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Theory of functions of intervals and applications to functions of a complex variable

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Theory of functions of intervals and applications to functions of a complex variable;

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1. INTRODUCTION. — In this work we shall establish two main theorems on functions of « intervals » and shall apply them, together with some other considerations, to problems of analyticity and to problems of representation of functions (not necessarily analytic) with the aid of « Cauchy double integrals »; furthermore, we shall study certain integral equations, leading to some more general representations of functions of a complex variable.

In the field of functions of intervals reference is made to *S. Saks* (1), *in the sequel referred to as* (S). *An interval* I in the Euclidean space R_m is defined in (S 57). In the plane of $z = x + iy$, that is in R_2, I is a rectangle with sides parallel to the axes. We shall use *regular sequences of nets* [*cf.* (S, 57)]. A *figure* is to denote a sum of a finite number of intervals. Functions of interval, F(I), complex- or real-valued will be supposed to be additive; that is,

$$F(I_1 + I_2) = F(I_1) + F(I_2)$$

for any pair of non overlapping intervals. Such functions are extended in an obvious way to figures. F(I) is said to be absolutely continous if $|R| =$ meas. $R < \delta(\varepsilon)$ (for figures R) implies that $|F(R)| < \varepsilon$. F(I) is continuous if the relation $|I| < \delta(\varepsilon)$ implies $|F(I)| < \varepsilon$.

(1) S. SAKS, *Theory of the integral*, Warszawa, 1937.

Clearly F is absolutely continuous (continuous) if and only if the real and imaginary parts of F are absolutely continuous (continuous).

In sections **2**, **3**, **4** the developments relate to the theory of continuous functions of intervals. Theorem **2**.2 constitutes an extension of a theorem of *Ridder*, while Theorem **4**.8 represents a generalization of a result of *Besicovitch* [cf. (S 193)]. In section **3** we generalize the notion of « length of a set ». We introduce *μ-length* (Definition **3**.1), some of the properties of which are established in Theorem **3**.6; μ-length is used in Theorem **4**.8 and in some subsequent results.

In section **5** we establish conditions under which a function $f(z)$, continuous in a domain G, is analytic. The theorems are **5**.3 and **5**.12. The first extends a result of *J. Wolff* and the other that of *Besicovitch* (S, 197).

In section **6** it is shown that for certain classes of continuous functions of a complex variable there are on hand representations of the form

$$\iint_G \frac{\varphi(J)\,dJ_1\,dJ_2}{J - z} + a(z)$$

[$a(z)$ analytic; $J = J_1 + iJ_2$]. Such representations are given (with the aid of Theorems **2**.15, **4**.25) in Theorems **6**.4, **6**.10, **6'**.6.

Integral equations of the form (**7**.2) are then studied. The main results are embodied in Theorems **7**.12, **8**.10, which present conditions under which (**7**.2) is equivalent to a regular Fredholm integral equation (**7**.3). These developments lead to certain further representations (**8**.14) for functions $f(z)$ belonging to a class R (Definition **7**.1) or to a class B_μ (Definition **8**.1).

Important developments, relating to the problems treated in sections **5**, **6**, are given by *V. S. Fédoroff* (1) along lines different from those of the present paper.

2. EXTENSION OF RIDDER'S THEOREM. — As in (S), a *normal sequence of nets* $N = \{N_k\}$ is a certain kind of regular sequence of nets N_k ($k = 1, 2, \ldots$), the intervals of N_k being closed and non

(1) V. S. FÉDOROFF, *Rec. Math. de Moscou*, t. 2 (44), 1937, pp. 521-541; this work contains an extensive bibliography.

overlapping; those of N_{k+1} are formed from those of N_k by subdivision [for details see (S, 188)]. $\underline{F}(x)$, $\overline{F}(x)$ and $F'(x)$ will denote *the ordinary lower, the ordinary upper derivates* and *the ordinary derivative*, respectively, of any given additive function F of intervals (S, 106); thus

$$(2.1)\qquad \underline{F}(x) = \text{l. b. } A, \qquad \overline{F}(x) = \text{u. b. } A,$$

where A denotes any number for which a "regular" sequence of closed intervals I_n, containing x and such that

$$(2.1\,a)\qquad \delta(I_n) = \text{diameter of } I_n \to 0,$$

exists so that

$$(2.1\,b)\qquad \lim_n \frac{F(I_n)}{|I_n|} = A.$$

A sequence of closed intervals is said to be "regular" with respect to a "parameter of regularity" in accordance with the familiar concept of Lebesgue (S, 106); whenever $\underline{F} = \overline{F}$, we have $F' = \underline{F} = \overline{F}$.

We shall establish the following generalization of a theorem of J. Ridder [cf. (S, 188)].

Theorem 2.2. — *Suppose* $N = \{N_k\}$ *is a normal sequence of nets; let* $g(x)$ *be summable in* R_m *and let* F *be a continuous additive function of an interval such that*

$$\text{(i)}\qquad (N)\,\underline{F}(x) > -\infty \quad \textit{at all points } x,$$

except those of a set D_0 = *sum of a denumerable infinity (at most) of hyperplanes* H_i, *parallel to either of the axes, and*

$$\text{(ii)}\qquad F'(x) \geq g(x) \qquad (\textit{almost everywhere}).$$

Then

$$(2.3)\qquad F(I) \geq \int_I g(x)\,dx \qquad (\textit{for intervals } I \subset P).$$

In (i) $(N)\overline{F}$ denotes the lower derivate, defined with the aid of the intervals of the nets (N).

In Ridder's theorem D_0 is a denumerable set of point.

Let P be the set, necessarily closed, so that, if p is a point of P, in every neighborhood of p there is an interval I so that

$$F(I) < \int_I g(x)\,dx. \tag{2.4}$$

For intervals I, for which $I^\circ \subset CP$ (I° = open interval) one has

$$F(I) \geqq \int_I g(x)\,dx. \tag{2.4a}$$

The latter fact is established in (S, 189).

If the theorem is not true P is non empty. *We shall now proceed under the supposition that* $P \neq 0$. As in (S), given integers k, h (>0), let

$$N_{k,h} = \Sigma I, \tag{2.5}$$

where the sum is over all I of the net N_k for which

$$F(I) > -h\,|I|. \tag{2.5a}$$

The product

$$N^h = \Pi_{k\geq h}^{\infty} N_{k,h}$$

is closed since $N_{k,h}$ is.

Let x denote any fixed point in CD_0; it will be established that

$$x \subset N^{h_0} \quad [h_0 = h_0(x) < +\infty]. \tag{2.6}$$

In fact, by (i)

$$\lim \frac{F(I)}{|I|} = -\gamma > -\infty$$

for intervals I of nets (N) not containing x; thus

$$\frac{F(I)}{|I|} \geqq -\gamma - \varepsilon \qquad [\text{a. e. f. I, of (N)}, \supset x].$$

Here and in the sequel (a. e. f.) *will mean "for all except a finite number of"*. — Accordingly, there exists an integer $h_0(x)$ ($>\gamma - \varepsilon$) sufficiently great so that

$$F(I) > -h_0(x)\,|I| \qquad [\text{for all I, of } N_k (k = h_0, h_0 + 1, \ldots), \supset x].$$

By (2.5) and (2.5 a) an interval I of N_k, involved in the above relation, is an interval of N_{k,h_0}. Such an I contains x; hence $x \subset N_{k,h_0}$: this being true for all $k \geqq h_0$, x will belong to $N_{h_0,h_0} N_{h_0+1,h_0} \ldots$. Whence (2.6) holds.

