. Thus in the *displaced* A_n vectors and tensors at $'x$ are just the vectors and tensors of A_n at $'x$, and the metric and connexion parameters are $a_{\alpha\beta}('x)$ and $\Gamma^\alpha_{\beta\gamma}('x)$. This displaced space will also be referred to as the *deform* of A_n.

If we introduce the notations $'a_{\alpha\beta}('x)$ and $'\Gamma^{\alpha}_{\beta\gamma}('x)$ for the metric and connexion parameters of the *transformed* A_n at $'x$, since the $'A_n$ is the same geometry as the A_n (except that it is attached to different points), we can consider $'a_{\alpha\beta}('x)$ and $'\Gamma^{\alpha}_{\beta\gamma}('x)$ as the *representatives* of A_n at the point $'x$.

A measure for the *deformation* of A_n is therefore obtained by comparing the metric and connexion parameters of the new Geometry (the *deform* of A_n) at $'x$, with the representatives of A_n at $'x$.

And since in this way we are going to consider the transform of A_n as being at the points of X_n, we shall replace the Latin suffixes of $'A_n$ by Greek ones in order to avoid a *formal* clash of suffixes. Thus we shall denote A^a_α and A^α_a by A^β_α and $\overline{A}^\alpha_\beta$ respectively, and similarly v^a, v_a, etc., will be replaced by $'v^\alpha$ and $'v_\alpha$, etc.

Thus the deformation of metric and connexion parameters will be measured by

$$(15)\qquad \delta a_{\alpha\beta} \equiv a_{\alpha\beta}('x) - 'a_{\alpha\beta}('x) = \epsilon\,[\xi^\gamma \partial_\gamma a_{\alpha\beta} + a_{\alpha\gamma}\partial_\beta \xi^\gamma + a_{\gamma\beta}\partial_\alpha \xi^\gamma],$$

which can be thrown into the tensor form

$$(16)\qquad \delta a_{\alpha\beta} = \epsilon\left[\nabla_\gamma a_{\alpha\beta} + a_{\alpha\gamma}\nabla_\beta \xi^\gamma + a_{\gamma\beta}\nabla_\alpha \xi^\gamma + 2S_{\alpha\gamma}^{\cdot\cdot\cdot\delta}\, a_{\delta\beta}\xi^\gamma + 2S_{\beta\gamma}^{\cdot\cdot\cdot\delta}\, a_{\alpha\delta}\xi^\delta\right].$$

Similarly

$$(17)\qquad \begin{aligned}\delta\Gamma^{\alpha}_{\beta\gamma} &\equiv \Gamma^{\alpha}_{\beta\gamma}('x) - '\Gamma^{\alpha}_{\beta\gamma}('x)\\ &= \epsilon\left[\partial_\gamma\partial_\beta \xi^\alpha + \Gamma^{\alpha}_{\delta\gamma}\partial_\beta \xi^\delta + \Gamma^{\alpha}_{\beta\delta}\partial_\gamma \xi^\delta - \Gamma^{\delta}_{\beta\gamma}\partial_\delta \xi^\alpha + \xi^\delta \partial_\delta \Gamma^{\alpha}_{\beta\gamma}\right],\end{aligned}$$

or, in tensor form

$$(18)\qquad \delta\Gamma^{\alpha}_{\beta\gamma} = \epsilon\left[\nabla_\gamma\nabla_\beta \xi^\alpha + R^{\alpha}_{\cdot\beta\gamma\delta}\,\xi^\delta + 2\nabla_\gamma\left(S_{\beta\delta}^{\cdot\cdot\alpha}\,\xi^\delta\right)\right].$$

In the sequel we shall frequently meet with a special kind of covariant derivation, in which the lower suffixes of the connexion parameters have been interchanged. We can call it the *conjugate* covariant derivation, and denote it by $\overset{*}{\nabla}$, where for an arbitrary contravariant vector v^α, we have

$$(19)\qquad \overset{*}{\nabla}_\gamma v^\alpha = \partial_\gamma v^\alpha + \Gamma^{\alpha}_{\gamma\beta} v^\beta.$$

Expressed in terms of ordinary covariant derivation, we evidently have

$$\overset{*}{\nabla}_\gamma v^\alpha = \nabla_\gamma v^\alpha + 2 S_{\gamma\beta}^{\cdot\cdot\alpha} v^\beta, \tag{20}$$

so that when the connexion parameters are symmetrical, the two operators coincide.

In consequence of (19), we can write (16) and (18) in the shorter forms

$$\delta a_{\alpha\beta} = \epsilon \left[\xi^\gamma \nabla_\gamma a_{\alpha\beta} + a_{\alpha\gamma} \overset{*}{\nabla}_\beta \xi^\gamma + a_{\gamma\beta} \overset{*}{\nabla}_\alpha \xi^\gamma\right], \tag{21}$$

$$\delta \Gamma^\alpha_{\beta\gamma} = \epsilon \left[\nabla_\gamma \overset{*}{\nabla}_\beta \xi^\alpha + R^\alpha_{\cdot\beta\gamma\delta} \xi^\delta\right]. \tag{22}$$

For a Reimann space V_n, these reduce to

$$\delta a_{\alpha\beta} = \epsilon \left[\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha\right], \tag{23}$$

$$\delta \left\{ {\alpha \atop \beta\gamma} \right\} = \epsilon \left[\nabla_\gamma \nabla_\beta \xi^\alpha + R^\alpha_{\cdot\beta\gamma\delta} \xi^\delta\right]. \tag{24}$$

It is easily proved that the $\delta\Gamma^\alpha_{\beta\gamma}$ in the case of a V_n is equal to the difference of the three-index symbols of Christoffel for the metric parameters $\bar{a}_{\alpha\beta} = a_{\alpha\beta} + \delta a_{\alpha\beta}$, and for the original ones. Moreover, in this case, since the whole Geometry is determined by the metric parameters, the deformation of a *structure* [1] tensor of V_n can be determined by merely calculating the tensor using the new metric parameters $\bar{a}_{\alpha\beta}$ (and consequently the new connexion parameters $\bar{\Gamma}^\alpha_{\beta\gamma}$), and subtracting the old tensor from the result. That the deformed space is also Reimannian is therefore immediately evident. By a definition of parallelism given in Dienes [1933, (iii)] however, it is seen that the deformed space shall always be of the same nature as the original space, so that any special properties possessed by the space will be preserved in deformation.

2. The definition of the deformation of individual vectors and tensors implies a comparison of vectors and tensors of the deform of A_n with those of $'A_n$ both at $'x$. Since however we usually attribute the deformation to the vectors and tensors of A_n at x, this implies a comparison between the pencils of tensors of A_n attached to x and $'x$ respectively. The simplest kind of correspondence is by

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(1) A structure tensor is one involving the connexion parameters of the space.

parallelism, by which we mean that in the displacement of het point x to $'x$, the vectors and tensors at x are carried along to $'x$ by parallel transport.

Thus, starting with a v^α say at x, its deformation due to the displacement (6) will be conveniently measured by

$$(1)\qquad \Delta v^\alpha \equiv v^\alpha(x\,\|\,'x) - {}'v^\alpha('x) = -\epsilon\, v^\beta\left(\Gamma^\alpha_{\beta\gamma}\xi^\gamma + \partial_\beta \xi^\alpha\right) = -\epsilon\, v^\beta\, \dot{\nabla}_\beta \xi^\alpha$$

and this is called the *direct deformation* (1).

In the case of a vector field $v^\alpha(x)$ we may also take $v^\alpha('x)$ as the vector corresponding to $v^\alpha(x)$. This leads to the *field deformation*

$$(2)\qquad \delta v^\alpha \equiv v^\alpha('x) - {}'v^\alpha('x) = \epsilon\,(\xi^\gamma \partial_\gamma v^\alpha - v^\gamma \partial_\gamma \xi^\alpha) = \epsilon\left[\xi^\gamma \nabla_\gamma v^\alpha - v^\gamma \dot{\nabla}_\gamma \xi^\alpha\right],$$

which, in virtue of (1), can also be written

$$(3)\qquad \delta v^\alpha = \epsilon\, \xi^\gamma \nabla_\gamma v^\alpha + \Delta v^\alpha.$$

If we have an individual vector v^α at x, a field can be defined between x and $'x$ along $x^\alpha + \epsilon\,\xi^\alpha$ by putting $\xi^\gamma \nabla_\gamma v^\alpha = 0$, i. e. a field can be created by parallel transport. In this way, from a purely mathematical point of view, the measuring process Δ appears as a special case of the δ process. We notice, however that the δ measure depends upon the existence of a field but not upon a definition of parallel transport in the space, whereas Δ implies parallelism but no field.

The definitions are readily extended to general tensors, so that

$$
\begin{aligned}
(4)\qquad \Delta T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} &\equiv T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}(x\,\|\,'x) - {}'T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}('x) \\
&= \epsilon\left[\left(\sum_{s=1}^{q} T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\gamma\ldots\beta_q}\,\partial_{\beta_s}\xi^\gamma - \sum_{s=1}^{p} T^{\alpha_1\ldots\gamma\ldots\alpha_p}_{\beta_1\ldots\beta_q}\,\partial_\gamma \xi^{\alpha_s}\right)\right. \\
&\qquad \left. + \xi^\delta\left(\sum_{s=1}^{q} T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\gamma\ldots\beta_q}\,\Gamma^\gamma_{\beta_s\delta} - \sum_{s=1}^{p} T^{\alpha_1\ldots\gamma\ldots\alpha_p}_{\beta_1\ldots\beta_q}\,\Gamma^{\alpha_s}_{\gamma\delta}\right)\right] \\
&= \epsilon\left[\sum_{s=1}^{q} T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\gamma\ldots\beta_q}\,\dot{\nabla}_{\beta_s}\xi^\gamma - \sum_{s=1}^{p} T^{\alpha_1\ldots\gamma\ldots\alpha_p}_{\beta_1\ldots\beta_q}\,\dot{\nabla}_\gamma \xi^{\alpha_s}\right],
\end{aligned}
$$

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(1) This difference Δ has been used by Hayden (1931) in his study of curves in a Riemann space.

and

$$(5)\quad \delta T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = \epsilon\left[\xi^\gamma \partial_\gamma T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} + \sum_{s=1}^{q} T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\gamma\ldots\beta_q}\,\partial_{\beta_s}\xi^\gamma - \sum_{s=1}^{p} T^{\alpha_1\ldots\gamma\ldots\alpha_p}_{\beta_1\ldots\beta_q}\,\partial_\gamma \xi^{\alpha_s}\right]$$

$$= \epsilon\left[\xi^\gamma \nabla_\gamma T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} + \sum_{s=1}^{q} T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\gamma\ldots\beta_q}\,\overset{*}{\nabla}_{\beta_s}\xi^\gamma - \sum_{s=1}^{p} T^{\alpha_1\ldots\gamma\ldots\alpha_p}_{\beta_1\ldots\beta_q}\,\overset{*}{\nabla}_\gamma \xi^{\alpha_s}\right],$$

which can also be written

$$(5')\quad \delta T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = \epsilon\left[\xi^\gamma \overset{*}{\nabla}_\gamma T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} + \sum_{s=1}^{q} T^{\alpha_1\ldots\alpha_q}_{\beta_1\ldots\gamma\ldots\beta_q}\,\nabla_{\beta_s}\xi^\gamma - \sum_{s=1}^{p} T^{\alpha_1\ldots\gamma\ldots\alpha_p}_{\beta_1\ldots\beta_q}\,\nabla_\gamma \xi^{\alpha_s}\right].$$

On comparing (4) and (5) we have

$$(6)\qquad \delta T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = \Delta T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} + \epsilon\xi^\gamma \nabla_\gamma T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}.$$

In particular, the δ measure for the deformation of $a_{\alpha\beta}$ as a tensor field coincides with (1.16), and in a metric space $\overset{*}{V}_n$, we have

$$(7)\qquad \Delta a_{\alpha\beta} = \delta a_{\alpha\beta} = \epsilon\left[\overset{*}{\nabla}_\alpha \xi_\beta + \overset{*}{\nabla}_\beta \xi_\alpha\right].$$

For structure tensors like $S_{\beta\gamma}^{\cdot\cdot\alpha}$ and $R^\alpha_{\cdot\beta\gamma\delta}$ formed of the metric and connexion parameters, also $S_{\beta\gamma}^{\cdot\cdot\alpha}('x)$ and $R^\alpha_{\cdot\beta\gamma\delta}('x)$ might be taken as the representatives at $'x$ [instead of $'S_{\beta\gamma}^{\cdot\cdot\alpha}('x)$ and $'R^\alpha_{\cdot\beta\gamma\delta}('x)$]. In this way we can define the *structural deformations :*

$$(8)\qquad DS_{\beta\gamma}^{\cdot\cdot\alpha} \equiv S_{\beta\gamma}^{\cdot\cdot\alpha}(x\,\|\,'x) - S_{\beta\gamma}^{\cdot\cdot\alpha}('x) = -\,\epsilon\,\xi^\delta \nabla_\delta S_{\beta\gamma}^{\cdot\cdot\alpha}$$

and

$$(9)\qquad DR^\alpha_{\cdot\beta\gamma\delta} \equiv R^\alpha_{\cdot\beta\gamma\delta}(x\,\|\,'x) - R^\alpha_{\cdot\beta\gamma\delta}('x) = -\,\epsilon\,\xi^\eta \nabla_\eta R^\alpha_{\cdot\beta\gamma\delta}.$$

We notice that

$$(10)\qquad \Delta T - DT \equiv [T(x\,\|\,'x) - 'T('x)] - [T(x\,\|\,'x) - T('x)] = \delta T.$$

These deformation operators δ, Δ, and D satisfy such formal rules of manipulation as the following

$$(11)\qquad \delta(T + T') = \delta T + \delta T',$$

$$(12)\qquad \delta(TT') = (\delta T)T' + T(\delta T').$$

If we take the product $v^\alpha w_\beta$ for example, we have

$$\partial(v^\alpha w_\beta) = v^\alpha('x)\, w_\beta('x) - v^\gamma w_\delta A^\alpha_\gamma \overline{A}^\delta_\beta$$
$$= w_\beta(\xi^\delta \partial_\delta v^\alpha - v^\delta \partial_\delta \xi^\alpha) + v^\alpha(\xi^\delta \partial_\delta w_\beta + w_\delta \partial_\beta \xi^\delta),$$

or

$$\partial(v^\alpha w_\beta) = (\partial v^\alpha)\, w_\beta + v^\alpha(\partial w_\beta).$$

Similar results hold for Δ and D.

If we consider the permutability of the operator ∂ with covariant derivation, we have, for a contravariant vector

$$\text{(13)} \qquad \partial(\nabla_\alpha v^\beta) - \nabla_\alpha(\partial v^\beta) = (\partial \Gamma^\beta_{\gamma\alpha})\, v^\gamma$$

and for a covariant vector

$$\text{(14)} \qquad \partial(\nabla_\alpha v_\beta) - \nabla_\alpha(\partial v_\beta) = -(\partial \Gamma^\gamma_{\beta\alpha}),$$

the extension to general tensors being obvious. The operators ∂ and covariant derivation will therefore be permutable for all tensors provided the equation

$$\text{(15)} \qquad \partial \Gamma^\alpha_{\beta\gamma} = 0,$$

is satisfied. This is the condition that the transformation (1.6) should define an isomorphic transformation of the space, as proved by Slebodzinski (1932, equs. 1).

3. The Geometry A_m on a point submanifold X_m given by

$$\text{(1)} \qquad x^\alpha = f^\alpha(u', \ldots, u^m) \qquad (m < n),$$

is usually determined by « projection » in the following manner. From $dx^\alpha = \partial_\lambda f^\alpha . du^\lambda$ we put $v^\alpha = v^\lambda \partial_\lambda f^\alpha$ expressing v^λ in the A_n frame. Hence

$$\text{(2)} \qquad B^\alpha_\lambda = \partial_\lambda f^\alpha,$$

are the first set of projection factors forming, as λ varies from 1 to m, the m contravariant, base vectors for the tangent plane. We complete it into an n-dimensional split frame A^α_a by taking $n - m$ vectors $C^\alpha_\rho(u', \ldots, u^m)$ subject to the only condition

$$\text{(3)} \qquad \| A^\alpha_a \| \neq 0,$$

where $A^\alpha_a \equiv (B^\alpha_\lambda, C^\alpha_\rho)$. The facet determined by the pseudo-normals C^α_ρ will be called the « span » of the submanifold at the point in question. The reciprocals are determined in the usual way by putting $A^a_\alpha \equiv (B^\alpha_\lambda, C^\alpha_\rho)$ and by requiring that

$$(4) \qquad A^a_\alpha A^\alpha_b = \delta^a_b \qquad \text{or} \qquad A^a_\alpha A^\beta_a = \delta^\beta_\alpha .$$

The quantities

$$B^\alpha_\beta = B^\alpha_\mu B^\mu_\beta, \qquad C^\alpha_\beta = C^\alpha_\rho C^\rho_\beta,$$

with

$$(5) \qquad B^\alpha_\beta + C^\alpha_\beta = \delta^\alpha_\beta,$$

have been used extensively by Schouten (1924) instead of the projection factors B^α_λ, C^α_ρ. We remark that $B^\alpha_\beta v^\beta$ is the A_n-component of the projection of v^α on A_m. Since

$$v^\alpha = v^\beta \delta^\alpha_\beta = v^\beta (B^\alpha_\beta + C^\alpha_\beta),$$

the conditions $v^\alpha = B^\alpha_\beta v^\beta$, or $C^\alpha_\beta v^\beta = 0$ express the fact that v^α lies in A_m, and $v^\alpha = C^\alpha_\beta v^\beta$ or $B^\alpha_\beta v^\beta = 0$ that v^α is pseudo-normal to A_m, or lies in the « span » of A_m.

The projected metrics in A_m and in its span A^ρ_m are given by

$$(6) \qquad b_{\lambda\mu} = a_{\alpha\beta} B^{\alpha\beta}_{\lambda\mu} \qquad \text{and} \qquad c_{\rho\sigma} = a_{\alpha\beta} C^{\alpha\beta}_{\rho\sigma}.$$

A system of split frames of the kind just defined leads to a fourfold connexion with the following projected connexion parameters [1],

$$(7) \quad \begin{cases} \text{(i)} \quad l^\lambda_{\mu\nu} = B^\lambda_\alpha \nabla_\nu B^\alpha_{(\mu)} = -\pi_{\mu\lambda\nu}, & \text{(ii)} \quad \lambda^\pi_{\rho\nu} = C^\pi_\alpha \nabla_\nu C^\alpha_{(\rho)} = -\pi_{\rho\pi\nu}, \\ \text{(iii)} \quad s^\lambda_{\mu\sigma} = B^\lambda_\alpha \nabla_\sigma B^\alpha_{(\mu)} = -\pi_{\mu\lambda\sigma}, & \text{(iv)} \quad \sigma^\pi_{\rho\sigma} = C^\pi_\alpha \nabla_\sigma C^\alpha_{(\rho)} = -\pi_{\rho\pi\sigma}, \end{cases}$$

where for orthogonal frames the π functions reduce to the corresponding γ functions of Ricci [Dienes, 1933 (ii)]. In this paper we

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(1) The quantites $\lambda^\pi_{\rho\nu}$ have only recently been used as connexion parameters [see Bortolotti (1931, form. 24) and Dienes (1932, form. 19)]. They have appeared in literature on the subject for many years however, and they appear for the Riemannian case as $\mu_{\pi\rho|\nu}$ in Ricci (1902, p. 357); $C^\nu_{\pi\rho}$ in Kühne (1903), $A^{\pi\rho}_\nu$ in Bortolotti (1928, form. 119), $v^{\pi\rho}_\nu$ in Schouten (1924, p. 200), and $R^\pi_{\rho\nu}$ in Lagrange (1926, p. 32).

shall only deal with projected connexion, in which case Table I in Dienes (1932, p. 268) simplifies, since $D = 0$, $E = 0$, $H = 0$, and $I = 0$. The equations of Gauss, Codazzi, and Kühne have consequently a simplified form, which we shall indicate in the next article.