Since every x of CD_0 is in some $N^{h_0}[h = h_0(x)]$, one has

$$CD_0 \subset \sum_{h=1}^{\infty} N^h$$

and

$$R_m = \sum_{h=1}^{\infty} N^h + D_0.$$

By hypothesis $D_0 = H_1 + H_2 + \ldots$; thus

$$(2.7) \qquad P \subset R_m = \sum_{i=1}^{\infty} F_i,$$

where $F_i = N^i + H_i$ is a closed set. Terefore by Baire's theorem there exists a portion SP, where S is an open sphere with a point P_0 of P for center, so that for some h

$$(2.8) \qquad 0 \neq SP \subset F_h = N^h + H_h.$$

We shall now prove that *a portion* SP *exists so that*

$$(2.9) \qquad 0 \neq SP \subset N^h.$$

If H_h does not contain P_0, then (2.9) is secured by taking the radius of S so small that H_h lies exterior S and on noting that P_0 is a point of P. Thus, consider the case when H_h contains P_0. The set $H_h.CN^h$ is open, when considered as a hyperplanar set (i. e. on H_h), since CN^h is open in R_m. *Let* P′ *be any point of* $H_h.CN^h$. Designate by I′ an interval on H_h so that I'^0 (i. e. the interior, on H_h, of I′) contains P′, while

$$I' \subset H_h.CN^h.$$

Let I be a non degenerate interval in S so that the intersection of I by H_h is I′, I′ not being a face of I, while

$$I \subset CN^h.$$

Such a m-dimensional interval I exists since the closed m-1-dimensional interval I′ lies in CN^h. We decompose I into non overlapping m-dimensional intervals I_1, I_2, I_3,

$$I = I_1 + I_2 + I_3$$

so that I' is not a face of either I_1, I_2, I_3 and so that I' constitutes the intersection of I_3 with H_h. One has

$$(\alpha) \qquad F(I) = F(I_1) + F(I_2) + F(I_3).$$

Since $SP \subset N^h + H_h$ [by (**2**.8)] and since the intervals I_1, I_2 have no points on H_h and lie in CN_h, we have

$$I_1 \subset CP, \qquad I_2 \subset CP.$$

Thus by virtue of (**2**.4 a)

$$(\beta) \qquad F(I_j) \geqq \int_{I_j} g(x)\, dx \qquad (j = 1, 2).$$

Let I_3 tend to I'; then by continuity of F it follows that

$$\lim F(I_3) = F(I') = 0.$$

Hence as a consequence of (α), (β)

$$F(I) \geqq \lim \int_{I_1 + I_2} g(x)\, dx = \int_I g(x)\, dx,$$

This relation will hold for all non degenerate m-dimensional intervals ${}_1I \subset I$, such that, ${}_1I^0 \supset P'$; by (β) it holds for all intervals in CN^h. Accordingly P' does not satisfy the conditions stated in connection with (**2**.4). Having established that $P' \subset CP$ and recalling that P' is any point of $H_h . CN^h$, we conclude that

$$(\gamma) \qquad H_h . CN^h \subset CP.$$

Now

$$SP \subset N^h + H_h = N^h + H_h . CN^h;$$

by (γ) no point of SP is in $H_h . CN^h$; hence $SP \subset N^h$. We finally recall that the center P_0 of S is a point of P. *Consequently* (**2**.9) *has been established.*

As in (S, 189) consider the function of intervals

$$(\mathbf{2}.10) \qquad H(I) = F(I) + h(I) \int_I |g(x)|\, dx;$$

here h is the integer involved in (**2**.9). *If* $I(\subset S)$ *is such that* $I^0 \subset CP$, then by (**2**.4*a*)

$$F(I) \geqq \int_I g(x)\,dx \geqq \int_I -|g(x)|\,dx, \qquad F(I) + \int_I |g(x)|\,dx \geqq 0$$

and, accordingly,

$$(2.11) \qquad H(I) \geqq h(I) \geqq 0.$$

Now consider an interval $I_\nu (\subset S)$ *belonging to one of the nets* N_ν, *for* $\nu \geqq h$, *such that* I_ν^0 *contains a point p of* P. Then by (**2**.9) *p* is a point of N^h; in view of the définition subsequent to (**2**.5*a*)

$$p \subset N_{\nu,h}.$$

As a consequence of the statement with respect to (**2**.5), (**2**.5*a*) I_ν is an interval of the set $N_{\nu,h}$ and

$$F(I_\nu) > -h\,|I_\nu|;$$

by virtue of (**2**.10) one thus has

$$(2.12) \qquad H(I_\nu) > \int_{I_\nu} |g(x)|\,dx \geqq 0.$$

Suppose now that I is any interval in S. One has

$$(2.13) \qquad I = \lim_\nu {}_\nu I,$$

where ${}_\nu I$ is the sum of all intervals ${}_\nu I_j$ of the net N_ν contained in I; ${}_\nu I$ is an interval $\subset$ I. The ${}_\nu I_j$ for which ${}_\nu I^0 \subset CP$, will be designated by ${}_\nu I'_j$; the other ${}_\nu I_j$, denoted by ${}_\nu I''_j$, will contain each a point of P in the interior; (**2**.11), (**2**.12) will apply to the ${}_\nu I'_j$, ${}_\nu I''_j$, respectively (for $\nu \geqq h$). Thus $H({}_\nu I) \geqq 0$. By (**2**.13) and since H(I) is continuous as a fuction of interval and thus vanishes on the faces of I one has

$$H(I) = \lim_\nu H({}_\nu I) \geqq 0$$

and

$$(2.14) \qquad F(I) \geqq -h\,|I| - \int_I |g(x)|\,dx$$

for all $I \subset S$. As in the analogous situation in (S, 189) the above implies that F(I) is of bounded variation and that, in view of (ii) of the Teorem,

$$F(I) \geqq \int_I F'(x)\,dx \geqq \int_I g(x)\,dx \qquad (\text{all } I \subset S).$$

This implies that $SP = 0$ [*cf.* (**2**.4) (**2**.4*a*)], which is contrary to (**2**.9). The Theorem is accordingly proved.

A consequence to Theorem (**2**.2) [compare with (S, 190)] is

THEOREM (**2**.15). — *Let* N *be a normal sequence of nets and* F(I) *be a continuous additive function for which*

$$\text{(i)} \qquad -\infty < (N)\,\underline{F}(x) \leqq (N)\,\overline{F}(x) < +\infty$$

at all points $x \subset CD_0$, *where* D_0 *is a sum of a denumerable infinity (at most) of hyperplanes* H_i *(parallel to either of the axes)*; *also suppose* $F'(x)$ *is summable. Then*

$$F(I) = \int_I F'(x)\,dx.$$

3. μ-LENGTH OF A SET. — Let $\mu(u) > 0$ for $u > 0$ and

$$\mu(u) \to 0$$

monotonically as $u \to 0$. We shall generalize the notion of length of a set in R_m, essentially due to *A. S. Besicovitch* [*cf.* (S, 53, 54)]. The μ-length, that we shall introduce, will reduce to that of Besicovitch for $\mu(u) = u^\alpha$ $(\alpha > 0)$.

DEFINITION (**3**.1). — *Given any set* E (*in the Eaclidean space* R_m), *let*

$$E = \sum_i E_i$$

be a partition of E *into a sequence of sets, possibly denumerable, no two of which have points in common. Let*

$$(\mathbf{3}.3) \qquad \Lambda^\varepsilon(E) = \text{l. b.} \sum_i \mu(\delta(E_i)) \qquad [\delta(E_i) = \textit{diameter of } E_i]$$

for all partitions (**3**.2) *for which* $\delta(E_i) < \varepsilon$. *The* μ-*length of* E *is* *defined as*

$$(3.4) \qquad \Lambda(E) = \lim_{\varepsilon} \Lambda^{\varepsilon}(E).$$

To justify the definition (3.4) we establish that

$$(3.5) \qquad \Lambda^{\varepsilon}(E) \leqq \Lambda^{\varepsilon'}(E) \qquad (\text{for } \varepsilon > \varepsilon' > 0).$$

In fact, given ξ (>0), there exists a decomposition $E = E_1 + E_2 + \ldots$, with $\delta(E_j) < \varepsilon'$, such that

$$\sum_j \mu(\delta(E_j)) \leqq \Lambda^{\varepsilon'}(E) + \xi,$$

Since $\delta(E_j) < \varepsilon$ one also has

$$\Lambda^{\varepsilon}(E) \leqq \sum_j \mu(\delta(E_j)).$$

Thus

$$\Lambda^{\varepsilon}(E) \leqq \Lambda^{\varepsilon'}(E) + \xi;$$

(3.5) will ensue on letting $\xi \to 0$.

We shall now prove the following [compare with (S, 53)].

THEOREM (3.6). — $\Lambda(E)$ *is an outer Caratheodory measure* [*in the sense of* (S, 43)]. *When* $\mu(u)$ *is such that*

$$\text{(i)} \qquad k(n) = \underset{\left[0 < u < \frac{1}{2n}\right]}{\text{u. b.}} \frac{\mu\left(\left(1 + \frac{1}{n}\right)u\right)}{\mu(u)} \to 1, \qquad (\textit{as } n \to +\infty),$$

then the outer measure Λ *is regular and, in fact, fort each set* E *there exists a set* H, *product of open sets containing* E, *so that* $\Lambda(H) = \Lambda(E)$.

Suppose $E \subset G$. Let $G = G_1 + G_2 + \ldots$, where $\delta(G_j) < \varepsilon$, be a partition of G so that

$$(3.7) \qquad \sum_j \mu(\delta(G_j)) \leqq \Lambda^{\varepsilon}(G) + \xi.$$

On writing $E_i = EG_i$, one obtains

$$\delta(E_i) \leqq \delta(G_i) < \varepsilon, \qquad \mu(\delta(E_i)) \leqq \mu(\delta(G_i));$$

there is a partition $E = E_1 + E_2 + \dots$ on hand for which

$$\Lambda^\varepsilon(E) \leqq \sum_i \mu(\delta(E_i)) \leqq \sum_i \mu(\delta(G_i)).$$

By (3.7)

$$\Lambda^\varepsilon(E) \leqq \Lambda^\varepsilon(G) + \xi$$

and, in the limit,

$$(\alpha_1) \qquad \Lambda^\varepsilon(E) \leqq \Lambda^\varepsilon(G) \qquad (\textit{whenever } E \subset G).$$

With $\{E_i\}$ denoting any sequence of sets, let

$$E_i = \sum_j E_{ij} \qquad [\delta(E_{ij}) < \varepsilon].$$

be a partition of E_i so that

$$\sum_j \mu(\delta(E_{ij})) \leqq \Lambda^\varepsilon(E^i) + 2^{-i}\xi;$$

then

$$(3.8) \qquad \sum_{ij} \mu(\delta(E_{ij})) \leqq \sum_i \Lambda^\varepsilon(E_i) + \xi.$$

Then the set $E_1 + E_2 + \dots$ has the decomposition

$$(3.9) \qquad \begin{cases} \sum_i E_i = E'_1 + E'_2 + E'_3 + \dots, & E'_1 = E_1, \\ E'_2 = E_2 - E'_1, \quad E'_3 = E_3 - (E'_1 + E'_2), & \dots, \end{cases}$$

where no two of the sets $E'_1, E'_2, \dots$ have points in common, while $E'_i \subset E_i$. Corresponding to the previously given partition of E_i we have the partition

$$E'_i = \sum_j E'_{ij}, \qquad E'_{ij} = E'_i E_{ij} \subset E_{ij};$$

for which

$$(3.10) \qquad \delta(E'_{ij}) \leqq \delta(E_{ij}), \qquad \mu(\delta(E'_{ij})) \leqq \mu(\delta(E_{ij})).$$

In view of (3.9), (3.10) the set $E_1 + E_2 + \dots$ has a partition

$$\sum_i E_i = \sum_{ij} E'_{ij} \qquad [\delta(E'_{ij}) < \varepsilon];$$

clearly

$$\Lambda^\varepsilon\left(\sum_i E_i\right) \leqq \sum_{ij} \mu(\delta(E'_{ij})) \leqq \sum_{ij} \mu(\delta(E_{ij})).$$

Hence as a consequence of (3.8)

$$\Lambda^\varepsilon\left(\sum_i E_i\right) \leqq \sum_i \Lambda^\varepsilon(E_i) + \xi$$

and, on letting $\xi \to 0$,

$$(\alpha_2) \qquad \Lambda^\varepsilon\left(\sum_i E_i\right) \leqq \sum_i \Lambda^\varepsilon(E_i)$$

for all sequences $\{E_i\}$.

Suppose the distance $\rho(E, G)$ between sets E and G is positive. With $\varepsilon (> 0)$ suitably small, any partition

$$E + G = \Sigma Q_i, \qquad \delta(Q_i) < \varepsilon,$$

will imply that either $Q_i \subset E$ or $Q_i \subset G$. For some Q_i ($i = 1, 2, \ldots$), with $\delta(Q_i) < \varepsilon$,

$$(3.11) \qquad \sum_i \mu(\delta(Q_i)) \leqq \Lambda^\varepsilon(E + G) + \xi.$$

On designating the Q_i, $\subset E$, by $E_1, E_2, \ldots$ and the Q_i, $\subset G$, by $G_1, G_2, \ldots$ we obtain

$$\sum_i \mu(\delta(Q_i)) = \sum_j \mu(\delta(E_j)) + \sum_j \mu(\delta(G_j)) \geqq \Lambda^\varepsilon(E) + \Lambda^\varepsilon(G),$$

inasmuch as $E = E_1 + E_2 + \ldots$, $G = G_1 + G_2 + \ldots$ are partitions with $\delta(E_j) < \varepsilon$, $\delta(G_i) < \varepsilon$. By (3.11) and the above

$$\Lambda^\varepsilon(E) + \Lambda^\varepsilon(G) \leqq \Lambda^\varepsilon(E + G) + \xi$$

and, in the limit,

$$(3.12) \qquad \Lambda^\varepsilon(E) + \Lambda^\varepsilon(G) \leqq \Lambda^\varepsilon(E + G),$$

Conversely, for some partitions

$$E = \Sigma E_i, \qquad G = \Sigma G_i \qquad (\delta(E_i), \delta(G_i) < \varepsilon),$$

where ε is sufficiently small, one has

$$\sum_i \mu(\delta(E_i)) \leqq \Lambda^\varepsilon(E) + \xi, \qquad \sum_i \mu(\delta(G_i)) \leqq \Lambda^\varepsilon(G) + \xi$$

and

$$(3.13) \qquad \sum_i \mu(\delta(E_i)) + \sum \mu(\delta(G_i)) \leqq \Lambda^\varepsilon(E) + \Lambda^\varepsilon(G) + 2\xi.$$

The sequence $(E_1, G_1, E_2, G_2, \ldots) = (Q_1, Q_2, \ldots)$ gives a partition

$$E + G = \Sigma Q_j \qquad (\delta(Q_j) < \varepsilon);$$

hence

$$\sum_j \mu(\delta(Q_j)) = \sum_i \mu(\delta(E_i)) + \sum_i \mu(\delta(G_i)) \geqq \Lambda^\varepsilon(E + G).$$

Therefore by virtue of (3.13)

$$\Lambda^\varepsilon(E + G) \leqq \Lambda^\varepsilon(E) + \Lambda^\varepsilon(G) + 2\xi.$$

We let here $\xi \to 0$ and, on taking note of (3.12), infer that

$$(\alpha_3) \qquad \Lambda^\varepsilon(E + G) = \Lambda^\varepsilon(E) + \Lambda^\varepsilon(G),$$

whenever $\rho(E, G) > 0$ (ε suitably small).

On letting $\varepsilon \to 0$, from (α_1), (α_2), (α_3) it follows that

$$(3.14) \qquad \Lambda(E) \leqq \Lambda(G) \qquad (\text{for } E \subset G),$$

$$(3.15) \qquad \Lambda\Big(\sum_i E_i\Big) \leqq \sum \Lambda(E_i) \qquad (\text{for all sequences } E_1, E_2, \ldots),$$

$$(3.16) \qquad \Lambda(E + G) = \Lambda(E) + \Lambda(G) \qquad [\text{when } \rho(E, G) > 0].$$

Hence μ-length Λ is an outer Caratheodory measure.