4. Relations between the fundamental tensors of A_m and A_n. — Having defined the connexion parameters in A_m and in A''_m, we shall introduce the first and second tensors of Eulerian Curvature $F^{\rho}_{\mu\nu}$ and $G^{\mu}_{\rho\nu}$ (1) of A_m in A_n as follows :

From the definitions of $l_{\mu\nu}^{\lambda}$ and $\lambda_{\rho\nu}^{\sigma}$, we have

$$B^{\lambda}_{\alpha} \nabla_{\nu} B^{\alpha}_{\mu} = 0 \qquad \text{and} \qquad C^{\sigma}_{\alpha} \nabla_{\nu} C^{\alpha}_{\rho} = 0.$$

so that we can write

$$\nabla_{\nu} B^{\alpha}_{\mu} = C^{\alpha}_{\beta} \nabla_{\nu} B^{\beta}_{\mu} = F^{\alpha}_{\mu\nu} = C^{\alpha}_{\rho} F^{\rho}_{\mu\nu} \tag{1}$$

and

$$\nabla_{\nu} C^{\alpha}_{\rho} = B^{\alpha}_{\beta} \nabla_{\nu} C^{\alpha}_{\rho} = G^{\alpha}_{\rho\nu} = B^{\alpha}_{\mu} G^{\mu}_{\rho\nu}. \tag{2}$$

We also define corresponding quantities $J^{\rho}_{\mu\sigma}$ and $K^{\mu}_{\rho\sigma}$ by the equations

$$\nabla_{\sigma} B^{\alpha}_{\mu} = C^{\alpha}_{\beta} \nabla_{\sigma} B^{\beta}_{\mu} = J^{\alpha}_{\mu\sigma} = C^{\alpha}_{\rho} J^{\rho}_{\mu\sigma} \tag{3}$$

and

$$\nabla_{\sigma} C^{\alpha}_{\rho} = B^{\alpha}_{\beta} \nabla_{\sigma} C^{\beta}_{\rho} = K^{\alpha}_{\rho\sigma} = B^{\alpha}_{\mu} K^{\mu}_{\rho\sigma}. \tag{4}$$

The fundamental equations connecting A_m and A_n are now the following

$$\bar{S}_{\mu\nu}^{\cdot\cdot\lambda} \equiv S_{\beta\gamma}^{\cdot\cdot\alpha} B^{\lambda\beta\gamma}_{\alpha\mu\nu} = S_{\mu\nu}^{\cdot\cdot\lambda}(l), \tag{5}$$

$$\bar{R}^{k}_{.\lambda\mu\nu} \equiv R^{\alpha}_{.\beta\gamma\delta} B^{k\beta\gamma\delta}_{\alpha\lambda\mu\nu} = R^{k}_{.\lambda\mu\nu}(l) + 2 F^{\sigma}_{\lambda[\mu} G^{k}_{|\sigma|\nu]}, \tag{6}$$

$$\bar{R}^{\rho}_{.\sigma\mu\nu} \equiv R^{\alpha}_{.\beta\gamma\delta} C^{\rho\beta}_{\alpha\sigma} B^{\gamma\delta}_{\mu\nu} = R^{\rho}_{.\sigma\mu\nu}(\lambda) + 2 G^{k}_{\sigma[\mu} F^{\rho}_{|k|\nu]}, \tag{7}$$

$$\bar{R}^{\rho}_{.\lambda\mu\nu} \equiv R^{\alpha}_{.\beta\gamma\delta} C^{\rho}_{\alpha} B^{\beta\gamma\delta}_{\lambda\mu\nu} = 2 \nabla_{[\nu} F^{\rho}_{|\lambda|\mu]} + 2 F^{\rho}_{\lambda k} S_{\mu\nu}^{\cdot\cdot k}, \tag{8}$$

$$\bar{R}^{\lambda}_{.\rho\mu\nu} \equiv R^{\alpha}_{.\beta\gamma\delta} B^{\lambda}_{\alpha} C^{\beta}_{\rho} B^{\gamma\delta}_{\mu\nu} = 2 \nabla_{[\nu} G^{\lambda}_{|\rho|\mu]} + 2 G^{\lambda}_{\rho k} S_{\mu\nu}^{\cdot\cdot k}. \tag{9}$$

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(1) These quantites correspond to $H_{\beta\gamma}^{\cdot\cdot\alpha}$ and $L_{\beta\cdot\gamma}^{\cdot\alpha}$ in Schouten's theory (1924, p. 159).

Actually $H_{\beta\gamma}^{\cdot\cdot\alpha} B^{\beta\gamma}_{\mu\nu} = F^{\alpha}_{\mu\nu} = C^{\alpha}_{\rho} F^{\rho}_{\mu\nu}$ and $L_{\beta\cdot\gamma}^{\cdot\alpha} B^{\mu}_{\alpha} C^{\beta}_{\rho} = - G^{\mu}_{\rho\gamma} = - B^{\nu}_{\gamma} G^{\mu}_{\rho\nu}$.

Equations (6) and (7) are the generalizations of the equations of Gauss and Kühne, while (8) and (9) are the generalizations of the equations of Codazzi.

In the latter part of the paper we shall need the following tensors, which we can call the conjugate Eulerian Curvature tensors

$$
(10)\quad \left\{
\begin{array}{ll}
\text{(i)} & \overset{*}{F}{}^{\rho}_{\mu\nu} = C^{\rho}_{\alpha}\overset{*}{\nabla}_{\nu} B^{\alpha}_{(\mu)} = F^{\rho}_{\mu\nu} - 2\bar{S}_{\mu\nu}{}^{..\rho} = F^{\rho}_{\nu\mu} \quad \text{where} \quad \bar{S}_{\mu\nu}{}^{..\rho} = S_{\beta\gamma}{}^{..\alpha} C^{\rho}_{\alpha} B^{\beta\gamma}_{\mu\nu}\ {}^{(1)}. \\
\text{(ii)} & \overset{*}{G}{}^{\mu}_{\rho\nu} = B^{\mu}_{\alpha}\overset{*}{\nabla}_{\nu} C^{\alpha}_{(\rho)} = G^{\mu}_{\rho\nu} - 2\bar{S}_{\rho\nu}{}^{..\mu}, \\
\text{(iii)} & \overset{*}{J}{}^{\rho}_{\mu\sigma} = C^{\rho}_{\alpha}\overset{*}{\nabla}_{\sigma} B^{\alpha}_{(\mu)} = J^{\rho}_{\mu\sigma} - 2\bar{S}_{\mu\sigma}{}^{..\rho}, \\
\text{(iv)} & \overset{*}{K}{}^{\mu}_{\rho\sigma} = B^{\mu}_{\alpha}\overset{*}{\nabla}_{\sigma} B^{\alpha}_{(\rho)} = K^{\mu}_{\rho\sigma} - 2\bar{S}_{\rho\sigma}{}^{..\mu}.
\end{array}
\right.
$$

with their corresponding A_n components, such as $\overset{*}{F}{}^{\alpha}_{\beta\gamma} = \overset{*}{F}{}^{\rho}_{\mu\nu} C^{\alpha}_{\rho} B^{\mu\nu}_{\beta\gamma}$. We also give the conjugates of formulae given by Dienes (1932, p. 270):

$$
(11)\quad \left\{
\begin{array}{llll}
\text{(i)} & \overset{*}{\nabla}_{\beta} B^{\alpha}_{\mu} = \overset{*}{F}{}^{\alpha}_{\mu\beta} + \overset{*}{J}{}^{\alpha}_{\mu\beta}, & \text{(ii)} & \overset{*}{\nabla}_{\beta} B^{\mu}_{\alpha} = -\overset{*}{G}{}^{\mu}_{\alpha\beta} - \overset{*}{K}{}^{\mu}_{\alpha\beta}. \\
\text{(iii)} & \overset{*}{\nabla}_{\beta} C^{\alpha}_{\rho} = \overset{*}{G}{}^{\alpha}_{\rho\beta} + \overset{*}{K}{}^{\alpha}_{\rho\beta}, & \text{(iv)} & \overset{*}{\nabla}_{\beta} C^{\rho}_{\alpha} = -\overset{*}{F}{}^{\rho}_{\alpha\beta} - \overset{*}{J}{}^{\rho}_{\alpha\beta}.
\end{array}
\right.
$$

5. THE DEFORMATION OF TENSOR SUBMANIFOLDS. — Consider now a neighbouring submanifold X_m given by

$$
(1)\qquad {}'x^{\alpha} = f^{\alpha}(u^1, \ldots, u^m) + \epsilon\,\xi^{\alpha}(u^1, \ldots, u^m)
$$

and repeat the construction given in Art. 3 in order to obtain the geometry $'A_m$.

From

$$
{}'dx^{\alpha} = (B^{\alpha}_{\lambda} + \epsilon\,\partial_{\lambda}\xi^{\alpha})\,du^{\lambda},
$$

we have

$$
(2)\qquad {}'B^{\alpha}_{\lambda} = B^{\alpha}_{\lambda} + \epsilon\,\partial_{\lambda}\xi^{\alpha}.
$$

The simplest way of assigning a span to $'X_m$ is to complete (1) into

$$
(3)\qquad {}'x^{\alpha} = F^{\alpha}(u^1, \ldots, u^n) + \epsilon\,\Xi^{\alpha}(u^1, \ldots, u^n)
$$

where

$$
(4)\qquad F^{\alpha}(u^1, \ldots, u^m, 0, \ldots, 0) = f^{\alpha}(u^1, \ldots, u^m)
$$

(1) The significance of the bar for the other cases will be sufficiently obvious from this example.

and

$$\Xi^{\alpha}(u^1, \ldots, u^m, 0, \ldots, 0) = \xi^{\alpha}(u^1, \ldots, u^m)$$

and by putting

$$(5) \qquad 'C^{\alpha}_{\rho} = C^{\alpha}_{\rho} + \epsilon\, \partial_{\rho} \xi^{\alpha},$$

where, for convenience, we write

$$(6) \qquad \partial_{\rho} \xi^{\alpha} = (\partial_{\rho} \Xi^{\alpha})_{u^{\sigma}=0}.$$

The reciprocal system is

$$(7) \qquad 'B^{\lambda}_{\alpha} = B^{\lambda}_{\alpha} - \epsilon\, B^{\lambda}_{\beta} \partial_{\alpha} \xi^{\beta}, \qquad 'C^{\rho}_{\alpha} = C^{\rho}_{\alpha} - \epsilon\, C^{\rho}_{\beta} \partial_{\alpha} \xi^{\beta}.$$

where

$$\partial_{\alpha} \xi^{\alpha} = B^{k}_{\alpha} \partial_{k} \xi^{\beta} + C^{\sigma}_{\alpha} \partial_{\sigma} \xi^{\beta}.$$

We also have

$$(8) \qquad \begin{cases} 'B^{\alpha}_{\beta} \equiv {}'B^{\lambda}_{\alpha}\, 'B^{\lambda}_{\beta} = B^{\alpha}_{\beta} + \epsilon\, B^{\gamma}_{\beta} \partial_{\gamma} \xi^{\alpha} - \epsilon\, B^{\alpha}_{\gamma} \partial_{\beta} \xi^{\gamma}, \\ 'C^{\alpha}_{\beta} \equiv {}'C^{\alpha}_{\rho}\, 'C^{\rho}_{\beta} = C^{\alpha}_{\beta} + \epsilon\, C^{\gamma}_{\beta} \partial_{\gamma} \xi^{\alpha} - \epsilon\, C^{\alpha}_{\gamma} \partial_{\beta} \xi^{\gamma}. \end{cases}$$

We notice that the projection factors $'B^{\alpha}_{\mu}$, $'C^{\alpha}_{\rho}$, $'B^{\lambda}_{\alpha}$, $'C^{\rho}_{\alpha}$, $'B^{\alpha}_{\beta}$, $'C^{\alpha}_{\beta}$ are the formal transforms of the corresponding factors treated as vectors and tensors of A_n, i. e. submanifold and span suffixes being ignored.

The displaced manifold $'A_m$, i. e. the *deform* of A_n will now be constructed by the following metric and connexion parameters

$$(9) \qquad \text{(i)} \quad 'b_{\lambda\mu} = a_{\alpha\beta}('x)\, 'B^{\alpha\beta}_{\lambda\mu}, \qquad \text{(ii)} \quad 'c_{\rho\sigma} = a_{\alpha\beta}('x)\, 'C^{\alpha\beta}_{\rho\sigma},$$

$$(10) \qquad \begin{cases} \text{(i)} & 'l^{\lambda}_{\mu\nu} = {}'B^{\lambda}_{\alpha}[\partial_{\nu}\, 'B^{\alpha}_{\mu} + \Gamma^{\alpha}_{\beta\gamma}('x)\, 'B^{\beta\gamma}_{\mu\nu}], \\ \text{(ii)} & '\lambda^{\pi}_{\rho\nu} = {}'C^{\pi}_{\sigma}[\partial_{\nu}\, 'C^{\alpha}_{\rho} + \Gamma^{\alpha}_{\beta\gamma}('x)\, 'C^{\beta}_{\rho}\, 'B^{\gamma}_{\nu}], \\ \text{(iii)} & 'S^{\lambda}_{\mu\sigma} = {}'B^{\lambda}_{\alpha}[\partial_{\sigma}\, 'B^{\alpha}_{\mu} + \Gamma^{\alpha}_{\beta\gamma}('x)\, 'B^{\beta}_{\mu}\, 'C^{\gamma}_{\sigma}], \\ \text{(iv)} & '\sigma^{\pi}_{\rho\sigma} = {}'C^{\pi}_{\alpha}[\partial_{\sigma}\, 'C^{\alpha}_{\rho} + \Gamma^{\alpha}_{\beta\gamma}('x)\, 'C^{\beta\gamma}_{\rho\sigma}]. \end{cases}$$

To obtain an image or representative of A_m at the points of $'A_m$, we notice that the point correspondence $'x \to x$ is established by identical values u', ..., u^m in (5.1) and (3.1). Therefore the simplest representative of A_m in $'A_m$ is obtained by taking vectors and tensors with identical components in the u-frames at x and $'x$ respectively as corresponding to each other, and by attaching the *same* metric ([1]) and connexion parameters to the u — frames at x and $'x$. This is possible since from the point of view of the variables $u^1, \dots u_m$, x and $'x$ are identical.

This method leads to

$$(12)\quad \begin{cases} \text{(i)} & \delta b_{\lambda\mu} \equiv {}'b_{\lambda\mu} - b_{\lambda\mu} = \epsilon\,[\xi^\delta \partial_\delta a_{\alpha\beta} + a_{\alpha\delta}\partial_\beta \xi^\delta + a_{\delta\beta}\partial_\alpha \xi^\delta]\,B^{\alpha\beta}_{\lambda\mu} = B^{\alpha\beta}_{\lambda\mu}\,\delta a_{\alpha\beta}, \\ \text{(ii)} & \delta c_{\rho\sigma} = C^{\alpha\beta}_{\rho\sigma}\,\delta a_{\alpha\beta}; \end{cases}$$

$$(11)\quad \begin{cases} \text{(i)} & \delta l^\lambda_{\mu\nu} \equiv {}'l^\lambda_{\mu\nu} - l^\lambda_{\mu\nu} = B^{\lambda\beta\gamma}_{\alpha\mu\nu}\,\delta\Gamma^\alpha_{\beta\gamma}, \qquad \text{(iv)}\quad \delta\lambda^\pi_{\rho\nu} = C^{\pi\beta}_{\alpha\rho} B^\gamma_\nu\,\delta\Gamma^\alpha_{\beta\gamma}, \\ \text{(iii)} & \delta S^\lambda_{\mu\sigma} = B^{\lambda\beta}_{\alpha\mu} C^\gamma_\sigma\,\delta\Gamma^\alpha_{\beta\gamma}, \qquad \text{(iv)}\quad \delta\sigma^\pi_{\rho\sigma} = C^{\pi\beta\gamma}_{\alpha\rho\sigma}\,\delta\Gamma^\alpha_{\beta\gamma}. \end{cases}$$

6. The deformation of vectors and tensors of A_m. — To study the deformation of tensors of A_m, we remark that the deformation of A_m is due to a displacement of A_n, so that to measure the deformation we have to express the tensors in their A_n-components and then apply the methods of the preceding articles.

For example, if we denote by a bar the A_n-component of a tensor of A_m, then a vector v^λ at x will appear as the vector $\bar{v}^\alpha = v^\lambda B^\alpha_\lambda$ of A_n, and the displacement will carry it to $'x$ by parallel transport. On the other hand, the *representative* of v^λ at the point $'x$ expressed in A_n components is $'\bar{v}^\alpha = v^\lambda\,'B^\alpha_\lambda$. Thus the deformation of v^λ is

$$(1)\quad \bar{v}^\alpha(x\|'x) - {}'\bar{v}^\alpha = -\epsilon\left(\Gamma^\alpha_{\beta\gamma} v^\lambda B^\beta_\lambda \xi^\gamma + v^\lambda \partial_\lambda \xi^\alpha\right) = -\epsilon\, v^\lambda B^\beta_\lambda \overset{*}{\nabla}_\beta \xi^\alpha = \Delta\bar{v}^\alpha.$$

For A_m however, only the projection on A_m is significant, so that we also have to introduce the deform $\bar{v}^\alpha(x\|'x)\,'B^\lambda_\alpha$ of v^λ in A_m with the corresponding measure

$$(2)\quad \Delta v^\lambda \equiv \bar{v}^\alpha(x\|'x)\,'B^\lambda_\alpha - v^\lambda = {}'B^\lambda_\alpha\,\Delta\bar{v}^\alpha = B^\lambda_\alpha\,\Delta\bar{v}^\alpha.$$

In the case of a vector field $\bar{v}^\alpha$ defined in X_n or at least in an n-dimensional neighbourhood $(X_m)_n$ of X_m, we can also take $v^\alpha('x)\,'B^\lambda_\alpha$ as the

---

([1]) Taking $b_{\lambda\mu}$ as the representative of $b_{\lambda\mu}$ at $'x$ is further justified by the fact that the simple *transform* of $a_{\alpha\beta}B^{\alpha\beta}_{\lambda\mu}$ is $'a_{\alpha\beta}\,'B^{\alpha\beta}_{\lambda\mu} = a_{\alpha\beta}B^{\alpha\beta}_{\lambda\mu} = b_{\lambda\mu}$.

displaced vector at $'x$, so that subtracting the representative of v^λ at $'x$ (i. e. v^λ itself), we have

$$\delta v^\lambda \equiv v^\alpha('x)'B^\lambda_\alpha - v^\lambda = B^\lambda_\alpha \delta \bar{v}^\alpha. \tag{3}$$

Both processes readily extend to covariant vectors and general tensors, leading to formulae like

$$\text{(i)}\quad \Delta v_\mu = B^\alpha_\mu \Delta \bar{v}_\alpha, \qquad \text{(ii)}\quad \delta v_\mu = B^\alpha_\mu \delta \bar{v}_\alpha; \tag{4}$$

$$\left\{\begin{array}{ll} \text{(i)} & \Delta v^\rho_{\mu\nu} \equiv \bar{v}^\alpha_{\beta\gamma}(x\|'x)\,'C^\rho_\alpha\,'B^{\beta\gamma}_{\mu\nu} - v^\rho_{\mu\nu} = C^\rho_\alpha B^{\beta\gamma}_{\mu\nu} \Delta \bar{v}^\alpha_{\beta\gamma}, \\ \text{(ii)} & \delta v^\rho_{\mu\nu} \equiv \bar{v}^\alpha_{\beta\gamma}('x)\quad 'C^\rho_\alpha\,'B^{\beta\gamma}_{\mu\nu} - v^\rho_{\mu\nu} = C^\rho_\alpha B^{\beta\gamma}_{\mu\nu} \delta \bar{v}^\alpha_{\beta\gamma}, \end{array}\right. \tag{5}$$

and, in general

$$\delta v^\rho_{\mu\nu} = \Delta v^\rho_{\mu\nu} + \epsilon \left(\xi^\delta \nabla_\delta \bar{v}^\alpha_{\beta\gamma}\right) C_\alpha B^{\beta\gamma}_{\mu\nu}. \tag{6}$$

We can write (3) in the form

$$\delta\left(B^\lambda_\alpha \bar{v}^\alpha\right) = B^\lambda_\alpha \delta \bar{v}^\alpha, \tag{7}$$

i. e. *with respect to the operator δ, the projection factors behave as constants.* This is due to the fact that the change resulting from replacing B by 'B has been accounted for in the construction of $\delta\bar{v}^\alpha$.

Applying the δ process to $b_{\lambda\mu}$ and $c_{\rho\sigma}$ we obtain (5.11). In a Riemann space V_n, $\delta b_{\lambda\mu}$ as given by (5.11) reduces to $dg'_{\lambda\mu}$ of Schouten (1928, form. 1). In order to obtain the $\delta g'_{\gamma\delta}$ of Schouten, we notice that, in a V_n,

$$a_{\alpha\beta}('x)\,'B^\alpha_\gamma\,'C^\beta_\delta - a_{\alpha\beta} B^\alpha_\gamma C^\beta_\delta = B^\alpha_\gamma C^\beta_\delta\, \delta a_{\alpha\beta} = \epsilon\, B^\alpha_\gamma C^\beta_\delta\, (\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) \tag{8}$$

which is Schouten's $\delta g'_{\gamma\delta}$

For mixed tensors of the type v^α_μ, the measure for *direct* deformation is given by

$$\left\{\begin{aligned} \Delta v^\alpha_\mu &\equiv \bar{v}^\alpha_\beta(x\|'x)\,'B^\beta_\mu - v^\beta_\mu A^\alpha_\beta \\ &= \left[\left(\bar{v}^\alpha_\beta - \epsilon\,\Gamma^\alpha_{\gamma\delta}\bar{v}^\gamma_\beta\xi^\delta + \epsilon\,\Gamma^\gamma_{\beta\delta}\bar{v}^\alpha_\gamma\xi^\delta\right)\left(B^\beta_\mu + \epsilon\,\partial_\mu\xi^\beta\right) - \left(\delta^\alpha_\beta + \epsilon\,\partial_\beta\xi^\alpha\right)v^\beta_\mu\right] \\ &= \epsilon\left[\bar{v}^\alpha_\beta\partial_\mu\xi^\beta - v^\beta_\mu\partial_\beta\xi^\alpha + \Gamma^\gamma_{\beta\delta}\bar{v}^\alpha_\gamma B^\beta_\mu\xi^\delta - \Gamma^\alpha_{\gamma\delta}\bar{v}^\gamma_\beta B^\beta_\mu\xi^\delta\right] \\ &= \epsilon\left[v^\alpha_\lambda B^\lambda_\beta \dot{\nabla}_\mu\xi^\beta - v^\beta_\mu\dot{\nabla}_\beta\xi^\alpha\right] \end{aligned}\right. \tag{9}$$

so that

(10) $$\Delta v^\alpha_\mu = B^\beta_\mu \Delta v^\alpha_\beta$$

and similarly

(11) $$\delta v^\alpha_\mu = B^\beta_\mu \delta \bar{v}^\alpha_\beta.$$

The rules of manipulation for sums and products readily extend to the deformation of a submanifold and its span. For example

$$\Delta(v^\rho w_\mu) = C^\rho_\alpha B^\beta_\mu \Delta(\bar{v}^\alpha \bar{w}_\beta),$$

and from $\Delta v^\rho = C^\rho_\alpha \Delta \bar{v}^\alpha$ and $\Delta w_\mu = B^\beta_\mu \Delta \bar{w}_\beta$, we have

$$w_\mu \Delta v^\rho + v^\rho \Delta w_\mu = C^\rho_\alpha B^\beta_\mu (\bar{w}_\beta \Delta \bar{v}^\alpha + \bar{v}^\alpha \Delta \bar{w}_\beta) = C^\rho_\alpha B^\beta_\mu \Delta(\bar{v}^\alpha \bar{w}_\beta),$$

by the rules for sums and products in the general space. Hence

(12) $$\Delta(v^\rho w_\mu) = v^\rho \Delta w_\mu + w_\mu \Delta v^\nu.$$

The same rules apply to contracted products. For example, if $u^\rho = v^\rho_\mu w^\mu$, then

$$\Delta u^\rho = C^\rho_\alpha \Delta \bar{u}^\alpha = -\epsilon\, C^\rho_\alpha \bar{u}^\beta \overset{*}{\nabla}_\beta \xi^\alpha = -\epsilon\, C^\rho_\alpha u^\sigma \overset{*}{\nabla}_\sigma \xi^\alpha,$$

and from

$$\Delta v^\rho_\mu = C^\rho_\alpha B^\beta_\mu \Delta \bar{v}^\alpha_\beta = \epsilon\, C^\rho_\alpha B^\beta_\mu (\bar{v}^\alpha_\gamma \overset{*}{\nabla}_\beta \xi^\gamma - \bar{v}^\gamma_\beta \overset{*}{\nabla}_\gamma \xi^\alpha),$$

and

$$\Delta w^\mu = -\epsilon\, B^\mu_\alpha \bar{w}^\beta \overset{*}{\nabla}_\beta \xi^\alpha;$$

we have

$$\Delta v^\rho_{\mu.} w^\mu + v^\rho_{\mu.} \Delta w^\mu = \epsilon \left[C^\rho_\alpha B^\beta_\mu \bar{v}^\alpha_\gamma w^\mu \overset{*}{\nabla}_\beta \xi^\gamma - C^\rho_\alpha B^\beta_\mu \bar{v}^\gamma_\beta w^\mu \overset{*}{\nabla}_\gamma \xi^\alpha - B^\mu_\beta \bar{w}^\beta v^\rho_\mu \overset{*}{\nabla}_\beta \xi^\alpha \right]$$

where the first and last terms in the bracket cancel one another, and thus

$$\Delta v^\rho_{\mu.} w^\mu + v^\rho_{\mu.} \Delta w^\mu = -\epsilon\, C^\rho_\alpha C^\gamma_\sigma v^\rho_\mu w^\mu \overset{*}{\nabla}_\gamma \xi^\alpha = -\epsilon\, C^\rho_\alpha u^\sigma \overset{*}{\nabla}_\sigma \xi^\alpha,$$

so that

(13) $$\Delta u^\rho = \Delta(v^\rho_\mu w^\mu) = \Delta v^\rho_{\mu.} w^\mu + v^\rho_{\mu.} \Delta w^\mu$$

and in the case of vector fields which can be defined in a small neighbourhood $(X_m)_n$ of X_m, the same result holds for δ.

Let us now consider a contravariant vector field v^α of the general space A_n. Since at every point of the submanifold the projection factors are defined, the field v^α will have components $\bar{v}^k = B^k_\alpha v^\alpha$ tangent to A_m. Hence treating $\bar{v}^k$ as a vector of the submanifold and remembering the property of the δ operation of leaving the projection factors unaltered, we have for the deformation

$$(14)\qquad \delta\bar{v}^k = B^k_\alpha\, \delta v^\alpha \qquad \text{or} \qquad \Delta\bar{v}^k = B^k_\alpha\, \Delta v^\alpha,$$

with corresponding results for the submanifold and span components of any tensor field of A_n, so that for instance

$$(15)\qquad \delta\bar{v}^\rho_{\mu\nu} = C^\rho_\alpha\, B^{\beta\gamma}_{\mu\nu}(\delta v^\alpha_{\beta\gamma}) \qquad \text{and} \qquad \Delta\bar{v}^\rho_{\mu\nu} = C^\rho_\alpha\, B^{\beta\gamma}_{\mu\nu}(\Delta v^\alpha_{\beta\gamma}).$$

Particular cases of this result are [5.11, (i) and (ii)]. Since the torsion tensors corresponding to the various connexion parameters, such as $S_{\mu\nu}^{\cdot\cdot\lambda} = l^\lambda_{[\mu\nu]}$ or $S_{\rho\sigma}^{\cdot\cdot\pi} = \sigma^\pi_{[\rho\sigma]}$ are all obtained from the torsion $S_{\beta\gamma}^{\cdot\cdot\alpha} = \Gamma^\alpha_{[\beta\gamma]}$ by projection, we can apply the deformation operators to them directly, and obtain either

$$(16)\qquad \delta S_{\mu\nu}^{\cdot\cdot\lambda} = B^{\lambda\beta\gamma}_{\alpha\mu\nu}\, \delta S_{\beta\gamma}^{\cdot\cdot\alpha} \qquad \text{or} \qquad \Delta S_{\mu\nu}^{\cdot\cdot\lambda} = B^{\lambda\beta\gamma}_{\alpha\mu\nu}\, \Delta S_{\beta\gamma}^{\cdot\cdot\alpha},$$

with corresponding results for the other torsion tensors.

The various projections of the $R^{\alpha}_{\cdot\beta\gamma\delta}$, such as $\bar{R}^{k}_{\cdot\lambda\mu\nu}$ or $\bar{R}^{\rho}_{\cdot\sigma\mu\nu}$ do not coincide with the corresponding intrinsic tensors $R^{k}_{\cdot\lambda\mu\nu}$ and $R^{\rho}_{\cdot\sigma\mu\nu}$. It is still true, however, that

$$(17)\qquad \delta\bar{R}^{k}_{\cdot\lambda\mu\nu} = B^{k\beta\gamma\delta}_{\alpha\lambda\mu\nu}\, \delta R^{\alpha}_{\cdot\beta\gamma\delta} \qquad \text{and} \qquad \delta\bar{R}^{\rho}_{\cdot\sigma\mu\nu} = C^{\rho\beta}_{\alpha\sigma}\, B^{\gamma\delta}_{\mu\nu}\, \delta R^{\alpha}_{\cdot\beta\gamma\delta}.$$

Part II. — Deformation of the fundamental tensors of the Submanifold.

7. Deformation of the Eulerian curvature tensors. — Let us now apply the preceding results to the first tensor of Eulerian curvature $F^\rho_{\mu\nu}$. Expressing this in A_n-components, we have (on omitting the bar) $F^\alpha_{\beta\gamma} = F^\rho_{\mu\nu} C^\alpha_\rho B^{\mu\nu}_{\beta\gamma}$. The $F^\alpha_{\beta\gamma}$ is now a tensor field defined only at the points of A_m, so that the δ operation is not applicable. We can however form the difference

$$(1)\qquad \Delta F^\alpha_{\beta\gamma} \equiv F^\alpha_{\beta\gamma}(x \,|\, 'x) - F^\alpha_{\beta\gamma},$$

where $'F^{\alpha}_{\beta\gamma} = F^{\rho}_{\mu\nu}{}'C^{\alpha}_{\rho}{}'B^{\mu\nu}_{\beta\gamma}$ is the representative of $F^{\rho}_{\mu\nu}$ at the new point $'x$-expressed in A_n-components. This difference gives

$$(2) \qquad \Delta F^{\alpha}_{\beta\gamma} = \epsilon\left[F^{\alpha}_{\delta\gamma}\overset{*}{\nabla}_{\beta}\xi^{\delta} + F^{\alpha}_{\beta\delta}\overset{*}{\nabla}_{\gamma}\xi^{\delta} - F^{\delta}_{\beta\gamma}\overset{*}{\nabla}_{\delta}\xi^{\alpha}\right].$$

We might, however, proceed otherwise in the case of a structure tensor like $F^{\rho}_{\mu\nu}$. Since the $'A_m$ is a submanifold of A_n, it will have a first tensor of Eulerian curvature in A_n, which we shall denote by $'F^{\rho}_{\mu\nu}$, where

$$(3) \qquad 'F^{\rho}_{\mu\nu} = {}'C^{\rho}_{\alpha}\left[\partial_{\nu}{}'B^{\alpha}_{\mu} + \Gamma^{\alpha}_{\beta\gamma}('x)\,'B^{\beta\gamma}_{\mu\nu}\right].$$

On expansion

$$'F^{\rho}_{\mu\nu} = F^{\rho}_{\mu\nu} + \epsilon\, C^{\rho}_{\alpha}\left[\partial_{\nu}\partial_{\mu}\xi^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}B^{\beta}_{\mu}\partial_{\nu}\xi^{\gamma} + \Gamma^{\alpha}_{\beta\gamma}B^{\gamma}_{\nu}\partial_{\mu}\xi^{\beta} - \Gamma^{\delta}_{\beta\gamma}B^{\beta\gamma}_{\mu\nu}\partial_{\delta}\xi^{\alpha} + B^{\beta\gamma}_{\mu\nu}\partial_{\delta}\Gamma^{\alpha}_{\beta\gamma}\xi^{\delta} - \partial_{\nu}B^{\delta}_{\mu}\,\partial_{\delta}\xi^{\alpha}\right],$$

so that from

$$\partial_{\nu}\partial_{\mu}\xi^{\alpha} = \partial_{\nu}B^{\delta}_{\mu}\,\partial_{\delta}\xi^{\alpha} + B^{\beta\gamma}_{\mu\nu}\partial_{\gamma}\partial_{\beta}\xi^{\alpha},$$

we obtain

$$'F^{\rho}_{\mu\nu} = F^{\rho}_{\mu\nu} + \epsilon\, C^{\rho}_{\alpha}B^{\beta\gamma}_{\mu\nu}\left[\partial_{\gamma}\partial_{\beta}\xi^{\alpha} + \Gamma^{\alpha}_{\delta\gamma}\partial_{\beta}\xi^{\delta} + \Gamma^{\alpha}_{\beta\delta}\partial_{\gamma}\xi^{\delta} - \Gamma^{\delta}_{\beta\gamma}\partial_{\delta}\xi^{\alpha} + \xi^{\delta}\partial_{\delta}\Gamma^{\alpha}_{\beta\gamma}\right],$$

i. e.

$$(4) \qquad 'F^{\rho}_{\mu\nu} = F^{\rho}_{\mu\nu} + C^{\rho}_{\alpha}B^{\beta\gamma}_{\mu\nu}\,\delta\Gamma^{\alpha}_{\beta\gamma}.$$

Now this $'F^{\rho}_{\mu\nu}$ can be taken as the representative, at the point $'x$, of the original $F^{\rho}_{\mu\nu}$ at x. So that, expressed in A_n-components, we take $'F^{\rho}_{\mu\nu}{}'C^{\alpha}_{\rho}{}'B^{\mu\nu}_{\beta\gamma}$ instead of $F^{\rho}_{\mu\nu}{}'C^{\alpha}_{\beta}{}'B^{\mu\nu}_{\beta\gamma}$ as the term of comparison, and thus obtain what we call the *total structural deformation*

$$(5) \qquad DF^{\alpha}_{\beta\gamma} \equiv F^{\alpha}_{\beta\gamma}(x \| 'x) - {}'F^{\rho}_{\mu\nu}{}'C^{\alpha}_{\rho}{}'B^{\mu\nu}_{\beta\gamma} = \Delta F^{\alpha}_{\beta\gamma} - C^{\alpha}_{\delta}B^{\eta\zeta}_{\beta\gamma}(\delta\Gamma^{\delta}_{\eta\zeta}).$$

We shall now prove that this expression for $DF^{\alpha}_{\beta\gamma}$ is equal to what Schouten (1928, p. 211, form. III) calls $\delta H_{\beta\gamma}^{\cdot\cdot\alpha}$ provided we assume (i) that A_n is a Riemann space V_n, and (ii) that in A_m (i. e. V_m) the pseudo-normals (or span base-vectors), are all perpendicular to the tangential base vectors.

It follows from (i) that, on replacing $R^{\alpha}_{\cdot\beta\gamma\delta}$ for the V_n by $K^{\alpha}_{\cdot\beta\gamma\delta}$,

$$\delta\Gamma^{\alpha}_{\beta\gamma} = \nabla_{\gamma}\nabla_{\beta}\xi^{\alpha} + K^{\alpha}_{\cdot\beta\gamma\delta}\xi^{\delta}$$

and from (ii) that $B^{\eta}_{\alpha} C^{\zeta}_{\beta} a_{\eta\zeta} = 0$, so that the δ operation applied to this gives

$$(6) \qquad B^{\eta}_{\alpha} C^{\zeta}_{\beta}\, \delta a_{\alpha\beta} = 0.$$

Schouten's formula, with the indices changed in accordance with the conventions here adopted, is

$$\begin{aligned}\delta H_{\beta\gamma}^{\cdot\cdot\alpha} = \epsilon \big[& H^{\eta\cdot\alpha}_{\cdot\beta} C^{\delta}_{\gamma} \nabla_{\eta} \xi_{\delta} + H^{\eta\cdot\alpha}_{\cdot\gamma} C^{\delta}_{\beta} \nabla_{\eta} \xi_{\delta} - H^{\delta\cdot\alpha}_{\cdot\beta} B^{\eta}_{\gamma} \nabla_{\eta} \xi_{\delta} \\ & - H_{\beta\gamma}^{\cdot\cdot\eta} b^{\alpha\delta} \nabla_{\eta} \xi_{\delta} - B^{\eta\zeta}_{\beta\gamma} C^{\alpha}_{\delta} K^{\delta}_{\cdot\eta\zeta\varepsilon} \xi^{\varepsilon} + C^{\alpha}_{\eta} (B^{\delta\varepsilon}_{\beta\gamma} \nabla_{\delta} B^{\zeta}_{\varepsilon}) + C^{\alpha}_{\delta} B^{\eta\zeta}_{\beta\gamma} \nabla_{\eta} \nabla_{\zeta} \xi^{\delta} \big]\end{aligned}$$

and this can be reduced, by re-arranging terms, to

$$\begin{aligned}\delta H_{\beta\gamma}^{\cdot\cdot\alpha} = C^{\alpha}_{\delta} B^{\eta\zeta}_{\beta\gamma} (\delta \Gamma^{\delta}_{\eta\zeta}) - \epsilon\, (& H_{\eta\gamma}^{\cdot\cdot\alpha} \nabla_{\beta} \xi^{\eta} + H_{\eta\beta}^{\cdot\cdot\alpha} \nabla_{\gamma} \xi^{\eta} - H_{\beta\gamma}^{\cdot\cdot\eta} \nabla_{\eta} \xi^{\alpha}) \\ & + C^{\eta}_{\gamma} b^{\delta\zeta} H_{\delta\beta}^{\cdot\cdot\alpha} \delta a_{\eta\zeta} + C^{\eta}_{\beta} b^{\delta\zeta} H_{\delta\gamma}^{\cdot\cdot\alpha} \delta a_{\eta\zeta} - H_{\beta\gamma}^{\cdot\cdot\eta} b^{\alpha\zeta} \delta a_{\eta\zeta}.\end{aligned}$$

The last three terms can easily be proved to vanish in virtue of (6), so that, since $H_{\beta\gamma}^{\cdot\cdot\alpha} = F^{\alpha}_{\beta\gamma}$, the $\delta H_{\beta\gamma}^{\cdot\cdot\alpha} = DF^{\alpha}_{\beta\gamma}$.