We now proceed under the condition (i) of the Theorem. The proof of the remainder of the Theorem we give following the pattern indicated in an analogous prof in (S, 53).

Given any set E, there exists a partition

$$(3.17) \qquad E = \sum_i E_i^{(n)}, \qquad \delta(E_i^{(n)}) < \frac{1}{2n},$$

so that

$$(3.17') \qquad \sum_i \mu(\delta(E_i^{(n)})) \leqq \Lambda^{\frac{1}{2n}}(E) + \frac{1}{n} \leqq \Lambda(E) + \frac{1}{n}.$$

For some open set $G_i^{(n)}, \supset E_i^{(n)}$, one has

$$(3.18) \qquad \delta(G_i^{(n)} \leqq \left(1 + \frac{1}{n}\right) \delta(E_i^{(n)})$$

and

$$O^{(n)} = \sum_i G_i^{(n)} \supset E, \qquad H = \prod_{n=1}^{\infty} O^{(n)} \supset E,$$

where H is a product of open sets. There is a partition of $O^{(n)}$ on hand,

$$O^{(n)} = \sum_i O_i^{(n)}, \qquad O_1^{(n)} = G_1^{(n)},$$
$$O_2^{(n)} = G_2^{(n)} - O_1^{(n)}, \qquad O_3^{(n)} = G_3^{(n)} - (O_1^{(n)} + O_2^{(n)}), \qquad \ldots,$$

where $O_i^{(n)} \subset G_i^{(n)}$. Since for a fixed n one has $H \subset O^{(n)}$, by (**3**.18), (**3**.17) we have a partition

$$H = \sum_i HO_i^{(n)}, \qquad \delta(HO_i^{(n)}) \leqq \delta(G_i^{(n)}) < \frac{1}{n}.$$

Thus as a consequence of the definition of $\Lambda^{\frac{1}{n}}(H)$ and by (**3**.18)

$$\Lambda^{\frac{1}{n}}(H) \leqq \sum_i \mu(\delta(HO_i^{(n)})) \leqq \sum_i \mu(\delta(G_i^{(n)})) \leqq \sum_i \mu\left(\left(1 + \frac{1}{n}\right)\delta(E_i^{(n)})\right).$$

Now (i) implies that

$$(\mathbf{3}.19) \qquad \mu\left(\left(1 + \frac{1}{n}\right)\delta(E_i^{(n)})\right) \leqq k(n)\,\mu(\delta(E_i^{(n)}))$$

in view of the inequality (**3**.17). Thus by (**3**.17′)

$$\Lambda^{\frac{1}{n}}(H) \leqq k(n) \sum_i \mu(\delta(E_i^{(n)})) \leqq k(n)\left(\Lambda(E) + \frac{1}{n}\right).$$

On letting $n \to \infty$, it is inferred that

$$\Lambda(H) \leqq \Lambda(E).$$

The reverse inequality holds by (**3**.14). Hence $\Lambda(H) = \Lambda(E)$.

Examples of functions $\mu(u)$ for which μ-length is regular are

$$u^{\alpha} \quad (\alpha > 0), \qquad \frac{1}{\log(u^{-1})}.$$

An example of a function $\mu(u)$ not satisfying (i) is $\exp.\left(-\frac{1}{u}\right)$.

4. Extension of Besicovitch's theorem. — Let (Q) denote a "binary" sequence of nets Q_k ($k=1, 2, \ldots$), défined as follows. With $x = x_1, \ldots, x_m$) denoting points in R_m, a net Q_k consists of closed "cubes" whose faces lie in the hyperplanes

$$x_i = p\,2^{-k} \qquad (i=1, \ldots m;\ p=0, \pm 1, \ldots).$$

Thus (Q) is a normal sequence of nets (a regular sequence of normal nets).

The Lemma in (S, 192) will be extended in the result.

Lemma (**4**.1). — *Suppose* $\mu(u)$, *used in the definition of* μ-*length* Λ *in accordance with Definition* (**3**.1), *satisfies the condition*

$$(\mathbf{4}.2)\qquad \lambda_m = \underset{\left[0<u\leqq\frac{1}{8}\right]}{\text{u. b.}} \frac{\mu(2mu)}{\mu(u)} < +\infty.$$

Given a set E, *an integer* k_0 *and* $\xi > 0$, *there exists cubes* $Q^1, Q^2, \ldots$ *be longing to the nets* Q_k ($k \geqq k_0$) *so that*

$$(\mathrm{i})\qquad \sum_k \mu(\delta(Q^n)) \leqq 2^m \lambda_m (\Lambda(E) + \xi);$$

(ii) *for every* $x \subset E$ *there exists* $k = k(x) \geqq k_0$ *so that all the cubes of* Q_k, *containing* x, *belong to* $\{Q^1, Q^2, \ldots\}$.

For some partition $E = E_1 + E_2 + \ldots$ $[\delta(E_i) < 2^{-k_0-1}]$ one has

$$(\mathbf{4}.3)\qquad \sum_i \mu(\delta(E_i)) \leqq \Lambda^{2^{-k_0-1}}(E) + \xi \leqq \Lambda(E) + \xi.$$

As in (S, 193), define an integer k_i so that

$$(\mathbf{4}.3')\qquad \frac{1}{2^{k_i}} < \delta(E_i) \geqq \frac{1}{2^{k_i+1}};$$

we have $k_i > k_0$ ($i = 1, 2, \ldots$); also we note that *each net* Q_{k_i} *has* 2^m *intervals at most containing points of* E_i. Let $Q^1, Q^2, \ldots$ be the totality of cubes obtained by picking out all the cubes of the net Q_{k_j} containing points of E_j ($j = 1, 2, \ldots$); the sequence $\{Q^1, Q^2, \ldots\}$ satisfies (ii).

By (**4**.3') $\delta(Q^n) \leqq m$ length of side of cube of net $Q_{k_i} = m\,2^{-k_i}$;

here i depends on n; thus, in view of the above italics,

$$(4.4)\qquad \sum_n \mu(\delta(Q^n)) \leq 2^m \sum_{i=1}^{\infty} \mu(m\,2^{-k_i}) = 2^m \sum_{i=1}^{\infty} \mu\left(2\,m\,\frac{1}{2^{k_i+1}}\right).$$

Now by (4.2)

$$\mu(2\,mu) \leq \lambda_m\,\mu(u) \qquad \left(0 < u \leq \frac{1}{8}\right);$$

hence, inasmuch as $2^{-k_i-1} \leq \frac{1}{8}$ $(i = 1, 2, \ldots)$,

$$\mu\left(2\,m\,\frac{1}{2^{k_i+1}}\right) \leq \lambda_m\,\mu\left(\frac{1}{2^{k_i+1}}\right).$$

Accordingly, as a consequence of (4.4) and (4.3′),

$$\sum_n \mu(\delta(Q^n)) \leq 2^m \lambda_m \sum_i \mu\left(\frac{1}{2^{k_i+1}}\right) \leq 2^m \lambda_m \sum_i \mu(\delta(E_i)).$$

The result (i) will ensue by virtue of (4.3). This completes the proof of the Lemma.

Definition (4.5). — *A function* F *of interval* I *will be said to satisfy* (μ^+) [*or* (μ^-)] *at a point* x *if*

$$(4.6)\qquad \underline{\lim}\,\frac{F(I)}{\mu(\delta(I))} \geq 0 \qquad \left[or\ \overline{\lim}\,\frac{F(I)}{\mu(\delta(I))} \leq 0\right]$$

for I *containing* x. F *satisfies* (μ) *if it satisfies both conditions* (4.6); *in this case*

$$(4.7)\qquad \lim \frac{F(I)}{\mu(\delta(I))} = 0 \qquad [as\ \delta(I) \to 0;\ I \supset x].$$

We shall now extend Besicovisth's theorem in (S 193).

Theorem (4.8). — *Let* F(I) *be continuous additive and satisfy* (μ) [*cf.* (4.7)] *at all points. Suppose* $\mu(u)$ *satisfies* (i) *of Theorem* (3.6) *and is such that*

$$(4.9)\qquad \lambda_m\,[\text{of }(4.2)] < +\infty, \qquad \lim_{u \to 0} \frac{u^m}{\mu(u)} = 0.$$

Let Λ *be the corresponding* μ*-length. Suppose that*

(i) $\qquad (Q)\,\underline{F}(x) > -\infty \qquad$ in CE, *where* E *is the sum of a denumerable infinity of sets of finite* μ*-length, and*

(ii) $(Q)\,\underline{F}(x) \geqq g(x)$ (*everywhere*; $g(x)$ *summable*).

Then

$$F(I_0) \geqq \int_{I_0} g(x)\,dx \tag{4.10}$$

for all intervals I_0.

It will suffice to prove (4.10) for a cube I_0 of a net Q_{k_0}.

Let V be a minor function of g, in accordance with (S, 190, 191); that is, V is an absolutely continuous function of an interval such that

$$\int_{I_0} g(x)\,dx \geqq V(I_0) > \int_{I_0} g(x)\,dx - \zeta; \tag{4.11}$$

$$\overline{V}(x) \leqq \overline{V}_0(x) \leqq g(x); \qquad \overline{V}_0(x) < +\infty. \tag{4.12}$$

As in (S) put

$$G(I) = F(I) - V(I) + \xi|I|. \tag{4.13}$$

Now $\overline{V}(x) = $ u. b. A, where A is any number such that

$$A = \overline{\lim}\,\frac{V(I_n)}{|I_n|}$$

for a regular sequence $\{I_n\} \to x$. Incidentally, *here and in the sequel when we designate* $I \to x$ *it is to be understood that* $I \supset x$ *and* $\delta(I) \to 0$. On the other hand, $(Q)\,\overline{V}(x)$ is $\overline{\lim}\,\frac{V(I)}{|I|}$ for cubes I of (Q) tending to x; that is, $(Q)\,\overline{V}$ is a number A. Thus

$$(Q)\,\overline{V}(x) \leqq \overline{V}(x), \tag{4.14}$$

By (ii) and (4.12) $(Q)\underline{F} \geqq g \geqq \overline{V}$ and

$$(Q)\,\underline{F} - \overline{V} \geqq 0. \tag{4.15}$$

We observe that

$$\frac{F(I)}{|I|} \geqq (Q)\,\underline{F}(x) - \varepsilon \qquad [\text{a. e. f. } I,\ \subset (Q), \to x],$$

for x in CE, and

$$\frac{V(I)}{|I|} \leqq (Q)\,\overline{V}(x) + \varepsilon \leqq \overline{V}(x) + \varepsilon \qquad [\text{a. e. f. } I,\ \subset (Q), \to x];$$

thus by (4.13)

$$\frac{G(I)}{|I|} \geq [(Q)\underline{F} - \varepsilon] - [\overline{V} + \varepsilon] + \xi \qquad [\text{a. e. f. } I, \subset (Q), \to x]$$

for x in CE; in view of (4.15) $\frac{G(I)}{|I|} \geq \xi - 2\varepsilon$ for intervals I as indicated above. Finally, on letting $\varepsilon \to 0$ we obtain

$$(4.16) \qquad (Q)\underline{G}(x) \geq \xi > 0 \qquad (\text{in CE}).$$

As a consequence of the last inequality (4.12)

$$(4.17) \qquad \frac{V(I)}{|I|} \leq B(x) < +\infty \qquad (\text{for all } I \supset x)$$

for all x; thus

$$(4.18) \qquad \frac{V(I)}{\mu(\delta(I))} = \frac{V(I)}{|I|}\,\frac{|I|}{\mu(\delta(I))} \leq B(x)\,\frac{|I|}{\mu(\delta(I))}.$$

Since

$$\frac{|I|}{\mu(\delta(I))} \leq \frac{\delta^m(I)}{\mu(\delta(I))},$$

in view of the second relation (4.9) we conclude that the last member in (4.18) tends to zero with $\delta(I)$. Hense $\overline{\lim}\,\frac{V(I)}{\mu(\delta(I))} \leq 0$ and V(I) satisfies (μ^-). Inasmuch as F satisfies (μ), we accordingly conclude that G (4.13) *satisfies* (μ^+).

By hypothesis

$$(4.19) \qquad E \subset \sum_{i=1}^{\infty} E_i, \qquad \Lambda(E_i) < +\infty \qquad (i = 1, 2, \ldots).$$

Let R_{in} be the set of points x such that for all I of the nets Q_k ($k \geq n$), containing x, one has

$$(4.20) \qquad \frac{G(I)}{\mu(\delta(I))} > -\xi q_i, \qquad q_i = \frac{1}{2^i[1 + \Lambda(E_i)]}.$$

Consequently CR_{in} is the set of points x such that for some I of Q_k ($k \geq n$), containing x, we have

$$(4.21) \qquad \frac{G(I)}{\mu(\delta(I))} \leq -\xi q_i.$$

Since G satisfies (μ^+)

$$\gamma = \underline{\lim} \frac{G(I)}{\mu(\delta(I))} \geq 0 \qquad (\text{for } I \rightarrow x).$$

Hence

$$\underline{\lim} \frac{G(I)}{\mu(\delta(I))} \geq \gamma \geq 0 \qquad [\text{for I}, \subset (Q), \rightarrow].$$

Thus

$$\frac{G(I)}{\mu(\delta(I))} > -\varepsilon \qquad [\text{a. e. f. I}, \subset (Q), \rightarrow x].$$

Whence

$$\frac{G(I)}{\mu(\delta(I))} > -\varepsilon \qquad [\text{for all I}, \subset Q_k(k \geq n(x, \varepsilon)), \rightarrow x].$$

Here we put $\varepsilon = \xi q_i$, $n(x, \varepsilon) = n_i(x)$ and infer that x is a point of $R_{i, n_i(x)}$. Accordingly

$$R_m = \sum_n R_{in}.$$

As in an analogous case in (S, 194) R_{in} is a product of open sets.

We have a decomposition of E_i without common points,

$$E_i = E_i \sum_n R_{i,n} = \sum_n E_{n,i}, \qquad E_{n,i} = E_i(R_{i,n} - R_{i,n-1}), \qquad R_{i,0} = 0.$$

Since the $R_{i,n}$ are Borel-measurable and the E_i are Λ-measurable, the $E_{n,i}$ are Λ-measurable. Clearly

$$(4.22) \qquad \Lambda(E_i) = \sum_n \Lambda(E_{in}).$$

Applying Lemma 4.1 (with $\xi = 2^{-n}$) to $E_{in}(\subset R_{in})$, corresponding to $i > 0$, $n > 0$ we find a sequence of cubes $\{Q_{in}^{(j)}\}(j = 1, 2, \ldots)$ of the nets $Q_k(k \geq n)$ so that

$$(\alpha) \qquad \sum_j \mu(\delta(Q_{in}^{(j)})) \leq 2^m \lambda_m [\Lambda(E_{in}) + 2^{-n}];$$

(β) for each x of E_{in} there exists $k = k(x) \geq n$ so that each $I. \subset Q_k$, containing x belongs to the sequence $\{Q_{in}^{(j)}\}$;
(γ) each $Q_{in}^{(j)}$ contains points of E_{in} (and hence of R_{in}) and conse-

quently [by (4.20)] satisfies

$$(4.23) \qquad \frac{G(Q_{in}^{(j)})}{\mu(\delta(Q_{in}^{(j)}))} > -\xi q_i.$$

As in (S) let I_0 be said to have *property* (A) if I_0 is a finite sum of non overlapping intervals I, where I is either a cube $Q_{in}^{(j)}$ or $G(I) > 0$. If R is a figure consisting of a finite sum of non overlapping $Q_{in}^{(j)}$, then by (4.23)

$$G(R) \geqq -\xi \sum_{inj} q_i \mu(\delta(Q_{in}^{(j)})).$$

On using (4.20), summing first with respect to i, taking account of (α), then summing with respect to n, in consequence of (4.22) we obtain

$$G(R) \geqq -2^m \lambda_m \xi.$$

Hence

$$(4.24) \qquad G(I) \geqq -2^m \lambda_m \xi \qquad [\textit{for all I with property (A)}];$$

in particular this will hold for $I = Q_{in}^{(j)}$.