As we have already pointed out the δ operation is not applicable to tensors of the submanifold, since when the new point $'x$ is no longer in the submanifold, there is no value defined for the *displaced* tensor $T('x)$. When dealing with a structure tensor, however, we can *define* the displaced tensor $T('x)$ as being the reconstructed tensor at $'x$. For the $F^{\rho}_{\mu\nu}$ for example, this would amount to taking $'F^{\rho}_{\mu\nu}$ as being $F^{\rho}_{\mu\nu}('x)$ and since the *representative* (i. e. the tensor with identical components in the u-frame) of $F^{\rho}_{\mu\nu}$ at $'x$ is $F^{\rho}_{\mu\nu}$ itself, we can write

$$(7) \qquad \delta F^{\rho}_{\mu\nu} \equiv {}'F^{\rho}_{\mu\nu} - F^{\rho}_{\mu\nu} = C^{\rho}_{\alpha} B^{\beta\gamma}_{\mu\nu} (\delta \Gamma^{\alpha}_{\beta\gamma})$$

and similarly

$$(8) \qquad \delta G^{\lambda}_{\rho\nu} \equiv {}'G^{\lambda}_{\rho\nu} - G^{\lambda}_{\rho\nu} = B^{\lambda}_{\alpha} C^{\beta}_{\rho} B^{\gamma}_{\nu} (\delta \Gamma^{\alpha}_{\beta\gamma}).$$

We remark that the δ here differs in one fundamental respect from the corresponding symbol for tensors of the general space. Whereas for the general space δ only requires a tensor field for its definition, here the connexion of the space is involved. This extension of the operator δ will, however, be found of use later.

The corresponding Δ and D deformations are measured by the difference between (i) the projection of the displaced tensor and its representative at $'x$; and (ii) the projection of the displaced tensor and the corresponding structure tensor of A_m at $'x$. Thus

$$\Delta F^{\rho}_{\mu\nu} \equiv F^{\alpha}_{\beta\gamma}(x \,\|\, 'x)\, 'C^{\rho}_{\alpha}\, 'B^{\beta\gamma}_{\mu\nu} - F^{\rho}_{\mu\nu} = C^{\rho}_{\alpha}\, B^{\beta\gamma}_{\mu\nu}\, \Delta F^{\alpha}_{\beta\gamma}, \tag{9}$$

$$DF^{\rho}_{\mu\nu} \equiv F^{\alpha}_{\beta\gamma}(x \,\|\, 'x) - 'F^{\rho}_{\mu\nu} = \Delta F^{\rho}_{\mu\nu} - C^{\rho}_{\alpha}\, B^{\beta\gamma}_{\mu\nu}\, \delta\Gamma^{\alpha}_{\beta\gamma} = C^{\rho}_{\alpha}\, B^{\beta\gamma}_{\mu\nu}\, DF^{\alpha}_{\beta\gamma}. \tag{10}$$

Taking now the mixed tensor $F^{\alpha}_{\mu\nu} = F^{\rho}_{\mu\nu} C^{\alpha}_{\rho} = \partial_{\nu} B^{\alpha}_{\mu} + \Gamma^{\alpha}_{\mu\gamma} B^{\beta\gamma}_{\mu\nu}$, we form the corresponding expression for $'A_m$,

$$'F^{\alpha}_{\mu\nu} = \partial_{\nu}\, 'B^{\alpha}_{\mu} + \Gamma^{\alpha}_{\beta\gamma}('x)\, 'B^{\beta\gamma}_{\mu\nu} = F^{\alpha}_{\nu\mu} + B^{\beta\gamma}_{\mu\nu}\, \delta\Gamma^{\alpha}_{\beta\gamma} + \epsilon\, F^{\delta}_{\mu\nu}\, \partial_{\delta}\, \xi^{\alpha}. \tag{11}$$

The difference $'F^{\alpha}_{\mu\nu} - F^{\alpha}_{\mu\nu}$, which coincides with Schouten's expression for $dH^{\cdot\cdot\alpha}_{\mu\nu}$ (1928, p. 211), is evidently not a tensor. If, however, we take the representative of $F^{\alpha}_{\mu\nu}$ at $'x$, namely $F^{\rho}_{\mu\nu}\, 'C^{\alpha}_{\rho}$ and form the difference $'F^{\alpha}_{\mu\nu} - F^{\rho}_{\mu\nu}\, 'C^{\alpha}_{\rho}$, we get a tensor

$$'F^{\alpha}_{\mu\nu} - F^{\rho}_{\mu\nu}\, 'C^{\alpha}_{\rho} = B^{\beta\gamma}_{\mu\nu}\, \delta\Gamma^{\alpha}_{\beta\gamma}, \tag{12}$$

which we call $\delta F^{\alpha}_{\mu\nu}$.

We can also define the deformations :

$$\Delta F^{\alpha}_{\mu\nu} \equiv F^{\alpha}_{\mu\nu}(x \,\|\, 'x) - F^{\rho}_{\mu\nu}\, C^{\alpha} = -\epsilon\, F^{\delta}_{\mu\nu}\, \overset{*}{\nabla}_{\delta}\, \xi^{\alpha} \tag{13}$$

(where $F^{\alpha}_{\mu\nu}$ is transported as a simple contravariant vector of A_n), and

$$DF^{\alpha}_{\mu\nu} \equiv F^{\alpha}_{\mu\nu}(x \,\|\, 'x) - 'F^{\alpha}_{\mu\nu} = -\epsilon\, F^{\delta}_{\mu\nu}\, \overset{*}{\nabla}_{\delta}\, \xi^{\alpha} - B^{\beta\gamma}_{\mu\nu}\, \delta\Gamma^{\alpha}_{\beta\gamma} = \Delta F^{\alpha}_{\mu\nu} - B^{\beta\gamma}_{\mu\nu}\, \delta\Gamma^{\alpha}_{\beta\gamma}. \tag{14}$$

To extend the δ operator to the tensor $F^{\alpha}_{\beta\gamma}$, which coincides with $H^{\cdot\cdot\alpha}_{\beta\gamma} (= B^{\eta\zeta}_{\beta\gamma} \nabla_{\eta} B^{\alpha}_{\rho})$ appearing in the works of Schouten, we remark that we can deal explicitly with this tensor in the form

$$F^{\alpha}_{\beta\gamma} = F^{\rho}_{\mu\nu}\, C^{\alpha}_{\rho}\, B^{\mu\nu}_{\beta\gamma} = C^{\alpha}_{\delta}\, B^{\zeta}_{\gamma} (\partial_{\zeta}\, B^{\delta}_{\beta} + \Gamma^{\delta}_{\eta\zeta}\, B^{\eta}_{\beta}). \tag{15}$$

Its *representative* at $'x$ will be its simple transform, which is

$$'C^{\alpha}_{\delta}\, 'B^{\zeta}_{\gamma} (\partial'_{\zeta}\, 'B^{\delta}_{\beta} + '\Gamma^{\delta}_{\eta\zeta}\, 'B^{\eta}_{\beta}) = F^{\alpha}_{\beta\gamma} + \epsilon \left[F^{\delta}_{\beta\gamma}\, \partial_{\delta}\, \xi^{\alpha} - F^{\alpha}_{\delta\gamma}\, \partial_{\beta}\, \xi^{\delta} - F^{\alpha}_{\beta\delta}\, \partial_{\gamma}\, \xi^{\delta} \right], \tag{16}$$

which coincides with $F^{\rho}_{\mu\nu}\, 'C^{\alpha}_{\rho}\, 'B^{\mu\nu}_{\beta\gamma}$.

Its *structural representative*, however, the $F^{\alpha}_{\beta\gamma}$ reconstructed for $'A_m$

at $'x$, is

$$(17)\qquad {}'C^{\alpha}_{\delta}\,{}'B^{\zeta}_{\gamma}\left[\partial'_{\zeta}\,{}'B^{\delta}_{\beta} + \Gamma^{\delta}_{\eta\zeta}('x)\,{}'B^{\eta}_{\beta}\right] = F^{\alpha}_{\beta\gamma} + (\delta\Gamma^{\delta}_{\eta\zeta})\,C^{\alpha}_{\delta}\,B^{\eta\zeta}_{\beta\gamma} + \epsilon\left[F^{\delta}_{\beta\gamma}\,\partial_{\delta}\,\xi^{\alpha} - F^{\alpha}_{\delta\gamma}\,\partial_{\beta}\,\xi^{\delta} - F^{\alpha}_{\beta\delta}\,\partial_{\gamma}\,\xi^{\delta}\right]$$

which coincides with $'F^{\rho}_{\mu\nu}\,'C^{\alpha}_{\rho}\,'B^{\mu\nu}_{\beta\gamma}$.

The obvious extension of the operator δ is therefore given by

$$(18)\qquad \delta F^{\alpha}_{\beta\delta} \equiv {}'F^{\rho}_{\mu\nu}\,{}'C^{\alpha}_{\rho}\,{}'B^{\mu\nu}_{\beta\gamma} - F^{\rho}_{\mu\nu}\,{}'C^{\alpha}_{\rho}\,{}'B^{\mu\nu}_{\beta\gamma} = (\delta\Gamma^{\delta}_{\eta\zeta})\,C^{\alpha}_{\delta}\,B^{\eta\zeta}_{\beta\gamma}.$$

In a similar manner we can give the corresponding results for the other tensors, such as $G^{\lambda}_{\rho\nu}$, $J^{\rho}_{\mu\sigma}$ and $K^{\lambda}_{\rho\sigma}$. We have for instance

$$(19)\qquad \begin{cases} \text{(i)} & \Delta G^{\dot{\lambda}}_{\rho\nu} = B^{\dot{\lambda}}_{\alpha}\,C^{\beta}_{\rho}\,B^{\gamma}_{\nu}\,\Delta G^{\alpha}_{\beta\gamma}, \\ \text{(ii)} & \Delta J^{\rho}_{\mu\sigma} = C^{\rho}_{\alpha}\,B^{\beta}_{\mu}\,C^{\gamma}_{\sigma}\,\Delta J^{\alpha}_{\beta\gamma}, \\ \text{(iii)} & \Delta K^{\dot{\lambda}}_{\rho\sigma} = B^{\dot{\lambda}}_{\alpha}\,C^{\beta\gamma}_{\rho\sigma}\,\Delta K^{\alpha}_{\beta\gamma}. \end{cases}$$

In this paper we assume throughout that in A_m, the connexion has been obtained by projection, so that

$$(20)\qquad D^{\dot{\lambda}}_{\mu\nu} = 0,\qquad E^{\pi}_{\rho k} = 0,\qquad H^{\dot{\lambda}}_{\mu\rho} = 0\qquad \text{and}\qquad I^{\pi}_{\rho\sigma} = 0.$$

A straighforward calculation shows however that

$$(21)\qquad {}'D^{\dot{\lambda}}_{\mu\nu} \equiv {}'B^{\dot{\lambda}}_{\alpha}\,\nabla_{\nu}\,{}'B^{\alpha}_{\mu} = D^{\dot{\lambda}}_{\mu\nu} + B^{\dot{\lambda}\beta\gamma}_{\alpha\mu\nu}\,\delta\Gamma^{\alpha}_{\beta\gamma} = D^{\dot{\lambda}}_{\mu\nu} + \delta l^{\dot{\lambda}}_{\mu\nu},$$

so that in general $'D^{\dot{\lambda}}_{\mu\nu}$ is different from zero. This happens becausse the connexion parameters $l^{\dot{\lambda}}_{\mu\nu}$ for $'A_m$ have not been obtained by projection. It follows that $\Delta D^{\dot{\lambda}}_{\mu\nu} = 0$, but $D.D^{\dot{\lambda}}_{\mu\nu} = - B^{\dot{\lambda}\beta\gamma}_{\alpha\mu\nu}\,\delta\Gamma^{\alpha}_{\beta\gamma}$ is in general not zero. The same remark applies to the tensors $E^{\pi}_{\rho k}$, $H^{\dot{\lambda}}_{\mu\sigma}$ and $I^{\pi}_{\rho\sigma}$ with corresponding formulae.

8. If we define $\bar{b}_{\lambda\mu} = b_{\lambda\mu} + \delta b_{\lambda\mu}$, where $\delta b_{\lambda\mu} = (\delta a_{\alpha\beta})\,B^{\alpha\beta}_{\lambda\mu}$ and the corresponding three index symbols $\bar{l}^{\lambda}_{\mu\nu} = \left\{ {\bar{\lambda} \atop \mu\nu} \right\}$, then

$$\bar{l}^{\dot{\lambda}}_{\mu\nu} - l^{\dot{\lambda}}_{\mu\nu} = B^{\dot{\lambda}\beta\gamma}_{\alpha\mu\nu}\,\delta\Gamma^{\alpha}_{\beta\gamma} = \delta l^{\dot{\lambda}}_{\mu\nu}$$

so that, in order to study the δ operator as applied to tensors occurring in the theory of a V_m in a V_n, we could, from the point of view of the *formal* results, disregard the point transformation altogether, and consider the $\bar{V}_m$ (consisting of the same points as V_m, but with the metric parameters $\overline{b}_{\lambda\mu}$) as immersed in the corresponding V_n. We could then introduce the operator $\overline{\nabla}$, where the $\overline{\nabla}$ would indicate covariant derivation using the *barred* connexion parameters $\overline{\Gamma}^{\alpha}_{\beta\gamma}$, $\overline{l}^{\lambda}_{\mu\nu}$, $\overline{\lambda}^{\rho}_{\sigma\mu}$, etc., so that for instance

$$\overline{\nabla}_{\nu} v^{\alpha}_{\mu} = \nabla_{\nu} v^{\alpha}_{\mu} + (\delta\Gamma^{\alpha}_{\beta\gamma})\, v^{\beta}_{\mu} B^{\gamma}_{\nu} - (\delta l^{\lambda}_{\mu\nu})\, v^{\alpha}_{\lambda},$$
$$\overline{\nabla}_{\nu} v^{\mu}_{\rho} = \nabla_{\nu} v^{\mu}_{\rho} + (\delta l^{\mu}_{\lambda\nu})\, v^{\lambda}_{\rho} - (\delta\lambda^{\sigma}_{\rho\nu})\, v^{\lambda}_{\sigma}.$$

This artifice, of considering a new space with a different set of metric coefficients $\overline{a}_{\alpha\beta}$, has been used by Bortolotti (1928) in his study of minimal submanifolds. It is a particular case of the theory of a point manifold to which has been assigned different metrics, and it has been developed systematically by Levi-Civita (1927, Ch. VIII) and Weitzenböck (1923, p. 352).

These remarks can be extended to general linearly connected spaces, in which the point manifolds are assigned two different sets of connexion parameters instead of two sets of metric parameters. If we calculate the projected connexion parameters and the various structure tensors for an A_m in an A_n, the differences between the quantities thus obtained and the original ones are equal to what we have already defined as the δ deformation of the quantities in question.

We have for example

$$\overline{l}^{\lambda}_{\mu\nu} - l^{\lambda}_{\mu\nu} = B^{\lambda}_{\alpha}\, \overline{\nabla}_{\nu} B^{\alpha}_{(\mu)} - B^{\lambda}_{\alpha} \nabla_{\nu} B^{\alpha}_{(\mu)} = B^{\lambda\beta\gamma}_{\alpha\mu\nu}\, \delta\Gamma^{\alpha}_{\beta\gamma} = \delta l^{\lambda}_{\mu\nu},$$

and

$$\overline{F}^{\rho}_{\mu\nu} - F^{\rho}_{\mu\nu} = C^{\rho}_{\alpha}\, \overline{\nabla}_{\nu} B^{\alpha}_{(\mu)} - C^{\rho}_{\alpha} \nabla_{\nu} B^{\alpha}_{(\mu)} = C^{\rho}_{\alpha} B^{\beta\gamma}_{\mu\nu} (\delta\Gamma^{\alpha}_{\beta\gamma}) = \delta F^{\rho}_{\mu\nu}.$$

This is also true for the extensions of δ to mixed tensors like $F^{\alpha}_{\mu\nu}$ and $F^{\alpha}_{\beta\gamma}$, so that

$$\overline{F}^{\alpha}_{\mu\nu} - F^{\alpha}_{\mu\nu} = \overline{\nabla}_{\nu} B^{\alpha}_{(\mu)} - \nabla_{\nu} B^{\alpha}_{(\mu)} = B^{\beta\gamma}_{\mu\nu} (\delta\Gamma^{\alpha}_{\beta\gamma}),$$
$$\overline{F}^{\alpha}_{\beta\gamma} - F^{\alpha}_{\beta\gamma} = C^{\alpha}_{\delta} B^{\eta\zeta}_{\beta\gamma} (\delta\Gamma^{\delta}_{\eta\zeta}).$$

9. Deformation of Riemannian Curvature Tensors. — To study structure tensors like $R^{k}_{.\lambda\mu\nu}$ or $R^{\rho}_{.\sigma\mu\nu}$ we can proceed in the following manner. Since they are expressible in terms of $R^{\alpha}_{.\beta\gamma\delta}$ and of the first and second tensors of Eulerian Curvature $F^{\rho}_{\mu\nu}$ and $G^{\mu}_{\rho\nu}$ by means of the Gauss and the Kühne equations, we can reconstruct these tensors at the points of $'A_m$, and then use the same artifice as we did in the case of $F^{\rho}_{\mu\nu}$, namely of defining the reconstructed tensor $'R^{k}_{.\lambda\mu\nu}$ as being the *displaced* tensor $R^{k}_{.\lambda\mu\nu}('x)$. The representative of $R^{k}_{.\lambda\mu\nu}$ at the points of A_m will be the tensor with identical components in the u-coordinates, *i.e.* $R^{k}_{.\lambda\mu\nu}$. Hence we have

$$(1)\quad \begin{aligned}\delta R^{k}_{.\lambda\mu\nu} &= {}'R^{k}_{.\lambda\mu\nu} - R^{k}_{.\lambda\mu\nu}\\ &= (\delta R^{\alpha}_{.\beta\gamma\delta})B^{k\beta\gamma\delta}_{\alpha\lambda\mu\nu} - F^{\sigma}_{\lambda\mu}\delta G^{k}_{\sigma\nu} - G^{k}_{\sigma\nu}\delta F^{\sigma}_{\lambda\mu} + F^{\sigma}_{\lambda\nu}\delta G^{k}_{\sigma\mu} + G^{k}_{\sigma\mu}\delta F^{\sigma}_{\lambda\nu}.\end{aligned}$$

Let us now consider the curvature tensor $\overline{\overline{R}}{}^{k}_{.\lambda\mu\nu}$ (where we introduce the double bar to distinguish it from $\overline{R}{}^{k}_{.\lambda\mu\nu} \equiv R^{\alpha}_{.\beta\gamma\delta}B^{k\beta\gamma\delta}_{\alpha\lambda\mu\nu}$), of A_m in A_n. By definition we have

$$(2)\qquad \overline{\overline{R}}{}^{k}_{.\lambda\mu\nu} = \partial_\nu \bar{l}^{k}_{\lambda\mu} - \partial_\mu \bar{l}^{k}_{\lambda\nu} + \bar{l}^{k}_{\iota\nu}\bar{l}^{\iota}_{\lambda\mu} - \bar{l}^{k}_{\iota\mu}\bar{l}^{\iota}_{\lambda\nu},$$

which, on replacing $\bar{l}^{k}_{\lambda\mu}$ by its value and on putting $\delta l^{\lambda}_{\mu\nu} = \epsilon\, L^{\lambda}_{\mu\nu}$, $\delta\Gamma^{\alpha}_{\beta\gamma} = \epsilon\, \Lambda^{\alpha}_{\beta\gamma}$ can be written

$$(3)\qquad \overline{\overline{R}}{}^{k}_{.\lambda\mu\nu} = R^{k}_{.\lambda\mu\nu} + \epsilon\,[\nabla_\nu L^{k}_{\lambda\mu} - \nabla_\mu L^{k}_{\lambda\nu} + 2S^{..\iota}_{\mu\nu}L^{k}_{\lambda\iota}].$$

On observing that $L^{k}_{\lambda\mu} = \Lambda^{\alpha}_{\beta\gamma}B^{k\beta\gamma}_{\alpha\lambda\mu}$ and that $S^{..\iota}_{\mu\nu} = S^{..\alpha}_{\beta\gamma}B^{\iota\beta\gamma}_{\alpha\mu\nu}$ we have

$$(4)\quad \begin{aligned}\overline{\overline{R}}{}^{k}_{.\lambda\mu\nu} = R^{k}_{.\lambda\mu\nu} &+ \epsilon\,[B^{k\beta\gamma\delta}_{\alpha\lambda\mu\nu}(\nabla_\delta \Lambda^{\alpha}_{\beta\gamma} - \nabla_\gamma \Lambda^{\alpha}_{\beta\delta} + 2S^{..\eta}_{\gamma\delta}\Lambda^{\alpha}_{\beta\zeta}B^{\zeta}_{\eta})]\\ &+ \epsilon\,\Lambda^{\alpha}_{\beta\gamma}[B^{\beta\gamma}_{\lambda\mu}\nabla_\nu B^{k}_{\alpha} - B^{\beta\gamma}_{\lambda\nu}\nabla_\mu B^{k}_{\alpha} + B^{k\beta}_{\alpha\nu}\nabla_\nu B^{\gamma}_{\mu}\\ &\qquad - B^{k\beta}_{\alpha\lambda}\nabla_\mu B^{\gamma}_{\nu} + B^{k\gamma}_{\alpha\mu}\nabla_\nu B^{\beta}_{\lambda} - B^{k\gamma}_{\alpha\nu}\nabla_\mu B^{\beta}_{\lambda}],\end{aligned}$$

and since by Dienes (1932, p. 269)

$$(5)\qquad \nabla_\nu B^{k}_{\alpha} = -G^{k}_{\alpha\nu} = -C^{\rho}_{\alpha}G^{k}_{\rho\nu},$$

the right hand side of (4) can be reduced to the form

$$(6)\quad \overline{\overline{R}}{}^{k}_{.\lambda\mu\nu} - R^{k}_{.\lambda\mu\nu} = (\delta R^{\alpha}_{.\beta\gamma\delta})B^{k\beta\gamma\delta}_{\alpha\lambda\mu\nu} - F^{\sigma}_{\lambda\mu}\delta G^{k}_{\sigma\nu} - G^{k}_{\sigma\nu}\delta F^{\sigma}_{\lambda\mu} + F^{\sigma}_{\lambda\nu}\delta G^{k}_{\sigma\mu} + G^{k}_{\sigma\mu}\delta F^{\sigma}_{\lambda\nu}.$$

so that the right hand side coincides exactly with that of (1), and is therefore $\delta R^{k}_{.\lambda\mu\nu}$.