Suppose I_0 (a cube of the net Q_{k_0}) *does not have the property* (A). Then $G(I_0) < -2^m \lambda_m \xi$. Accordingly for a sequence $\{I_p\}$ of cubes belonging to various nets, we have

$$I_0 \supset I_1 \supset I_2 \supset \ldots, \qquad G(I_p) < -2^m \lambda_m \xi.$$

Let x_0 be the common point of the I_p. If $x_0 \subset E$, then by (4.19) and the decomposition preceding (4.22) x_0 belongs to some E_{ni}: thus by (β) amongst the I_p there are some $Q_{in}^{(j)}$; in view of (4.24), where we replace I by one of these $Q_{in}^{(j)}$, a contradiction arises. If $x_0 \subset CE$, then by (4.16) $\underline{\lim} \frac{G(I)}{|I|} \doteq \sigma > 0$ (for $I. \subset (Q), \to x_0$); furthermore, $\underline{\lim} \frac{G(I_p)}{|I_p|} \geqq \sigma$. Hence $\frac{G(I_p)}{|I_p|} \geqq \frac{\sigma}{2}$ (a. e. f. p.) and $G(I_p) > 0$ (all $p \geqq p_0$); this again presents a contradiction. Hence I_0 has the property (A); by virtue of (4.24) (with $I = I_0$) and (4.13), (4.11) an inequality is obtained for $F(I_0)$ which, on letting $\xi \to 0$, yields (4.10). *The Theorem is accordingly proved.*

A corollary to Theorem 4.8 is as follows.

Theorem 4.25. — *Let a continuous additive* F(I) *satisfy condition* (μ) (**4**.7), *while* $\mu(u)$ *satisfies* (i) *of Theorem* **3**.6 *and* (**4**.9). *Let*

$$(4.26)\qquad -\infty < (Q)\underline{F} \leqq (Q)\overline{F} < +\infty \qquad [in\ (E)],$$

where $E = E_1 + E_2 + \ldots$, *with* $\Lambda(E_j) < +\infty$ ($j = 1, 2, \ldots$). *Then*

$$(4.27)\qquad F(I) = \int_I (Q)F'(x)\,dx$$

for all intervals I *in any portion of* R_m *in which* $(Q)F'(x)$ *exists* (*almost everywhere*) *and is summable.*

Under conditions of this Theorem F(I) will be absolutely continuous in the indicated portion of R_m; accordingly in (**4**.27) (Q)F′ may be replaced by F′.

5. Conditions for analyticity. — *In this section we let*

$$(5.1)\qquad f(z) = u + iv \qquad (z = x + iy)$$

be a function continuous in the complex variable z *for* z *in a domain* G.

Correspondingly there is on hand a function of an interval (rectangle)

$$(5.2)\qquad J(I) = \int_{(I)} f(z)\,dz = J_1(I) + iJ_2(I),$$

continuous and additive, as a function of I. As is well known, $f(z)$ is analytic in G if F(I) = 0 for all I in G.

We extend a theorem of J. Wolff (S, 196) as follows.

Theorem 5.3. — *$f(z)$ is analytic in* G, *if*

$$(\text{i})\qquad \underline{\lim}\,\frac{1}{|I|}\left|\int_{(I)} f(z)\,dz\right| = 0 \qquad (I \supset z_0)$$

for almost all z_0 *in* G *and if*

$$(\text{ii})\qquad \overline{\lim}\,\frac{1}{|I|}\left|\int_{(I)} f\,dz\right| < +\infty \qquad (I \supset z_0),$$

except at most at points z_0 *of a set* E *consisting of a denumerable infinity of rectilinear segments parallel to the axes.*

In consequence of (ii)

$$(5.4)\qquad -\infty < \underline{J}_{1,s}(z) = \underline{\lim}\frac{J_1(I)}{|I|} \leq \overline{\lim}\frac{J_1(I)}{|I|} = \overline{J}_{1,s}(x) < +\infty$$

(for $I \supset z$) in $G - E$. It is known [*cf.* (S, 141)] that, if F is an additive function of an interval, on has

$$(5.5)\qquad F' = (\underline{F}_s = \overline{F}_s =) F'_s$$

almost everywhere in any set in which the derivates $\underline{F}_s$, $\overline{F}_s$ are finite. Since $m(E) = 0$, (5.4) implies that the strong derivates of J_1 are finite almost everywhere in G. Hence by (5.5) $J'_{1,s} = J'_1$ almost everywhere in G and

$$(5.6)\qquad \lim\frac{J_1(I)}{|I|} = J'_1 \qquad (I \rightarrow z)$$

for almost all z in G. In view of (i) $\frac{|J(I_n)|}{|I_n|}$ tends to zero, for some sequence $I_1, I_2, \ldots$, tending to z, almost everywhere in G; thus

$$(5.7)\qquad \lim\frac{J_1(I_n)}{|I_n|} = 0 = J'_{1,}$$

for I_n and z as stated above. Hence

$$(5.8)\qquad J'_1 = 0 \qquad \text{(almost everywhere in G).}$$

Let (N) denote a normal sequence of nets as in section **2**. The numbers $(N)\underline{J}_1(z)$, $(N)\overline{J}_1(z)$ are derivates formed with the aid of special sequences of intervals, tending to z, while the strong derivates are formed with the aid of arbitrary sequences of intervals tending to z; hence

$$\underline{J}_{1,s} \leq (N)\underline{J}_1 \leq (N)\overline{J}_1 \leq \overline{J}_{1,s}.$$

Thus by virtue of (5.4)

$$(5,8')\qquad -\infty < (N)\underline{J}_1 \leq (N)\overline{J}_1 < +\infty$$

in $G - E$. *Consequently* (i) *of Theorem* **2**.15 *holds for* J_1 (*with* $D_0 = E$). In view of (5.8) the ordinary derivative J'_1 is summable in G. Thus Theorem 2.15 applies and one obtains

$$J_1(I) = \iint_I J'_1(z)\, dx\, dy = 0.$$

Similarly it is shown that $J_2(I) = 0$. Whence $J(I) = 0$ for all I in G; $f(z)$ is analytic. Our theorem is proved.

Let (N) be a binary (normal) sequence of nets [Def. in (S, 191)], consisting of squares. We shall prove the following modification of Theorem 5.3

Theorem 5.3'. — *$f(z)$ is analytic in G, if (i), (ii) of Theorem 5.3 hold, as stated, where I denotes squares of the above (N).*

In fact, bi (ii) of Theorem 5.3'

$$-\infty < (N)\underline{J}_\nu \leq (N)\overline{J}_\nu < +\infty \qquad (\text{in } G - E;\ \nu = 1, 2).$$

Now in (S; 192) it has been noted that, with (N) denoting a binary sequence, (N)F' exists almost everywhere where

$$(N)\underline{F} > -\infty, \qquad \text{or} \quad (N)\overline{F} < +\infty.$$

Thus the above implies that

$$(N)J'_\nu \quad (\nu = 1, 2) \text{ exist almost everywhere in } G.$$

We note that (i) of Theorem 5.3' implies that for almost all z_0 in G there exists a sequence $\{I_n\}$ of squares of (N), such that $I_n \supset z_0$, $\delta(I_n) \to 0$, so that

$$\lim \frac{1}{|I_n|} J_\nu(I_n) = 0 \qquad (\nu = 1, 2).$$

Clearly $(N)J'_\nu(z_0) = 0$ almost everywhere in G. By Theorem 2.15, applied with the binary sequence (N), one has

$$J_\nu(I) = \iint_I (N)J'_\nu(z)\, dx\, dy = 0, \qquad J(I) = 0$$

for all intervals I in G. The conclusion of the Theorem ensues.