The same procedure applied to the tensor $R^{\rho}_{.\sigma\mu\nu}$ would give us

$$(7)\quad \delta R^{\rho}_{.\sigma\mu\nu} = (\delta R^{\alpha}_{.\beta\gamma\delta})C^{\rho\beta}_{\alpha\sigma}B^{\gamma\delta}_{\mu\nu} + F^{\rho}_{\lambda\mu}\delta G^{\lambda}_{\sigma\nu} + G^{\lambda}_{\sigma\nu}\delta F^{\rho}_{\lambda\mu} - F^{\rho}_{\lambda\nu}\delta G^{\lambda}_{\sigma\mu} - G^{\lambda}_{\sigma\mu}\delta F^{\rho}_{\lambda\nu}.$$

The results given in equations (1) and (7) of this article show that for structure tensors of A_m which are connected with the corresponding tensors of the general space by means of the Gauss and Kühne equations, all we have to do is to apply the δ operation to the whole equation, as if it were an ordinary symbol of derivation except that the projection factors are to be regarded as constants with respect to δ. The extension of the operator δ to Eulerian Curvature tensors as introduced in Article **7** therefore proves a convenient one.

If we apply the direct deformation method, we have

$$(8) \qquad \begin{aligned} \Delta R^{k}_{.\lambda\mu\nu} &\equiv \overline{R}^{\alpha}_{.\beta\gamma\delta}\,(x \,\|\, {}'x)' B^{k\beta\gamma\delta}_{\alpha\lambda\mu\nu} - R^{k}_{.\lambda\mu\nu} \\ &= B^{k\beta\gamma\delta}_{\alpha\lambda\mu\nu}(\Delta\overline{R}^{\alpha}_{.\beta\gamma\delta}), \end{aligned}$$

and it can be written explicitly in terms of $R^{k}_{.\lambda\mu\nu}$ itself in the form

$$(9) \quad \Delta R^{k}_{.\lambda\mu\nu} = \epsilon\left[B^{\iota}_{\alpha}\left(R^{k}_{.\iota\mu\nu}\,\overset{*}{\nabla}_{\lambda}\xi^{\alpha} + R^{k}_{.\lambda\iota\nu}\,\overset{*}{\nabla}_{\mu}\xi^{\alpha} + R^{k}_{.\lambda\mu\iota}\,\overset{*}{\nabla}_{\nu}\xi^{\alpha}\right) - B^{k}_{\alpha}R^{\iota}_{.\lambda\mu\nu}\,\overset{*}{\nabla}_{\iota}\xi^{\alpha}\right].$$

Similarly

$$(10) \qquad \Delta R^{\rho}_{.\sigma\mu\nu} = C^{\rho\beta}_{\alpha\sigma}\,B^{\gamma\delta}_{\mu\nu}\,\Delta\overline{R}^{\alpha}_{.\beta\gamma\delta},$$

which can be written explicitly as

$$(11) \quad \Delta R^{\rho}_{.\sigma\mu\nu} = \epsilon\left[B^{\iota}_{\alpha}\left(R^{\rho}_{.\sigma\mu\iota}\,\overset{*}{\nabla}_{\nu}\xi^{\alpha} + R^{\rho}_{.\sigma\iota\mu}\,\overset{*}{\nabla}_{\mu}\xi^{\alpha}\right) + C^{\pi}_{\alpha}R^{\rho}_{.\pi\mu\nu}\,\overset{*}{\nabla}_{\sigma}\xi^{\alpha} - C^{\rho}_{\alpha}\,R^{\pi}_{.\sigma\mu\nu}\,\overset{*}{\nabla}_{\pi}\xi^{\alpha}\right].$$

10. Let us consider the first of the Codazzi equations in the form

$$(1) \qquad \nabla_{\nu}F^{\rho}_{\lambda\mu} - \nabla_{\mu}F^{\rho}_{\lambda\nu} = R^{\alpha}_{.\beta\gamma\delta}C^{\rho}_{\alpha}B^{\beta\gamma\delta}_{\lambda\mu\nu} - 2\,S^{\cdot\cdot k}_{\mu\nu}F^{\rho}_{\lambda k}.$$

In view of our extensions of the operator δ, its application to the right hand side of this equation would give us

$$(2) \qquad (\delta R^{\alpha}_{.\beta\gamma\delta})\,C^{\rho}_{\alpha}\,B^{\beta\gamma\delta}_{\lambda\mu\nu} - 2(S^{\cdot\cdot k}_{\mu\nu}\,\delta F^{\rho}_{\lambda k} + F^{\rho}_{\lambda k}\,\delta S^{\cdot\cdot k}_{\mu\nu}).$$

In order to prove that the δ applied to the left hand side will lead to the same result, let us remark first that

$$(3) \qquad \nabla_{\nu}F^{\rho}_{\lambda\mu} = C^{\rho}_{\alpha}\,B^{\beta\gamma\delta}_{\lambda\mu\nu}\,\nabla_{\delta}F^{\alpha}_{\beta\gamma},$$

a fact which can easily be verified. Consequenly, remembering that the δ operator leaves the projection factors unaltered, we have

$$(4) \qquad \delta(\nabla_{\nu}F^{\rho}_{\lambda\mu}) = C^{\rho}_{\alpha}\,B^{\beta\gamma\delta}_{\lambda\mu\nu}\,.\,\delta(\nabla_{\delta}F^{\alpha}_{\beta\gamma}).$$

Further, on applying the formula (2,13) for the interchangeability of δ and covariant dérivation, we have

$$(5)\qquad \delta(\nabla_\delta F^\alpha_{\beta\gamma}) - \nabla_\delta(\delta F^\alpha_{\beta\gamma}) = (\delta\Gamma^\alpha_{\eta\delta})\, F^\eta_{\beta\gamma} - (\delta\Gamma^\eta_{\beta\delta})\, F^\alpha_{\eta\gamma} - (\delta\Gamma^\eta_{\gamma\delta})\, F^\alpha_{\beta\eta}.$$

Finally, on putting $\delta\Gamma^\alpha_{\beta\gamma} = \epsilon\, \Lambda^\alpha_{\beta\gamma}$ and $\delta F^\alpha_{\beta\gamma} = \epsilon\, (\Lambda^\delta_{\eta\zeta}) C^\alpha_\delta B^{\eta\zeta}_{\beta\gamma}$ we get on simplification

$$(6)\qquad \delta(\nabla_\nu F^\rho_{\lambda\mu}) = \epsilon\, C^\rho_\alpha B^{\beta\gamma\delta}_{\lambda\mu\nu} \nabla_\delta \Lambda^\alpha_{\beta\gamma} + C^\rho_\alpha B^\beta_\lambda C^\gamma_\sigma (\delta\Gamma^\alpha_{\beta\gamma})\, F^\sigma_{\mu\nu} + \delta\lambda^\rho_{\sigma\mu}.F^\sigma_{\lambda\nu} + \delta\lambda^\rho_{\sigma\nu}.F^\sigma_{\lambda\mu}$$
$$- \delta l^\iota_{\lambda\nu}.F^\rho_{\iota\mu} - \delta l^\iota_{\lambda\mu}\, F^\rho_{\iota\nu} - \delta l^\iota_{\mu\nu}\, F^\rho_{\lambda\iota}.$$

On interchanging μ and ν and subtracting the result from (6), we get

$$(7)\qquad \delta[\nabla_\nu F^\rho_{\lambda\mu} - \nabla_\mu F^\rho_{\lambda\nu}] = \epsilon\, C^\rho_\alpha B^{\beta\gamma\delta}_{\lambda\mu\nu} [\nabla_\delta \Lambda^\alpha_{\beta\gamma} - \nabla_\gamma \Lambda^\alpha_{\beta\delta}]$$
$$- 2 F^\rho_{\lambda\iota} S_{\mu\nu}^{\cdot\cdot\iota} + 2(\delta\Gamma^\alpha_{\beta\gamma})\, C^\rho_\alpha B^\beta_\lambda C^\gamma_\sigma S_{\mu\nu}^{\cdot\cdot\sigma},$$

which, in view of the fact that $\delta R^\alpha_{\cdot\beta\gamma\delta} = \epsilon\, [\nabla_\delta \Lambda^\alpha_{\beta\gamma} - \nabla_\gamma \Lambda^\alpha_{\beta\delta} + 2 S_{\gamma\delta}^{\cdot\cdot\eta} \Lambda^\alpha_{\beta\eta}]$ can be written

$$(8)\qquad \delta[\nabla_\nu F^\rho_{\lambda\mu} - \nabla_\mu F^\rho_{\lambda\nu}] = (\delta R^\alpha_{\cdot\beta\gamma\delta})\, C^\rho_\alpha B^{\beta\gamma\delta}_{\lambda\mu\nu} - 2(F^\rho_{\lambda\iota}\, \delta S_{\mu\nu}^{\cdot\cdot\iota} + S_{\mu\nu}^{\cdot\cdot\iota}\, \delta F^\rho_{\lambda\iota}),$$

the right hand side of which coincides with (2).

We have therefore proved that we can apply the δ operator to both sides of the Codazzi equations just as we have done for the Gauss and the Kühne equations.

It can easily be verified that the expression (2) is equal to $2\, \overline{\nabla}_{[\nu} \overline{F}^\rho_{|\lambda|\mu]}$ where, to first order

$$(9)\qquad \overline{\nabla}_\nu \overline{F}^\rho_{\lambda\mu} = \overline{\nabla}_\nu (F^\rho_{\lambda\mu} + \delta F^\rho_{\lambda\mu}) = \overline{\nabla}_\nu F^\rho_{\lambda\mu} + \nabla(\delta F^\rho_{\lambda\mu}),$$

and

$$(10)\qquad \overline{\nabla}_\nu F^\rho_{\lambda\mu} = \nabla_\nu F^\rho_{\lambda\mu} + (\delta\lambda^\rho_{\sigma\nu})\, F^\sigma_{\lambda\mu} - (\delta l^\iota_{\lambda\nu})\, F^\rho_{\iota\mu} - (\delta l^\iota_{\mu\nu})\, F^\rho_{\lambda\iota}.$$

11. Let us now consider the curvature tensor $R^k_{\cdot\lambda\mu\nu}$ for a Riemann space. In this case, since $B^k_\alpha = b^{k\nu} a_{\alpha\beta} B^\beta_\nu$ we have

$$\nabla_\mu B^k_\alpha = b^{k\nu} a_{\alpha\beta} \nabla_\mu B^\beta_\nu = b^{k\nu} a_{\alpha\beta} F^\beta_{\nu\mu}$$

and since $\nabla_\nu B^\lambda_\alpha = - G^\lambda_{\alpha\nu}$, we can write

$$(1)\qquad G^\lambda_{\alpha\nu} = - b^{\lambda\nu} a_{\alpha\beta} F^\beta_{\nu\mu}.$$

Further, on recalling that a term of the form $F^{\sigma}_{\lambda\mu}G^{k}_{\sigma\nu}$ occurring in the Gauss equation can be written as $F^{\alpha}_{\lambda\mu}G^{k}_{\alpha\nu}$, and therefore, in virtue of (1) as $-b^{k\iota}a_{\alpha\beta}F^{\alpha}_{\lambda\mu}F^{\beta}_{\iota\nu}$ we can write the Gauss equation in the form

$$R^{k}_{.\lambda\mu\nu}=R^{k}_{.\beta\gamma\delta}B^{\alpha\beta\gamma\delta}_{\alpha\lambda\mu\nu}+b^{\iota k}a_{\alpha\beta}(F^{\alpha}_{\lambda\mu}F^{\beta}_{\iota\nu}-F^{\alpha}_{\lambda\nu}F^{\beta}_{\iota\mu}). \tag{2}$$

On contracting this with b_{k_0}, and changing indices, we get

$$R_{k\lambda\mu\nu}=R_{\alpha\beta\gamma\delta}B^{\alpha\beta\gamma\delta}_{k\lambda\mu\nu}+a_{\alpha\beta}(F^{\alpha}_{\lambda\mu}F^{\beta}_{k\nu}-F^{\alpha}_{\lambda\nu}F^{\beta}_{k\mu}). \tag{3}$$

To find the $\delta R_{k\lambda\mu\nu}$ we have only to apply the operator δ to the whole equation (3) as it stands, taking account of the extension of δ to $F^{\alpha}_{\lambda\mu}$. This gives

$$\begin{aligned}\delta R_{k\lambda\mu\nu}=(\delta R_{\alpha\beta\gamma\delta})B^{\alpha\beta\gamma\delta}_{k\lambda\mu\nu}+\delta a_{\alpha\beta}(F^{\alpha}_{\lambda\mu}F^{\beta}_{k\nu}-F^{\alpha}_{\lambda\nu}F^{\beta}_{k\mu})\\ +a_{\alpha\beta}(F^{\alpha}_{\lambda\mu}\,\delta F^{\beta}_{k\nu}+F^{\beta}_{k\nu}\,\delta F^{\alpha}_{\lambda\mu}-F^{\alpha}_{\lambda\nu}\,\delta F^{\beta}_{k\mu}-F^{\beta}_{k\mu}\,\delta F^{\alpha}_{\lambda\nu}),\end{aligned} \tag{4}$$

which is equivalent to the expression obtained by Schouten (1928).

We could also have obtained the same result by applying the operator δ to the equation $R_{k\lambda\mu\nu}=R^{\iota}_{.\lambda\mu\nu}b_{k\iota}$ giving

$$\delta R_{k\lambda\mu\nu}=b_{k\iota}\,\delta R^{\iota}_{.\lambda\mu\nu}+R^{\iota}_{.\lambda\mu\nu}\,\delta b_{k\iota}, \tag{5}$$

where $\delta R^{\iota}_{.\lambda\mu\nu}$ and $\delta b_{k\iota}$ are to have values already given for them.

Since the δ operator can be applied to products and contracted products, we can use it directly to obtain the contractions of the $R^{k}_{.\lambda\mu\nu}$ tensor. Let us take first the so-called Ricci tensor ([1]) given by

$$R_{\lambda\nu}=R_{k\lambda\mu\nu}b^{k\mu}=R^{k}_{.\lambda\mu\nu}. \tag{6}$$

From this we have

$$\delta R_{\lambda\nu}=R_{k\lambda\mu\nu}\,\delta b^{k\mu}+b^{k\mu}\,\delta R_{k\lambda\mu\nu}, \tag{7}$$

so that, on putting in the value of $\delta R_{k\lambda\mu\nu}$ from (4), and putting $\delta b^{\mu k}$ in the form

$$\delta b^{\mu k}=-b^{\mu\iota}b^{\nu k}B^{\alpha\beta}_{\nu\iota}\,\delta a_{\alpha\beta}, \tag{8}$$

([1]) In view of the difference in notation, since $R^{k}_{.\lambda\mu\nu}$ is $R_{\mu\nu\lambda}^{\cdot\cdot\cdot k}$ in Schouten's notation, the corresponding Ricci tensor would appear as $R_{\nu\lambda}$, with the indices interchanged.

we have

$$(9) \qquad \begin{aligned} \delta R_{\lambda\nu} = {} & b^{k\mu}(\delta R_{\alpha\beta\gamma\delta})\, B^{\alpha\beta\gamma\delta}_{k\lambda\mu\nu} + b^{k\mu}\,\delta a_{\alpha\beta}\,(F^{\alpha}_{\lambda\mu}\,\delta F^{\beta}_{k\nu} - F^{\alpha}_{\lambda\nu}\, F^{\beta}_{k\mu}) \\ & + b^{k\mu} a_{\alpha\beta}[F^{\alpha}_{\lambda\mu}\,\delta F^{\beta}_{k\nu} + F^{\beta}_{k\nu}\,\delta F^{\alpha}_{k\mu} - F^{\alpha}_{\lambda\nu}\,\delta F^{\beta}_{k\mu} - F^{\beta}_{k\mu}\,\delta F^{\alpha}_{\lambda\nu}] \\ & - R_{k\lambda\mu\nu}\, b^{\mu\iota}\, b^{k\nu}\, B^{\alpha\beta}_{\iota\nu}\,\delta a_{\alpha\beta}, \end{aligned}$$

to which Schouten's expression for the corresponding tensor can be reduced.

If on the other hand we take the $\delta R^{k}_{.\lambda\mu\nu}$ as given in (9,1) and contract it with respect to the indices k and μ we obtain

$$(10) \quad \delta R^{\mu}_{.\lambda\mu\nu} = (\delta R^{\alpha}_{.\beta\gamma\delta})\, B^{\gamma}_{\alpha} B^{\beta\delta}_{\lambda\nu} - F^{\sigma}_{\lambda\mu}\delta G^{\mu}_{\sigma\nu} - G^{\mu}_{\sigma\nu}\,\delta F^{\sigma}_{\lambda\mu} + F^{\sigma}_{\lambda\nu}\,\delta G^{\mu}_{\sigma\mu} + G^{\mu}_{\sigma\mu}\,\delta F^{\sigma}_{\lambda\nu}$$

which appears to differ materially from the right hand side of (9). If however, we make systematic use of equation (1) of this article, of the Gauss equation, and of

$$\delta R_{\alpha\beta\gamma\delta} = a_{\alpha\eta}\,\delta R^{\eta}_{.\beta\gamma\delta} + R^{\eta}_{.\beta\gamma\delta}\,\delta a_{\alpha\eta},$$

it is not difficult to prove that the right hand side of (10) does in effect coincide with that of (9). Hence we can write $\delta R_{\lambda\nu}$ for $\delta R^{\mu}_{.\lambda\mu\nu}$ and (10) gives the δ deformation of the Ricci tensor independently of any use of the metric parameters.

If finally we wish to determine the deformation of the invariant R of the submanifold for the infinitesimal deformation, we have only to apply the operator δ to the equation

$$(11) \qquad R = R_{\lambda\nu}\, b^{\lambda\nu},$$

giving

$$(12) \qquad \begin{aligned} \delta R & = b^{\lambda\nu}\,\delta R_{\lambda\nu} + R_{\lambda\nu}\,\delta b^{\lambda\nu} \\ & = b^{\lambda\nu}\,\delta R_{\lambda\nu} - b^{\lambda\iota}\, b^{\nu k} B^{\alpha\beta}_{\iota k}\,\delta a_{\alpha\beta}.R_{\lambda\nu}, \end{aligned}$$

where we imagine $\delta R_{\lambda\nu}$ replaced by its value from (9).

12. Geodesic and mininal submanifolds. — A submanifold is *geodesic* when the first tensor of Eulerian Curvature vanishes, i.e. when $F^{\rho}_{\mu\nu} = 0$.

The condition that, as a result of the infinitesimal deformation, the resulting submanifold $'A_m$ should also be geodesic is therefore that

$$\delta F^{\rho}_{\mu\nu} = 0, \tag{1}$$

or that

$$(\delta \Gamma^{\alpha}_{\beta\gamma}) C^{\rho}_{\alpha} B^{\beta\gamma}_{\mu\nu} = 0. \tag{2}$$

Since $F^{\rho}_{\mu\nu} = 0$ this can be written at length in the form

$$C^{\rho}_{\alpha} [\nabla_{\nu} \nabla_{\mu} \xi^{\alpha} + R^{\alpha}_{.\beta\gamma\delta} B^{\beta\gamma}_{\mu\nu} \xi^{\delta} + 2 B^{\beta}_{\mu} \xi^{\delta} \nabla_{\nu} S_{\beta\delta}^{..\alpha} + 2 B^{\beta}_{\mu} S_{\beta\delta}^{..\alpha} \nabla_{\nu} \xi^{\delta}] = 0, \tag{3}$$

which for a Riemann space becomes

$$C^{\rho}_{\alpha} [\nabla_{\nu} \nabla_{\mu} \xi^{\alpha} + K^{\alpha}_{.\beta\gamma\delta} B^{\beta\gamma}_{\mu\nu} \xi^{\delta}] = 0, \tag{4}$$

where $K^{\alpha}_{.\beta\gamma\delta}$ is the expression for $R^{\alpha}_{.\beta\gamma\delta}$ when the space is aV_n. This generalization is already given by Schouten (1928, p. 213) for Levi-Civita's equations of geodesic deviation.

Let us consider again the minimal submanifolds immersed in a metric space in which autoparallels are lines of extremal length, so that the minimal submanifolds of this space coincide with those of the Riemann space determined by the metric parameters.

The condition for a minimal submanifold is that

$$b^{\mu\nu} F^{\rho}_{\mu\nu} = 0, \tag{5}$$

so that the déformed submanifold is also minimal provided

$$F^{\rho}_{\mu\nu} \, \delta b^{\mu\nu} + b^{\mu\nu} \, \delta F^{\rho}_{\mu\nu} = 0, \tag{6}$$

i. e. provided

$$b^{\mu\nu} C^{\rho}_{\alpha} [\nabla_{\nu} \nabla_{\mu} \xi^{\alpha} + R^{\alpha}_{.\beta\gamma\delta} B^{\beta\gamma}_{\mu\nu} \xi^{\delta} + 2 B^{\beta}_{\mu} \nabla_{\nu} (S_{\beta\delta}^{..\alpha} \xi^{\delta}) - 2 b^{\mu\iota} b^{\nu\lambda} B^{\alpha\beta}_{\iota\lambda} F^{\rho}_{\mu\nu} \nabla_{\alpha} \xi_{\beta} = 0. \tag{7}$$

But in this case the relation

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ {}^{\,\alpha}_{\beta\gamma} \right\} + S_{\beta\gamma}^{..\alpha} - a^{\alpha\delta} (S_{\beta\gamma\delta} + S_{\beta\delta\gamma}),$$

between the connexion parameters and the three-index symbols of Christoffel takes the simple form

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ {}^{\,\alpha}_{\beta\gamma} \right\} + S_{\beta\gamma}^{..\alpha}, \tag{8}$$

and consequently

$$(9) \qquad R^{\alpha}_{.\beta\gamma\delta} = K^{\alpha}_{.\beta\gamma\delta} + \nabla_{\delta} S_{\beta\gamma}^{..\alpha} - \nabla_{\gamma} S_{\beta\delta}^{..\alpha} + S_{\eta\gamma}^{..\alpha} S_{\beta\delta}^{..\eta} - S_{\eta\delta}^{..\alpha} S_{\beta\gamma}^{..\eta},$$

so that using (8) and (9) and replacing covariant derivation with $\Gamma^{\alpha}_{\beta\gamma}$ as connexion parameters by derivation with $\{ {}^{\alpha}_{\beta\gamma} \}$ as parameters, we get the equation which Bortolotti (1928, p. 176, form. 95) has given for the Riemann case.

Part III. — Tangential and " Span " Deformations.

13. From this point onwards we shall consider the two special cases in which the infinitesimal displacement $\epsilon\, \xi^{\alpha}(u)$ undergone by the points of the submanifold, is either definitely in the tangent plane to A_m at every point, or else in the pseudo-normal « plane » (which can in our terminology be described as being « in the span »). In the case of a tangential deformation, where ξ^{α} can be written in the form $B^{\alpha}_{\lambda} \eta^{\lambda}$, since the new point P′ will now be in the tangent plane of A_m at P, it will also, to first order terms in ϵ, be in the A_m itself. Hence P′ will have coordinates in the u-system, and, from the point of view of the submanifold, we could regard the infinitesimal displacement as producing an *intrinsic* deformation of the A_m, expressed by

$$(1) \qquad {}'u^{\lambda} = u^{\lambda} + \epsilon\, \eta^{\lambda}.$$

The fundamental tensors of A_m (considered independently of the surrounding space A_n) will undergo changes due to the transformation (1), corresponding to the changes undergone by the fundamental tensors of A_n due to an infinitesimal transformation (${}'x^{\alpha} = x^{\alpha} + \xi \epsilon^{\alpha}$) of the A_n.

When, however, the tensors (or more precisely the tensor fields) of the submanifold are related to those of A_n by laws, such as projection (in the case of the metric and the torsion tensors) or the Gauss and Kühne equations (in the case of the curvature tensors), then there is usually a discrepancy between the results obtained by considering the tensors of A_m as undergoing an intrinsic deformation determined by (1), — and those obtained by finding the particular form taken by the results of the preceding articles on putting $\xi^\alpha = B^\alpha_\lambda \eta^\lambda$.

We shall now consider this discrepancy, denoting by δ_s the symbol corresponding to the intrinsic deformation which a fundamental tensor of A_m would undergo in virtue of the transformation (1).

We shall also obtain the results of the preceding sections in the case where ξ^α is in the span, i. e. where $\xi^\alpha = C^\alpha_\rho \zeta^\rho$. We cannot in this case introduce an intrinsic point deformation corresponding to (1), although certain results have a form suggesting the use of an operation δ_n for the « span » deformation corresponding to δ_s for the tangential deformation.

When dealing with a V_m in a V_n, we shall also see that a constant infinitesimal displacement in a direction normal to V_m, will produce in the metric tensor of V_m a deformation which is expressible in terms of the second fundamental form of a V_m in a V_n.

14. Tangential deformations for vectors of A_m and of A^p_m. — We have proved in Art. **6** that when we have a tensor field such as $v^\alpha_{\beta\gamma}$ in A_n, the deformations of its A_m — and A^p_m — components are given by such formulae as

$$(1)\qquad \delta\bar{v}^\rho_{\mu\nu} = C^\rho_\alpha B^{\beta\gamma}_{\mu\nu}(\delta v^\alpha_{\beta\gamma}).$$

Now let us consider the particular case where $\xi^\alpha = B^\alpha_\lambda \eta^\lambda$ as applied to the A_m components of a contravariant vector field of A_n. By (1) we have

$$(2)\qquad \delta\bar{v}^k = \epsilon\left[\xi^\delta \nabla_\delta v^\alpha - \bar{v}^\delta \overset{*}{\nabla}_\delta \xi^\alpha\right] B^k_\alpha = \epsilon\left[\eta^\lambda \nabla_\lambda v^\alpha - v^\delta \overset{*}{\nabla}_\delta (B^\alpha_\lambda \eta^\lambda)\right] B^k_\alpha.$$

On reduction this becomes

$$(3)\qquad \delta\bar{v}^k = \epsilon\left[\eta^\lambda \nabla_\lambda \bar{v}^k - \bar{v}^\lambda \overset{*}{\nabla}_\lambda \eta^k\right] - \epsilon\, \bar{v}^\sigma\left[\overset{*}{\nabla}_\sigma \eta^k - G^k_{\sigma\lambda}\eta^\lambda\right].$$

But the expression $\epsilon\left[\eta^\lambda \nabla_\lambda \bar{v}^k - \bar{v}^\lambda \overset{*}{\nabla}_\lambda \eta^k\right]$ gives the deformation which a vector field of A_m of components $\bar{v}_k$ would undergo under an infinitesimal transformation (**1**, 1), and therefore by definition

$$(4)\qquad \delta_s \bar{v}^k = \epsilon\left[\eta^\lambda \nabla_\lambda \bar{v}^k - \bar{v}^\lambda \overset{*}{\nabla}_\lambda \eta^k\right].$$

Hence we can write (3) in the form

$$(5) \qquad \delta\bar{v}^k = \partial_s \bar{v}^k - \epsilon\, \bar{v}^\sigma \left(\dot{\nabla}_\sigma \eta_l{}^k - G^k_{\sigma\lambda}\eta_l{}^\lambda\right).$$

If the field v^α of A_n happens to be tangential to A_m at the points of A_m, then by definition $\bar{v}^\sigma = 0$, and we have

$$(6) \qquad \delta v^{\bar{k}} = \partial_s v^{\bar{k}}.$$

For a covariant vector field v_α of A_n, the deformation of its A_m components will be

$$(7) \qquad \delta\bar{v}_k = \epsilon \left[\xi^\delta \nabla_\delta v_\alpha + v_\delta \dot{\nabla}_\alpha \xi^\delta\right] B^\alpha_k,$$

and this can be reduced to the form

$$(8) \qquad \delta\bar{v}_k = \epsilon \left[\eta_l{}^\lambda \nabla_\lambda \bar{v}_k + v_\delta \dot{\nabla}_\alpha \xi^\delta\right] B^\alpha_k,$$

so that

$$(9) \qquad \delta\bar{v}_k = \partial_s \bar{v}_k.$$

We remark that in this case there is no discrepancy between the deformation of $\bar{v}_k$ treated as a covariant vector field of A_m under the transformation (**1**.1), and the corresponding deformation obtained by taking $\xi^\alpha = B^\alpha_\lambda \eta_l{}^\lambda$ in the general result. We notice that this is true whether v_α is tangential to A_m or not, since $\bar{v}_\sigma$ does not enter into the result.

Treating the span components in the same way, we have

$$(10) \qquad \delta\bar{v}^\rho = \epsilon \left[\xi^\delta \nabla_\delta v^\alpha - v^\delta \dot{\nabla}_\delta \xi^\alpha\right] C^\rho_\alpha = \epsilon \left[\eta_l{}^\lambda \nabla_\lambda \bar{v}^\rho - \eta_l{}^\lambda v^\alpha \nabla_\lambda C^\rho_\alpha + B^\alpha_\lambda \eta_l{}^\lambda v^\delta \dot{\nabla}_\delta C^\rho_\alpha\right],$$

and applying the formula

$$(11) \qquad \dot{\nabla}_\delta C^\rho_\alpha = -\dot{F}^\rho_{\alpha\delta} - \dot{J}^\rho_{\alpha\delta},$$

we get

$$(12) \qquad \delta\bar{v}^\sigma = \epsilon \left[\eta_l{}^\lambda \nabla_\lambda \bar{v}^\rho - \eta_l{}^\lambda \bar{v}^\sigma \dot{J}^\rho_{\lambda\sigma}\right],$$

so that no tangential components of v^α appear.

For the covariant components, however, we have the result

$$(13) \qquad \delta\bar{\nu}_\rho = \epsilon\left[\eta^\lambda \nabla_\lambda \bar{\nu}_\rho + \eta^\lambda \bar{\nu}_\sigma \overset{*}{J}{}^\sigma_{\lambda\rho} + \bar{\nu}_\mu\left(\overset{*}{\nabla}_\rho \eta^\mu - G^\mu_{\rho\lambda}\eta^\lambda\right)\right],$$

which, when $\bar{\nu}_\mu = 0$, reduces to

$$(14) \qquad \delta\bar{\nu}_\rho = \epsilon\left[\eta^\lambda \nabla_\lambda \bar{\nu}_\rho + \eta^\lambda \bar{\nu}_\sigma \overset{*}{J}{}^\sigma_{\lambda\rho}\right],$$

an expression similar to (12) obtained for contravariant components.

15. Span deformations for vectors of A_m and of A''_m. — We quote the corresponding results for the case in which $\xi^\alpha = C^\alpha_\rho \zeta^\rho$. We have

$$(1) \qquad \delta\bar{\nu}^k = \epsilon\left[\zeta^\sigma \nabla_\sigma \bar{\nu}^k - \zeta^\sigma \bar{\nu}^\mu \overset{*}{G}{}^k_{\sigma\mu}\right],$$

$$(2) \qquad \delta\bar{\nu}_k = \epsilon\left[\zeta^\sigma \nabla_\sigma \bar{\nu}_k + \zeta^\sigma \bar{\nu}^\mu \overset{*}{G}{}^\mu_{\sigma k}\right] + \epsilon\,\bar{\nu}_\rho\left[\overset{*}{\nabla}_k \zeta^\rho - J^\sigma_{\mu\rho}\zeta^\rho\right],$$

which, when $\bar{\nu}_\rho = 0$, reduces to

$$(2') \qquad \delta\bar{\nu}_k = \epsilon\left[\zeta^\sigma \nabla_\sigma \bar{\nu}_k + \zeta^\sigma \bar{\nu}_\mu \overset{*}{G}{}^\mu_{\sigma k}\right],$$

$$(4) \qquad \delta\bar{\nu}^\rho = \epsilon\left[\zeta^\sigma \nabla_\sigma \bar{\nu}^\rho - \bar{\nu}^\sigma \overset{*}{\nabla}_\sigma \zeta^\rho\right] - \epsilon\,\bar{\nu}^\mu\left[\overset{*}{\nabla}_\mu \zeta^\rho - \overset{*}{J}{}^\rho_{\mu\sigma}\zeta^\sigma\right],$$

which when, $\bar{\nu}^\mu = 0$, reduces to

$$(4') \qquad \delta\bar{\nu}^\rho = \epsilon\left[\zeta^\sigma \nabla_\sigma \bar{\nu}^\rho - \bar{\nu}^\sigma \overset{*}{\nabla}_\sigma \zeta^\rho\right],$$

$$(5) \qquad \delta\bar{\nu}_\rho = \epsilon\left[\zeta^\sigma \nabla_\sigma \bar{\nu}_\rho - \bar{\nu}_\sigma \overset{*}{\nabla}_\rho \zeta^\sigma\right].$$

On considering (4′) and (5) we notice that the forms of the right hand sides correspond exactly to those of $\delta_s\bar{\nu}^k$ and $\delta_s\bar{\nu}_k$ for tangential deformations. We can therefore conveniently introduce the notation

$$(6) \qquad \delta_n\bar{\nu}_\rho = \epsilon\left[\zeta^\sigma \nabla_\sigma \nu_\rho - \bar{\nu}_\sigma \overset{*}{\nabla}_\rho \zeta^\sigma\right]$$

so that (4) and (5) can be written

$$(7) \qquad \delta\bar{\nu}_\rho = \delta_n\bar{\nu}_\rho - \epsilon\,\bar{\nu}^\mu\left(\overset{*}{\nabla}_\mu \zeta^\rho - J^\rho_{\mu\sigma}\zeta^\sigma\right);$$

and

$$(8) \qquad \delta\bar{\nu}_\rho = \delta_n\bar{\nu}_\rho.$$

16. Application to the components of any tensor field of A_n. — Having dealt with every kind of index that can occur, we can write down, on the basis of the above results, the expressions for the deformation of any mixed components of a general tensor field of X_n. In fact we might use the Aronhold-Clebsch symbols for any components such as

$$(1)\qquad \overline{A}^{\lambda_1\ldots\lambda_m\,\sigma_1\ldots\sigma_s}_{\mu_1\ldots\mu_n\,\rho_1\ldots\rho_t} = \underset{1}{\overline{A}}{}^{\lambda_1}\ldots\underset{m}{\overline{A}}{}^{\lambda_m}\overset{1}{\overline{A}}_{\mu_1}\ldots\overset{n}{\overline{A}}_{\mu_n}\underset{1}{\overline{A}}{}^{\sigma_1}\ldots\underset{s}{\overline{A}}{}^{\sigma_s}\overset{1}{\overline{\Lambda}}_{\rho_1}\ldots\overset{t}{\overline{\Lambda}}_{\rho_t},$$

and use the symbolical relationship

$$(2)\qquad \delta\overline{A}^{\lambda_1\ldots\sigma_s}_{\mu_1\ldots\rho_t} = \sum_{i=1}^{m}\underset{1}{\overline{A}}{}^{\lambda_1}\ldots\delta\underset{i}{\overline{A}}{}^{\lambda_i}\ldots\underset{m}{\overline{A}}{}^{\lambda_m}\ldots\overset{t}{\overline{\Lambda}}_{\rho_t} + \ldots + \sum_{q=1}^{t}\underset{1}{\overline{A}}{}^{\lambda_1}\ldots\overset{1}{\overline{\Lambda}}_{\rho_1}\ldots\delta\overset{q}{\overline{\Lambda}}_{\rho_q}\ldots\overset{t}{\overline{\Lambda}}_{\rho_t},$$

so that we can treat each index as corresponding to an « ideal factor ».