In the above Theorem one may replace (ii) by the requirement that

$$\overline{\lim} \frac{1}{|I|} \left| \int_{(I)} f\, dz \right| < +\infty$$

for z_0 in $G - E$, where I denotes *any* square containing $z_0 (\to z_0)$.

Before we apply section 4 to functions of a complex variable we shall investigate the connection between property (μ) of J_1, J_2 and the "modulus of continuity" of $f(z)$.

If $f(z)$ is not uniformly continuous in G, we replace G by any domain (open), whose closure lies in G; in such a domain $f(z)$ will be uniformly continuous and, for z, z' in it, one will have

$$(5.9) \qquad |f(z') - f(z)| \leqq \eta(|z' - z|),$$

where $\eta(u)$ ($\eta > 0$ for $u > 0$; $\eta \to 0$ monotonically with u) is "modulus of continuity" of f. *We shall consider the case when $f(z)$ is uniformly continuous in G; thus, in G we shall have (5.9).*

Consider an interval I, whose left lower vertex is $\mathcal{J} = \mathcal{J}_1 + i\mathcal{J}_2$ and whose horizontal and vertical sides are of length a and b, respectively. By (5.2) one has

$$|J(I)| \leqq \left| \int_{x=\mathcal{J}_1}^{\mathcal{J}_1+a} [f(x + i\mathcal{J}_2) - f(x + i(\mathcal{J}_2 + b))]\, dx \right| + \left| \int_{y_2=\mathcal{J}_2}^{\mathcal{J}_2+b} [f(\mathcal{J}_1 + iy) - f(\mathcal{J}_1 + a + iy)]\, dy \right|.$$

Hence in view of (5.9)

$$|J(I)| \leqq a\eta(b) + b\eta(a) \leqq 2\delta(I)\eta(\delta(I))$$

and

$$\frac{|J(I)|}{\mu(\delta(I))} \leqq \frac{2\delta(I)\eta(\delta(I))}{\mu(\delta(I))} = 2\omega(\delta(I)).$$

On taking account of Definition 4.5 we thus infer the following.

Lemma 5.10. — $J_1(I)$, $J_2(I)$ [*cf* (5.2) *satisfy the condition* (μ), *if the modulus of continuity* $\eta(u)$ *of* f *is of the form*

$$(5.11) \qquad \eta(u) = \omega(u) \frac{\mu(u)}{u},$$

where $\omega(u)$ *tends to zero with* u *sufficiently fast, so that* $\eta(u) \to 0$ *with* u.

The following is our extension of Besicovitch's theorem (S, 197).

Theorem 5.12. — *Let* μ, *used in the definition of* μ*-length* Λ, *satisfy* (i) *of Theorem* 5.6, (4.2), (4.9) (*with* $m=2$). *Suppose that the modulus of continuity* η [*cf.* (5.9)] *of* $f(z)$ *is of the form* (5.11). *Then* $f(z)$ *is analytic in* G, *if* $f^{(1)}(z)$ *exists almost everywhere in* G *and if*

$$(5.13) \qquad \overline{\lim_{h\to 0}}\left|\frac{f(z+h)-f(z)}{h}\right| < +\infty$$

in G — E, *where*

$$(5.14) \qquad E=\sum_{j=1}^{\infty} E_j, \qquad \Lambda(E_j)<+\infty \qquad (j=1, 2, \ldots).$$

If z_0 is a point at which $f^{(1)}(z_0)$ exists, then as indicated in (S, 196), one has

$$(5.15) \qquad \frac{J_1(I)}{|I|},\ \frac{J_2(I)}{|I|}\to 0 \qquad (\textit{any} \text{ square } I\to z_0).$$

Let now z_0 be a point in G — E; then by (5.13) [also *cf.* (S, 190)]

$$\frac{|J_1(I)|}{|I|},\quad \frac{|J_2(I)|}{|I|}\leqq q(z_0)$$

for *all* sufficiently small squares I containing z_0. Accordingly

$$(5.16) \qquad -\infty<(Q)\underline{J}_1(z_0)\leqq(Q)\overline{J}_1(z_0)<+\infty$$

in G — E. Now (for z_0 in G — E)

$$(Q)\underline{J}_1(z_0)=\underline{\lim}\frac{J_1(I)}{|I|} \qquad (\text{for squares I of } (Q), \to z_0)$$

and

$$\frac{J_1(I_n)}{|I_n|}\to(Q)\underline{J}_1(z_0)$$

for some sequence of squares $\{I_n\}$, of (Q), $\to z_0$; hence by (5.15) $(Q)\underline{J}_1(z_0)=0$ at every point of G — E at which $f^{(1)}$ exists; since $f^{(1)}$ exists almost everywhere in G and $m(E)=0$, we have $(Q)\underline{J}_1(z_0)=0$ almost everywhere in G. A similar result will hold for $(Q)\overline{J}_1(z_0)$. Hence

$$(Q)J'_1=0 \qquad (\text{almost everywhere in G}).$$

In view of the assumption regarding the modulus of continuity of $f(z)$ it follows by Lemma 5.10 that $J_1(I)$ satisfies the condition (μ).

By (5.16), (5.17) from Theorem 4.25 we accordingly infer that

$$J_1(I)=\iint_I (Q) J'_1(z)\,dx\,dy=0$$

for all intervals I in G. Similarly it is shown that $J_2(I)=0$. Thus J(I) vanishes on all I in G. The theorem is proved.

We shall give some examples of functions μ, η that may be used in the Theorem. Let μ be termed *admissible* if μ satisfies the conditions of the Theorem. Furthermore, in the remainder of this section limits will be understood to be attained monotonically.

(5.18). *The function* $\mu(u)=u\lambda(u)$ *is admissible, if* $\lambda(u)(\geqq\lambda_0>0)$ *is monotone non decreasing, as* $u\to 0$, *and is such that* $\mu(u)\to 0$: *in this case no condition is required on the modulus of continuity* $\eta(u)$: *thus, when* $\lambda(u)=1$, *Theorem* 5.12 *reduces to Besicovitch's theorem* (S, 197).

In fact

$$(\alpha_1)\qquad 1\leqq\frac{\mu(\nu u)}{\mu(u)}=\nu\frac{\lambda(\nu u)}{\lambda(u)}\leqq\nu\qquad(\text{for } \nu\geqq 1,\ u>0).$$

Hence $k(n)$ of (i) of the Theorem 3.6 satisfies

$$1\leqq k(n)\leqq 1+\frac{1}{n},\qquad \lim_{n\to\infty} k(n)=1;$$

(α_1), with $\nu=4$, will imply (4.2). Moreover,

$$\frac{u^2}{\mu(u)}=\frac{u}{\lambda(u)}\leqq\frac{u}{\lambda_0};$$

whence (4.9) holds. Accordingly μ is admissible. By (5.11)

$$\omega(u)=\frac{u}{\mu(u)}\eta(u)=\frac{\eta(u)}{\lambda(u)}\leqq\frac{\eta(u)}{\lambda_0};$$

$\eta(u)\to 0$ in consequence of the continuity of $f(z)$; hence $\omega(u)\to 0$ with u; thus no condition on η is needed.

(5.19). *The function* $\mu(u)=\dfrac{u^2}{o(u)}$ *is admissible provided*

$$(5.19\,a)\qquad o(u)\to 0,\qquad \frac{u}{o(u)}\to 0\qquad(\text{with } u);$$

correspondingly in the Theorem one may use any modulus of the form

$$\eta(u) = \omega(u)\frac{u}{o(u)}, \tag{5.19 b}$$

where $\omega(u) < o) \to$ *with* u.