We can accordingly write down immediatly, for *tangential deformations*

$$(3)\qquad \delta b_{\mu\nu} = \delta_s b_{\mu\nu},$$

$$(4)\qquad \delta S_{\mu\nu\lambda} = \delta_s S_{\mu\nu\lambda},$$

$$(5)\qquad \delta\overline{R}_{k\lambda\mu\nu} = \delta_s\overline{R}_{k\lambda\mu\nu}.$$

Further, in the case of the metric tensor $b^{\mu\nu}$, we have

$$(6)\qquad \delta b^{\mu\nu} = \delta_s b^{\mu\nu} - \epsilon\left[b^{\mu\sigma}\left(\overset{*}{\nabla}_\sigma\eta^\nu - G^{\nu}_{\sigma\lambda}\eta^\lambda\right) + b^{\sigma\nu}\left(\overset{*}{\nabla}_\sigma\eta^\mu - G^{\mu}_{\sigma\lambda}\eta^\lambda\right)\right],$$

But if in this case the span vectors $C^{\alpha}_{(\rho)}$ have been determined as being *orthogonal* to the $B^{\alpha}_{(\lambda)}$, we have $b^{\mu\sigma} = A^{\alpha\beta}B^{\mu}_{\alpha}C^{\sigma}_{\beta} = 0$ so that we have

$$(6')\qquad \delta b^{\mu\nu} = \delta_s b^{\mu\nu}.$$

This result for contravariant indices is however a special property of the metric tensor only. In general, mixed tensor components have forms as follow

$$(7)\qquad \delta S_{\mu\nu}{}^{\cdot\lambda} = \delta_s S_{\mu\nu}{}^{\cdot\lambda} - \epsilon\,\overline{S}_{\mu\nu}{}^{\cdot\sigma}\left(\overset{*}{\nabla}_\sigma\eta^\lambda - G^{\lambda}_{\sigma k}\eta^k\right),$$

$$(8)\qquad \delta\overline{R}^{k}_{\cdot\lambda\mu\nu} = \delta_s\overline{R}^{k}_{\cdot\lambda\mu\nu} - \epsilon\,\overline{R}^{\sigma}_{\cdot\lambda\mu\nu}\left(\overset{*}{\nabla}_\sigma\eta^k - G^{k}_{\sigma\iota}\eta^\iota\right),$$

$$(9)\qquad \delta\overline{R}^{\rho}_{\cdot\sigma\mu\nu} = \epsilon\Big[\eta^\lambda\nabla_\lambda\overline{R}^{\rho}_{\cdot\sigma\mu\nu} + \overline{R}^{\rho}_{\cdot\sigma\lambda\nu}\overset{*}{\nabla}_\mu\eta^\lambda + \overline{R}^{\rho}_{\cdot\sigma\mu\lambda}\overset{*}{\nabla}_\nu\eta^\lambda$$
$$+ \eta^\lambda\overline{R}^{\rho}_{\cdot\pi\mu\nu}\overset{*}{J}{}^{\pi}_{\lambda\sigma} - \eta^\lambda\overline{R}^{\pi}_{\cdot\sigma\mu\nu}\overset{*}{J}{}^{\rho}_{\lambda\pi} + \overline{R}^{\rho}_{\cdot\lambda\mu\nu}\left(\overset{*}{\nabla}_\sigma\eta^\lambda - G^{\lambda}_{\sigma\iota}\eta^\iota\right)\Big].$$

And for *Span* deformations, we have

$$(10)\qquad \delta b_{\mu\nu} = \epsilon\left[\zeta^\sigma\nabla_\sigma b_{\mu\nu} + \zeta^\sigma\left(b_{\iota\nu}\overset{*}{G}{}^{\iota}_{\sigma\mu} + b_{\mu\iota}\overset{*}{G}{}^{\iota}_{\sigma\nu}\right)\right].$$

But $\nabla_\sigma b_{\mu\nu} = \nabla_\sigma(a_{\alpha\beta} B^{\alpha\beta}_{\mu\nu})$, and if we are dealing with a metric space, this gives

$$\nabla_\sigma b_{\mu\nu} = a_{\alpha\beta}(B^\alpha_\mu J^\beta_{\nu\sigma} + B^\alpha_\nu J^\beta_{\mu\sigma}), \tag{11}$$

But $J^\beta_{\nu\sigma} = C^\beta_\rho J^\rho_{\nu\sigma}$ and hence we have $\nabla_\sigma b_{\mu\nu} = 0$, giving

$$\delta b_{\mu\nu} = \epsilon\left[\zeta^\sigma\left(b_{\iota\nu}\overset{*}{G}{}^\iota_{\sigma\mu} + b_{\mu\iota}\overset{*}{G}{}^\iota_{\sigma\nu}\right)\right]. \tag{10'}$$

If we are dealing with a Riemann Geometry, this reduces to

$$\delta b_{\mu\nu} = \epsilon\,\zeta^\sigma(b_{\iota\nu}G^\iota_{\sigma\mu} + b_{\mu\iota}G^\iota_{\sigma\nu}), \tag{11'}$$

so that, on expressing ζ^σ in A_n components, and using the relation (**11**.1) we get

$$\delta b_{\mu\nu} = -2\,\epsilon\,\xi^\alpha a_{\alpha\beta}F^\beta_{\mu\nu} = -2\,\epsilon\,\zeta^\rho a_{\alpha\beta}C^\alpha_\rho F^\beta_{\mu\nu}.$$

Now $(\delta b_{\mu\nu})\,du^\mu du^\nu = {}'ds^2 - ds^2$ and if we take ζ^ρ to be contant, and call $\epsilon\zeta^\rho = h^{(\rho)}$ say, then we can write (¹)

$$\frac{'ds^2 - ds^2}{-2h^{(\rho)}} = F^\rho_{\mu\nu}\,du^\mu du^\nu.$$

Returning now to the general case, we have the following results foı other tensors

$$\delta b^{\mu\nu} = \epsilon\,\zeta^\sigma\left[\nabla_\sigma b^{\mu\nu} - \left(b^{\mu\iota}\overset{*}{G}{}^\nu_{\sigma\iota} + b^{\iota\nu}\overset{*}{G}{}^\mu_{\sigma\iota}\right)\right], \tag{12}$$

$$\begin{aligned}\delta S_{\mu\nu}^{\cdot\cdot k} = \epsilon\Big[&\zeta^\sigma\nabla_\sigma S_{\mu\nu}^{\cdot\cdot k} + \zeta^\sigma\left(S_{\lambda\nu}^{\cdot\cdot k}\overset{*}{G}{}^\lambda_{\sigma\mu} + S_{\mu\lambda}^{\cdot\cdot k}\overset{*}{G}{}^\lambda_{\sigma\nu} - S_{\mu\nu}^{\cdot\cdot\lambda}\overset{*}{G}{}^k_{\sigma\lambda}\right) \\ &+ \overline{S}_{\sigma\nu}^{\cdot\cdot k}\left(\overset{*}{\nabla}_\mu\zeta^\sigma - J^\sigma_{\mu\rho}\zeta^\rho\right) + \overline{S}_{\mu\sigma}^{\cdot\cdot k}\left(\overset{*}{\nabla}_\nu\zeta^\sigma - J^\sigma_{\nu\rho}\zeta^\rho\right)\Big],\end{aligned} \tag{13}$$

$$\begin{aligned}\delta R^k_{\cdot\lambda\mu\nu} = \epsilon\Big[&\zeta^\sigma\nabla_\sigma\overline{R}^k_{\cdot\lambda\mu\nu} + \zeta^\sigma\big(\overline{R}^k_{\cdot\lambda\iota\nu}\overset{*}{G}{}^\iota_{\sigma\mu} + \overline{R}^k_{\cdot\lambda\mu\iota}\overset{*}{G}{}^\iota_{\sigma\nu} + \overline{R}^k_{\cdot\iota\mu\nu}\overset{*}{G}{}^\iota_{\sigma\lambda} - \overline{R}^\iota_{\cdot\lambda\mu\nu}\overset{*}{G}{}^k_{\nu\iota}, \\ &+ \overline{R}^k_{\cdot\lambda\sigma\nu}\left(\overset{*}{\nabla}_\mu\zeta^\sigma - J^\sigma_{\mu\rho}\zeta^\rho\right) \\ &+ \overline{R}^k_{\cdot\lambda\mu\sigma}\left(\overset{*}{\nabla}_\nu\zeta^\sigma - J^\sigma_{\nu\rho}\zeta^\rho\right) + \overline{R}^k_{\cdot\sigma\mu\nu}\left(\overset{*}{\nabla}_\lambda\zeta^\sigma - J^\sigma_{\lambda\rho}\zeta^\rho\right)\Big]\end{aligned} \tag{14}$$

(¹) This equation appears in Bompiani (1921, p. 1132) in the form

$$\frac{\overline{ds}^2 - ds^2}{-2h} = \omega^{(h)}_{rs}\,du^r du^s$$

and the quadratic form appearing on the right hand side is called by him « th second fundamental form of the submanifold V_m relative to the normal ξ_h ».

and

$$(15)\quad \delta\overline{R}^{\rho}_{.\sigma\mu\nu} = \epsilon\Big[\zeta^{\pi}\nabla_{\pi}\overline{R}^{\rho}_{.\sigma\mu\nu} + \overline{R}^{\rho}_{.\pi\mu\nu}\overset{*}{\nabla}_{\sigma}\zeta^{\pi} - \overline{R}^{\pi}_{.\sigma\mu\nu}\overset{*}{\nabla}_{\pi}\zeta^{\rho} + \zeta^{\pi}\big(\overline{R}^{\rho}_{.\sigma\lambda\nu}\overset{*}{G}{}^{\lambda}_{\pi\mu} + \overline{R}^{\rho}_{.\sigma\mu\lambda}\overset{*}{G}{}^{\lambda}_{\pi\nu}\big) + \overline{R}^{\rho}_{.\sigma\pi\nu}\big(\overset{*}{\nabla}_{\mu}\zeta^{\pi} - J^{\pi}_{\mu\tau}\zeta^{\tau}\big) + \overline{R}^{\rho}_{.\sigma\mu\pi}\big(\overset{*}{\nabla}_{\nu}\zeta^{\pi} - J^{\pi}_{\nu\tau}\zeta^{\tau}\big) - \overline{R}^{\lambda}_{.\sigma\mu\nu}\big(\overset{*}{\nabla}_{\lambda}\zeta^{\rho} - J^{\rho}_{\lambda\tau}\zeta^{\tau}\big)\Big].$$

17. Deformation of the connexion in a Tangential deformation. — In order to obtain the particular forms taken by $\delta l^{\lambda}_{\mu\nu}$, $\delta\lambda^{\rho}_{\sigma\nu}$, $\delta F^{\rho}_{\mu\nu}$, $\delta G^{\lambda}_{\rho\nu}$ for tangential deformations, we must first of all consider the form taken by $\delta\Gamma^{\alpha}_{\beta\gamma}$ when $\xi^{\alpha} = B^{\alpha}_{\iota}\eta^{\iota}$. Considering first of all the term $\nabla_{\alpha}\overset{*}{\nabla}_{\beta}\xi^{\alpha}$ occurring in the expression for $\delta\Gamma^{\alpha}_{\beta\gamma}$ we have

$$\overset{*}{\nabla}_{\beta}\xi^{\alpha} = \overset{*}{\nabla}_{\beta}(B^{\alpha}_{\iota}\eta^{\iota}) = \eta^{\iota}\overset{*}{\nabla}_{\beta}B^{\alpha}_{\iota} + B^{\alpha}_{\iota}\overset{*}{\nabla}_{\beta}\eta^{\iota} = \eta^{\iota}\big(\overset{*}{F}{}^{\alpha}_{\beta} + \overset{*}{J}{}^{\alpha}_{\iota\beta}\big) + B^{\alpha}\overset{*}{\nabla}_{\beta}\eta^{\iota}.$$

Forming the ordinary covariant derivation of this, we have

$$(1)\quad \nabla_{\gamma}\overset{*}{\nabla}_{\beta}\xi^{\alpha} = \big(\overset{*}{F}{}^{\alpha}_{\iota\beta} + \overset{*}{J}{}^{\alpha}_{\iota\beta}\big)\nabla_{\gamma}\eta^{\iota} + \eta^{\iota}\nabla_{\gamma}\big(\overset{*}{F}{}^{\alpha}_{\iota\beta} + \overset{*}{J}{}^{\alpha}_{\iota\beta}\big) + \big(F^{\alpha}_{\iota\gamma} + J^{\alpha}_{\iota\gamma}\big)\nabla_{\beta}\eta^{\iota} + B^{\alpha}_{\iota}\nabla_{\gamma}\overset{*}{\nabla}_{\beta}\eta^{\iota},$$

Now

$$(\mathrm{i})\quad \delta l^{\lambda}_{\mu\nu} = B^{\lambda\beta\gamma}_{\alpha\mu\nu}(\delta\Gamma^{\alpha}_{\beta\gamma}) = B^{\lambda\beta\gamma}_{\alpha\mu\nu}\big(\nabla_{\gamma}\overset{*}{\nabla}_{\beta}\xi^{\alpha} + R^{\alpha}_{.\beta\gamma\delta}\xi^{\delta}\big),$$

so treating the two terms separately, we have from (1),

$$B^{\lambda\beta\gamma}_{\alpha\mu\nu}\big(\nabla_{\gamma}\overset{*}{\nabla}_{\beta}\xi^{\alpha}\big) = \eta^{\iota}B^{\lambda\beta}_{\alpha\mu}\nabla_{\nu}\big(\overset{*}{F}{}^{\alpha}_{\iota\beta} + \overset{*}{J}{}^{\alpha}_{\iota\beta}\big) + B^{\beta\gamma}_{\mu\nu}\nabla_{\gamma}\overset{*}{\nabla}_{\beta}\eta^{\lambda}$$

where

$$B^{\lambda\beta}_{\alpha\mu}\nabla_{\nu}\overset{*}{F}{}^{\alpha}_{\iota\beta} = B^{\lambda}_{\alpha}\nabla_{\nu}\overset{*}{F}{}^{\alpha}_{\iota\mu} = -\overset{*}{F}{}^{\alpha}_{\iota\mu}\nabla_{\nu}B^{\lambda}_{\alpha} = +\overset{*}{F}{}^{\rho}_{\iota\mu}G^{\lambda}_{\rho\nu},$$

and

$$B^{\lambda\beta}_{\alpha\mu}\nabla_{\nu}\overset{*}{J}{}^{\alpha}_{\iota\beta} = 0.$$

Using again the relation

$$B^{\beta\gamma}_{\mu\nu}\nabla_{\gamma}\overset{*}{\nabla}_{\beta}\eta^{\lambda} = \nabla_{\nu}\overset{*}{\nabla}_{\mu}\eta^{\lambda} - F^{\rho}_{\mu\nu}\overset{*}{\nabla}_{\rho}\eta^{\lambda}$$

and applying the Gauss equation to the term involving $R^{\alpha}_{.\beta\gamma\delta}$ we have finally

$$(2)\quad \delta l^{\lambda}_{\mu\nu} = \epsilon\big(\nabla_{\nu}\overset{*}{\nabla}_{\mu}\eta^{\lambda} + R^{\lambda}_{.\mu\nu\iota}\eta^{\iota}\big) - \epsilon F^{\rho}_{\mu\nu}\big(\overset{*}{\nabla}_{\rho}\eta^{\lambda} - G^{\lambda}_{\rho\iota}\eta^{\iota}\big).$$

But the expression in the first bracket is evidently the intrinsic deformation which would be undergone by $l^{\lambda}_{\mu\nu}$ in an infinitesimal transformation ($'u^\lambda = u^\lambda + \epsilon\,\eta^\lambda$) of the submanifold. Hence we can write

$$(3)\qquad \delta l^\lambda_{\mu\nu} = \delta_s l^\lambda_{\mu\nu} - \epsilon\, F^\rho_{\mu\nu}\left(\overset{*}{\nabla}_\rho \eta^\lambda - G^\lambda_{\rho\iota}\eta^\iota\right).$$

For

$$(\mathrm{ii})\qquad \delta\lambda^\rho_{\rho\nu} = C^{\rho\beta}_{\alpha\sigma} B^\gamma_\nu (\delta\Gamma^\alpha_{\beta\gamma})$$

using (1), we have

$$C^{\rho\beta}_{\alpha\sigma} B^\gamma_\nu\left(\nabla_\gamma \overset{*}{\nabla}_\beta \xi^\alpha\right) = \overset{*}{J}{}^\rho_{\iota\sigma}\nabla_\nu \eta^\iota + \eta^\iota C^{\rho\beta}_{\alpha\sigma}\nabla_\nu\left(\overset{*}{F}{}^\alpha_{\iota\beta} + \overset{*}{J}{}^\alpha_{\iota\beta}\right) + F^\rho_{\iota\nu}\overset{*}{\nabla}_\sigma \eta^\iota$$

where

$$C^{\rho\beta}_{\alpha\sigma}\nabla_\nu \overset{*}{F}{}^\alpha_{\iota\beta} = C^\beta_\sigma \nabla_\nu \overset{*}{F}{}^\rho_{\iota\beta} = -\overset{*}{F}{}^\rho_{\iota\lambda} G^\lambda_{\sigma\nu},$$

and

$$C^{\rho\beta}_{\alpha\sigma}\nabla_\nu \overset{*}{J}{}^\alpha_{\iota\beta} = C^\beta_\sigma \nabla_\nu \overset{*}{J}{}^\rho_{\iota\beta} = \nabla_\nu \overset{*}{J}{}^\rho_{\iota\sigma}.$$

Using again the Kühne equation for the term involving $R^{\alpha}_{.\beta\gamma\delta}$ we have finally

$$(4)\qquad \delta\lambda^\rho_{\sigma\nu} = \epsilon\,\nabla_\nu\left(\overset{*}{J}{}^\rho_{\iota\sigma}\eta^\iota\right) + \epsilon\,\eta^\iota R^\rho_{.\sigma\nu\iota} + \epsilon\, F^\rho_{\lambda\nu}\left(\overset{*}{\nabla}_\sigma \eta^\lambda - G^\lambda_{\sigma\iota}\eta^\iota\right).$$

Treating the deformations of the tensors of Eulerian Curvature in the same way, the

$$\delta F^\rho_{\mu\nu} = C^\rho_\alpha B^{\beta\gamma}_{\mu\nu}(\delta\Gamma^\alpha_{\beta\gamma})$$

on using (1), gives

$$\delta F^\rho_{\mu\nu} = \overset{*}{F}{}^\rho_{\iota\mu}\nabla_\nu \eta^\iota + C^\rho_\alpha B^\beta_\mu \nabla_\nu\left(\overset{*}{F}{}^\alpha_{\iota\beta} + \overset{*}{J}{}^\alpha_{\iota\beta}\right) + F^\rho_{\iota\nu}\overset{*}{\nabla}_\mu \eta^\iota + \overline{R}{}^\rho_{.\mu\nu\iota}\eta^\iota$$

where

$$C^\rho_\alpha B^\beta_\mu \nabla_\nu \overset{*}{F}{}^\alpha_{\iota\beta} = B^\beta_\mu \nabla_\nu \overset{*}{F}{}^\rho_{\iota\beta} = \nabla_\nu \overset{*}{F}{}^\rho_{\iota\mu};$$

and

$$C^\rho_\alpha B^\beta_\mu \nabla_\nu \overset{*}{J}{}^\alpha_{\iota\beta} = B^\beta_\mu \nabla_\nu \overset{*}{J}{}^\rho_{\iota\beta} = -\overset{*}{J}{}^\rho_{\iota\beta}\nabla_\nu B^\beta_\mu = -\overset{*}{J}{}^\rho_{\iota\sigma} F^\sigma_{\mu\nu}.$$

On applying the Codazzi equation to the term $\overline{R}{}^\rho_{.\mu\nu\iota}$ we have

$$\overline{R}{}^\rho_{.\mu\nu\iota} = \nabla_\iota F^\rho_{\mu\nu} - \nabla_\nu F^\rho_{\mu\iota} + 2F^\rho_{\mu\lambda}S_{\nu\iota}^{..\lambda},$$

so that on reduction, we have

$$(5)\qquad \delta F^\rho_{\mu\nu} = \epsilon\left[\eta^\iota\nabla_\iota F^\rho_{\mu\nu} + F^\rho_{\iota\nu}\overset{*}{\nabla}_\mu \eta^\iota + F^\rho_{\mu\iota}\overset{*}{\nabla}_\nu \eta^\iota - \eta^\iota F^\sigma_{\mu\nu}\overset{*}{J}{}^\rho_{\iota\sigma}\right].$$

Now if A_m is a totally geodesic subspace of A_n, then $F^{\rho}_{\mu\nu} = 0$ over the subspace. We can therefore conclude from (5) that *for a tangential deformation, a totally geodesic subspace is deformed into a totally geodesic subspace.*

Finally we have

$$\delta G^{\lambda}_{\rho\nu} = B^{\lambda}_{\alpha} C^{\beta}_{\rho} B^{\gamma}_{\nu} (\delta \Gamma^{\alpha}_{\beta\gamma})$$

which, on using (1) gives

$$\delta G^{\lambda}_{\rho\nu} = \epsilon \left[\eta^{\iota} B^{\lambda}_{\alpha} C^{\beta}_{\rho} \nabla_{\nu} \left(\overset{*}{F}{}^{\alpha}_{\iota\beta} + \overset{*}{J}{}^{\alpha}_{\iota\beta} \right) + C^{\beta}_{\rho} B^{\gamma}_{\nu} \nabla_{\gamma} \overset{*}{\nabla}_{\beta} \eta^{\lambda} + \overline{R}^{\lambda}_{\cdot\rho\nu\iota} \eta^{\iota} \right]$$

where

$$B^{\lambda}_{\alpha} C^{\beta}_{\rho} \nabla_{\nu} \overset{*}{F}{}^{\alpha}_{\iota\beta} = 0;$$

$$B^{\lambda}_{\alpha} C^{\beta}_{\rho} \nabla_{\nu} \overset{*}{J}{}^{\alpha}_{\iota\beta} = B^{\lambda}_{\alpha} \nabla_{\nu} \overset{*}{J}{}^{\alpha}_{\iota\rho} = - \overset{*}{J}{}^{\alpha}_{\iota\rho} \nabla_{\nu} B^{\lambda}_{\alpha} = + \overset{*}{J}{}^{\sigma}_{\iota\rho} G^{\lambda}_{\sigma\nu},$$

and

$$C^{\beta}_{\rho} B^{\gamma}_{\nu} \nabla_{\gamma} \overset{*}{\nabla}_{\beta} \eta^{\lambda} = \nabla_{\nu} \overset{*}{\nabla}_{\rho} \eta^{\lambda} - G^{\iota}_{\rho\nu} \overset{*}{\nabla}_{\iota} \eta^{\lambda}.$$

Finally, applying the Codazzi equation to the term $\overline{R}^{\lambda}_{\cdot\rho\nu\iota}$ we have

$$\delta G^{\lambda}_{\rho\nu} = \epsilon \left[\nabla_{\nu} \left(\overset{*}{\nabla} \eta^{\lambda} - G^{\lambda}_{\rho\iota} \eta^{\iota} \right) + G^{\lambda}_{\rho\iota} \overset{*}{\nabla}_{\nu} \eta^{\iota} - G^{\iota}_{\rho\nu} \overset{*}{\nabla}_{\iota} \eta^{\lambda} + \eta^{\iota} \nabla_{\iota} G^{\lambda}_{\rho\nu} + G^{\lambda}_{\sigma\nu} \overset{*}{J}{}^{\sigma}_{\iota\rho} \eta^{\iota} \right].$$

18. Deformation of the connexion in a span deformation. — In this case

$$\xi^{\alpha} = C^{\alpha}_{\sigma} \zeta^{\sigma},$$

so that

$$\overset{*}{\nabla}_{\beta} \xi^{\alpha} = C^{\alpha}_{\sigma} \overset{*}{\nabla}_{\beta} \zeta^{\sigma} + \zeta^{\sigma} \overset{*}{\nabla}_{\beta} C^{\alpha}_{\sigma} = C^{\alpha}_{\sigma} \overset{*}{\nabla}_{\beta} \zeta^{\sigma} + \zeta^{\sigma} \left(\overset{*}{G}{}^{\alpha}_{\sigma\beta} + \overset{*}{K}{}^{\alpha}_{\sigma\beta} \right),$$

and hence

$$\nabla_{\gamma} \overset{*}{\nabla}_{\beta} \xi^{\alpha} = (G^{\alpha}_{\sigma\gamma} + K^{\alpha}_{\sigma\gamma}) \overset{*}{\nabla}_{\beta} \zeta^{\sigma} + C^{\alpha}_{\sigma} \overset{*}{\nabla}_{\gamma} \nabla_{\beta} \zeta^{\sigma} + \left(\overset{*}{G}{}^{\alpha}_{\sigma\beta} + \overset{*}{K}{}^{\alpha}_{\sigma\beta} \right) \nabla_{\gamma} \zeta^{\sigma} + \zeta^{\sigma} \nabla_{\gamma} \left(\overset{*}{G}{}^{\alpha}_{\sigma\beta} + \overset{*}{K}{}^{\alpha}_{\sigma\beta} \right), \tag{1}$$

and so

$$\delta l^{\lambda}_{\mu\nu} = \epsilon \left[G^{\lambda}_{\sigma\nu} \overset{*}{\nabla}_{\mu} \zeta^{\sigma} + \overset{*}{G}{}^{\lambda}_{\sigma\mu} \nabla_{\nu} \zeta^{\sigma} + \zeta^{\sigma} B^{\lambda\beta}_{\alpha\mu} \nabla_{\nu} \left(\overset{*}{G}{}^{\alpha}_{\sigma\beta} + \overset{*}{K}{}^{\alpha}_{\sigma\beta} \right) + \overline{R}^{\lambda}_{\cdot\mu\nu\sigma} \zeta^{\sigma} \right].$$

But

$$B^{\lambda\beta}_{\alpha\mu} \nabla_{\nu} \overset{*}{G}{}^{\alpha}_{\sigma\beta} = \nabla_{\nu} \overset{*}{G}{}^{\lambda}_{\sigma\mu} \qquad \text{and} \qquad B^{\lambda\beta}_{\alpha\mu} \nabla_{\nu} \overset{*}{K}{}^{\alpha}_{\sigma\beta} = - \overset{*}{K}{}^{\lambda}_{\sigma\rho} F^{\rho}_{\mu\nu}$$

giving

$$\delta l^{\lambda}_{\mu\nu} = \epsilon \left[\nabla_{\nu} \left(\overset{*}{G}{}^{\lambda}_{\sigma\mu} \zeta^{\sigma} \right) + G^{\lambda}_{\sigma\nu} \overset{*}{\nabla}_{\mu} \zeta^{\sigma} - F^{\rho}_{\mu\nu} \overset{*}{K}{}^{\lambda}_{\sigma\rho} \zeta^{\sigma} + \zeta^{\sigma} \overline{R}^{\lambda}_{\cdot\mu\nu\sigma} \right]. \tag{2}$$

Similarly we have

$$\delta\lambda^{\rho}_{\sigma\nu} = \epsilon\left[C^{\beta}_{\sigma} B^{\gamma}_{\nu} \nabla_{\gamma} \overset{*}{\nabla}_{\beta} \zeta^{\rho} + \zeta^{\pi} C^{\rho\beta}_{\alpha\sigma} \nabla_{\nu}\left(\overset{*}{G}{}^{\alpha}_{\pi\beta} + \overset{*}{K}{}^{\alpha}_{\pi\beta}\right) + \overline{R}^{\rho}_{.\sigma\nu\pi} \zeta^{\pi}\right],$$

which can be reduced, in virtue of

$$C^{\beta}_{\sigma} B^{\gamma}_{\nu} \overset{*}{\nabla}_{\gamma} \nabla_{\beta} \zeta^{\rho} = \nabla_{\nu} \overset{*}{\nabla}_{\sigma} \zeta^{\rho} - G^{\mu}_{\sigma\nu} \overset{*}{\nabla}_{\mu} \zeta^{\rho},$$

$$C^{\rho\beta}_{\alpha\sigma} \nabla_{\nu} \overset{*}{G}{}^{\alpha}_{\pi\beta} = 0 \qquad \text{and} \qquad C^{\rho\beta}_{\alpha\sigma} \nabla_{\nu} \overset{*}{K}{}^{\alpha}_{\pi\beta} = \overset{*}{K}{}^{\mu}_{\pi\beta} F^{\rho}_{\mu\nu},$$

to

(3) $$\delta\lambda^{\rho}_{\sigma\nu} = \epsilon\left[\nabla_{\nu} \overset{*}{\nabla}_{\sigma} \zeta^{\rho} - G^{\mu}_{\sigma\nu} \overset{*}{\nabla} \zeta^{\rho} + \zeta^{\pi} \overset{*}{K}{}^{\mu}_{\pi\rho} F^{\rho}_{\mu\nu} + \overline{R}^{\rho}_{.\sigma\nu\pi} \zeta^{\pi}\right].$$

We quote the corresponding results for the two tensors of Eulerian Curvature

(4) $$\delta F^{\rho}_{\mu\nu} = \epsilon\left[\nabla_{\nu} \overset{*}{\nabla}_{\mu} \zeta^{\rho} - F^{\sigma}_{\mu\nu} \overset{*}{\nabla}_{\sigma} \zeta^{\rho} + \zeta^{\sigma} F^{\rho}_{\iota\nu} G^{\iota}_{\sigma\mu} + \overline{R}^{\rho}_{.\mu\nu\sigma} \zeta^{\sigma}\right],$$

(5) $$\delta G^{\lambda}_{\rho\nu} = \epsilon\left[\nabla_{\nu}\left(\overset{*}{K}{}^{\lambda}_{\sigma\rho} \zeta^{\sigma}\right) + G^{\lambda}_{\sigma\nu} \overset{*}{\nabla}_{\rho} \zeta^{\sigma} - \zeta^{\sigma} \overset{*}{G}{}^{\lambda}_{\sigma\mu} G^{\mu}_{\rho\nu} + \overline{R}^{\lambda}_{.\rho\nu\sigma} \zeta^{\sigma}\right].$$

19. Deformation of the Riemannian Curvature Tensors for Tangential and Span deformations. — **Let us first consider the changes in the tensors $R^{k}_{.\lambda\mu\nu}$ and $R^{\rho}_{.\sigma\mu\nu}$ for the case $\xi^{\alpha} = B^{\alpha}_{\iota} \eta^{\iota}$. We have the general result**

(1) $$\delta R^{k}_{.\lambda\mu\nu} = \delta \overline{R}^{k}_{.\lambda\mu\nu} - F^{\sigma}_{\lambda\mu} \delta G^{k}_{\sigma\nu} - G^{k}_{\sigma\nu} \delta F^{\sigma}_{\lambda\mu} + F^{\sigma}_{\lambda\nu} \delta G^{k}_{\sigma\mu} + G^{k}_{\sigma\mu} \delta F^{\sigma}_{\lambda\nu}$$

where the complete expression for $\delta \overline{R}^{k}_{.\lambda\mu\nu}$ for a tangential deformation is

(2) $$\delta \overline{R}^{k}_{.\lambda\mu\nu} = \epsilon\left[\eta^{\iota} \nabla_{\iota} \overline{R}^{k}_{.\lambda\mu\nu} + \overline{R}^{k}_{.\iota\mu\nu} \overset{*}{\nabla}_{\lambda} \eta^{\iota} + \overline{R}^{k}_{.\lambda\iota\nu} \nabla_{\mu} \eta^{\iota} + \overline{R}^{k}_{.\lambda\mu\iota} \overset{*}{\nabla}_{\nu} \eta^{\iota} - \overline{R}^{\iota}_{.\lambda\mu\nu} \nabla_{\iota} \eta^{k}\right] - \epsilon\, \overline{R}^{\sigma}_{.\lambda\mu\nu}\left(\overset{*}{\nabla}_{\sigma} \eta^{k} - G^{k}_{\sigma\iota} \eta^{\iota}\right).$$

If we insert this value of $\delta \overline{R}^{k}_{.\lambda\mu\nu}$ in (1), and apply the Gauss equation to each term, afterwards inserting the values of $\delta F^{\sigma}_{\lambda\mu}$ and $\delta G^{k}_{\sigma\nu}$, we have, on putting for brevity

(3) $$Y^{k}_{\sigma} = \overset{*}{\nabla}_{\sigma} \eta^{k} - G^{k}_{\sigma\iota} \eta^{\iota}$$

that

(4) $$\delta R^{k}_{.\lambda\mu\nu} = \epsilon\left[\eta^{\iota} \nabla_{\iota} R^{k}_{.\lambda\mu\nu} + R^{k}_{.\iota\mu\nu} \overset{*}{\nabla}_{\lambda} \eta^{\iota} + R^{k}_{.\lambda\iota\nu} \overset{*}{\nabla}_{\mu} \eta^{\iota} + R^{k}_{.\lambda\mu\iota} \nabla_{\nu} \eta^{\iota} - R^{\iota}_{.\lambda\mu\nu} \overset{*}{\nabla}_{\iota} \eta^{k}\right] - \epsilon\, \overline{R}^{\sigma}_{.\lambda\mu\nu} Y^{k}_{\sigma} + \epsilon\left[F^{\sigma}_{\lambda\nu} \nabla_{\mu} Y^{k}_{\sigma} - F^{\sigma}_{\lambda\mu} \nabla_{\nu} Y^{k}_{\sigma}\right].$$

But the expression inside the first bracket is evidently $\partial_s R^{k}_{.\lambda\mu\nu}$ and hence, on applying the Codazzi equation to the term involving $\overline{R}^{\sigma}_{.\lambda\mu\nu}$ we have finally

$$(5) \qquad \delta R^{k}_{.\lambda\mu\nu} = \partial_s R^{k}_{.\lambda\mu\nu} + 2\epsilon\left[\nabla_{[\mu}\left(F^{\sigma}_{|\lambda|\nu]} Y^{k}_{\sigma}\right) - Y^{k}_{\sigma} F^{\sigma}_{\lambda\iota} S_{\mu\nu}^{\cdot\cdot\iota}\right].$$

This result could also have been obtained by the following method. We have from

$$(6) \qquad \delta R^{k}_{.\lambda\mu\nu} = \nabla_\nu(\delta l^{k}_{\lambda\mu}) - \nabla_\mu(\delta l^{k}_{\lambda\nu}) + 2 S_{\mu\nu}^{\cdot\cdot\iota}\delta l^{k}_{\lambda\iota}$$

and

$$(7) \qquad \partial_s R^{k}_{.\lambda\mu\nu} = \nabla_\nu(\partial_s l^{k}_{\lambda\mu}) - \nabla_\mu(\partial_s l^{k}_{\lambda\nu}) + 2 S_{\mu\nu}^{\cdot\cdot\iota}\partial_s l^{k}_{\lambda\iota}$$

so that, on using

$$\delta l^{k}_{\lambda\mu} = \partial_s l^{k}_{\lambda\mu} - \epsilon F^{\sigma}_{\lambda\mu} Y^{k}_{\sigma}$$

and substituting in (6), we have the result already given in (5).

For the $\delta R^{\rho}_{.\sigma\mu\nu}$ we have the general result

$$(8) \qquad \delta R^{\rho}_{.\sigma\mu\nu} = \delta\overline{R}^{\rho}_{.\sigma\mu\nu} + F^{\rho}_{\lambda\mu}\,\delta G^{\lambda}_{\sigma\nu} + G^{\lambda}_{\sigma\nu}\,\delta F^{\rho}_{\lambda\mu} - F^{\rho}_{\lambda\nu}\,\delta G^{\lambda}_{\sigma\mu} - G^{\lambda}_{\sigma\mu}\,\delta F^{\rho}_{\lambda\nu}$$

where $\delta\overline{R}^{\rho}_{.\sigma\mu\nu}$ is given in (**16**.9). On inserting their values for $\delta\overline{R}^{\rho}_{.\sigma\mu}$ $\delta F^{\rho}_{\lambda\mu}$, $\delta G^{\lambda}_{\sigma\nu}$ in (8), and applying systematically the Kühne and Codazzi equations, we obtain

$$(9) \quad \delta R^{\rho}_{.\sigma\mu\nu} = \epsilon\left[\eta^{\iota}\nabla_\iota R^{\rho}_{.\sigma\mu\nu} + R^{\rho}_{.\sigma\iota\nu}\overset{*}{\nabla}_\mu \eta^{\iota} + R^{\rho}_{.\sigma\mu\iota}\overset{*}{\nabla}_\nu \eta^{\iota} + \eta^{\iota} R^{\rho}_{.\pi\mu\nu}\overset{*}{J}{}^{\pi}_{\iota\sigma} - \eta^{\iota} R^{\pi}_{.\sigma\mu\nu}\overset{*}{J}{}^{\rho}_{\iota\pi}\right]$$
$$- 2\epsilon\left[\nabla_{[\mu}\left(F^{\rho}_{|\lambda|\nu]} Y^{\lambda}_{\sigma}\right) - Y^{\lambda}_{\sigma} S_{\mu\nu}^{\cdot\cdot\iota} F^{\rho}_{\lambda\iota}\right].$$

The corresponding formulae for the $\delta R^{k}_{.\lambda\mu\nu}$ and $\delta R^{\rho}_{.\sigma\mu\nu}$ for the span deformations can be obtained in the same way, but in this case the formulae become considerably more complicated than in the case of tangential deformations, and they are not so interesting.

Added in proof. — Since this paper was presented in 1933, a comprehensive treatment of deformation problems has been published by Schouten and van Kampen, « *Beiträge zur Theorie der Deformation* » *Prac. Mat. Fiz.*, s. XLI, Warsaw, 1933, p. 1-19.

In that paper the deformed projection factors corresponding to B^{λ}_{α} and C^{α}_{λ} are not the simple *transforms* as in this paper and consequently $\delta B^{\lambda}_{\alpha}$ and $\delta C^{\lambda}_{\alpha}$ do not vanish. In fact :

$$\delta B^{\lambda}_{\alpha} = 2\, b^{\lambda\mu}\, B^{\beta}_{\mu}\, C^{\gamma}_{\alpha}\, \nabla_{(\beta}\varepsilon_{\gamma)}$$

and

$$\delta C^{\alpha}_{\rho} = -\, 2\, b^{\lambda\mu}\, B^{\alpha\gamma}_{\lambda\mu}\, C^{\delta}_{\rho}\, \nabla_{(\gamma}\varepsilon_{\delta)}.$$

A very convenient consequence of this choice of deformed projection factors is that the connection parameters $\overline{l}^{\lambda}_{\mu\nu}$ on $'A_m$ will be obtained from those of A_n by projection, with

$$\delta l^{\lambda}_{\mu\nu} = B^{\lambda\beta\gamma}_{\alpha\mu\nu}(\delta\Gamma^{\alpha}_{\beta\gamma}) + C^{\delta}_{\rho}(\delta B^{\lambda}_{\delta})F^{\rho}_{\mu\nu}, \qquad \delta\lambda^{\rho}_{\sigma\nu} = C^{\rho\beta}_{\alpha\sigma}B^{\gamma}_{\nu}(\delta\Gamma^{\alpha}_{\beta\gamma}) - C^{\delta}_{\sigma}(\delta B^{\mu}_{\delta})F^{\rho}_{\mu\nu},$$
$$\delta F^{\rho}_{\mu\nu} = C^{\rho}_{\alpha}\, B^{\beta\gamma}_{\mu\nu}(\delta\Gamma^{\alpha}_{\beta\gamma}), \qquad \delta G^{\lambda}_{\rho\nu} = B^{\lambda}_{\alpha}\, C^{\beta}_{\rho}\, B^{\gamma}_{\nu}(\delta\Gamma^{\alpha}_{\beta\gamma}) - C^{\alpha}_{\rho}\, \nabla_{\nu}(\delta B^{\lambda}_{\alpha}).$$

Some formulae in Part III of the present paper will then take a simpler form, such as (**17**.3) which becomes

$$\delta l^{\lambda}_{\mu\nu} = \delta_s l^{\lambda}_{\mu\nu}$$

and (**19**.5), which becomes

$$\delta R^{k}_{\cdot\lambda\mu\nu} = \delta_s R^{k}_{\cdot\lambda\mu\nu}.$$

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