To establish the above we first not that

$$1 \leq \frac{\mu(\nu u)}{\mu(u)} = \nu^2 \frac{o(u)}{o(\nu u)} \leq \nu^2 \qquad (\text{for } \nu \geq 1,\ u > 0). \tag{α_2}$$

Hence, on letting $\nu = 1 + n^{-1}$, one obtains

$$1 \leq k(n) \leq \left(1 + \frac{1}{n}\right)^2, \qquad \lim_{n \to \infty} k(n) = 1;$$

(α_2) also implies (4.2). Moreover,

$$\frac{u^2}{\mu(u)} = o(u) \to 0 \qquad (\text{with } u)$$

by hypothesis, which implies (4.9). Thus μ is admissible. Finally, (5.19 b) is inferred from (5.11).

The set E, involved in the Theorem, contains by hypothesis all the points at which $\overline{\lim}\left|\frac{(f(z+h) - f(z))}{h}\right|$ is infinite; in this sense E may be termed " singular ". In view of the condition (5.14), satisfied by E, one may assert that in a certain sense the following holds. The faster $\mu(u) \to 0$ (subject to the conditions of Theorem) the " greater " one may allow the singular set to be. Accordingly, the case described in (5.18) presents nothing essentially different from the situation involved in Besicovitch's theorem (S, 197). By the same token the case presented in (5.19) is essentially distinct from that in the theorem in (S, 197).

6. Representations of functions of a complex variable. — We write

$$\mathfrak{J} = \mathfrak{J}_1 + i\mathfrak{J}_2, \qquad J(I) = \int_{(I)} f(z)\,dz = J_1 + iJ_2 \tag{6.1}$$

and recall the definition of upper and lower derivates and derivatives

$$(N)\underline{J}_\nu,\ (N)\overline{J}_\nu,\ (N)J',\qquad (Q)\underline{J}_\nu,\ (Q)\overline{J}_\nu,\ (Q)J'_\nu,$$
$$\underline{J}_\nu,\ \overline{J}_\nu,\ J'_\nu \qquad (\nu = 1, 2),$$

with respect to a normal sequence of nets (N), with respect to a binary sequence of nets (Q) and ordinary, respectively.

We shall make use of the following theorem of N. Théodoresco ([1]).

If $\varphi(\mathfrak{J})$ is measurable and $|\varphi(\mathfrak{J})|$ is bounded almost everywhere in a domain G then, on letting

$$(6,2) \qquad g(z) = -\frac{1}{2\pi i}\iint_G \frac{\varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2}{\mathfrak{J} - z},$$

one obtains

$$(6.3) \qquad \int_{(I)} g(z)\, dz = \iint_I \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2$$

for intervals I in G (we stated the theorem in a slightly different form). *Here and in the sequel an expression like " $|F(\mathfrak{J})|$ is bounded almost everywhere in G " is to signify that*

$$|F(\mathfrak{J})| \leq B \qquad (\textit{in } G - G_0),$$

where $mG_0 = 0$.

We shall now prove.

THEOREM 6.4. — *Suppose* (i) $f(z)$ *is continuous in* G *and*

$$(6.5) \qquad \overline{\lim}\, \frac{1}{|I|}\left|\int_{(I)} f(z)\, dz\right| < +\infty \qquad (\textit{intervals } I \supset z)$$

for all z *in* G, *except perhaps on a set* D_0 *consisting of a denumerable infinity of rectilinear segments parallel to the axes: furthermore, assume* (ii) *that* $|J'(\mathfrak{J})|$ *is bounded almost everywhere in* G. *Then* $f(z)$ *will have a representation*

$$(6.6) \qquad f(z) = -\frac{1}{2\pi i}\iint_G \frac{J'(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2}{\mathfrak{J} - z} + a(z)$$

[$a(z)$ *analytic in* G].

Form the function

$$(6.7) \qquad q(z) = f(z) + \frac{1}{2\pi i}\iint_G \frac{J'(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2}{\mathfrak{J} - z}.$$

$J'(\mathfrak{J})$ is measurable in any case; thus in view of (ii) and by

([1]) N. THÉODORESCO, *La dérivée aréolaire et ses applications à la physique mathématique* (*Thèse*, Paris, 1931).

Théodoresco's theorem

$$(6.8)\qquad \int_{(I)} q(z)\,dz = \int_{(I)} f(z)\,dz - \int_{(I)} \left[-\frac{1}{2\pi i} \iint_G \frac{J'(\mathfrak{z})\,d\mathfrak{z}_1\,d\mathfrak{z}_2}{\mathfrak{z} - z} \right] dz$$
$$= J(I) - \iint_I J'(\mathfrak{z})\,d\mathfrak{z}_1\,d\mathfrak{z}_2 \equiv F(I).$$

If it is shown that $F(I) = 0$ for all intervals I in G, then $q(z)$ is analytic and the theorem is proved. Now (6.5) implies that

$$(6.9)\qquad -\infty < (N)\underline{J}_\nu \leqq (N)\overline{J}_\nu < +\infty$$

in $G - D_0$; this fact has been established subsequent to Theorem 5.3 [text from (5.4) to (5.8′)]. Inequalities (6 9), together with the condition (ii) which implies summability of J′ on every I in G, enable application of Theorem 2.15; accordingly

$$J_\nu(I) = \iint_I J'_\nu(\mathfrak{z})\,d\mathfrak{z}_1\,d\mathfrak{z}_2 \quad (\nu = 1, 2), \qquad J(I) = \iint_I J'(\mathfrak{z})\,d\mathfrak{z}_1\,d\mathfrak{z}_2$$

and $F(I) = 0$.

Another representation is as follows.

THEOREM 6.10. — *Let $\mu(u)$ be and "admissible" function in the sense of the preceding sections. Suppose $f(z)$ satisfies*

$$(i)\qquad \overline{\lim_{h \to 0}} \left| \frac{f(z+h) - f(z)}{h} \right| < +\infty$$

for z in G except perhaps on a set $E = E_1 + E_2 + \ldots$, where the E_j are each of finite μ-length; assume the modulus of continuity $\eta(u)$ of $f(z)$ of the form

$$(ii)\qquad \eta(u) = \omega(u)\frac{\mu(u)}{u} \qquad [\eta(u),\ \omega(u) \to 0 \textit{ with } u).$$

Then, provided (iii) $|(Q)J'(\mathfrak{z})|$ *is bounded almost everywhere in* G, *one has a representation*

$$(6.11)\qquad f(z) = -\frac{1}{2\pi i} \iint \frac{(Q)J'(\mathfrak{z})\,d\mathfrak{z}_1\,d\mathfrak{z}_2}{\mathfrak{z} - z} + a(z)$$

[*$a(z)$ analytic in* G].

We form again $q(z)$ as in (6.7), where J′ is replaced by (Q)J′.

By (**6**.2), (**6**.3) and (iii) we obtain

$$(\mathbf{6}.12)\qquad \int_{(I)} q(z)\,dz = F(I) \equiv J(I) - \iint_I (Q)\,J'(\mathfrak{J})\,d\mathfrak{J}_1\,d\mathfrak{J}_2.$$

It will suffice to prove that $F(I) = 0$ for all intervals I in G.

Subsequent to Theorem **5**.12 it has been established that (**5**.13) implies (**5**.16). Thus in view of (i),

$$(\alpha)\qquad -\infty < (Q)\underline{J}_\nu \leq (Q)\overline{J}_\nu < +\infty \qquad (\text{in } G - E).$$

As a consequence of (ii) and Lemma **5**.10

(β) $J_\nu(I)$ satisfies condition (μ) (Def. **4**.5).

By virtue of (iii)

(γ) $(Q)J'_\nu$ is summable on every I in G.

By (α), (β), (γ) from Theorem **4**.25 we infer

$$J_\nu(I) = \iint_I (Q)J'_\nu(\mathfrak{J})\,d\mathfrak{J}_1\,d\mathfrak{J}_2 \quad (\nu = 1, 2), \qquad J(I) = \iint_I QJ'(\mathfrak{J})\,d\mathfrak{J}_1\,d\mathfrak{J}_2.$$

Hence F(I) (**6**.12) vanishes on all I in G.

In the representations (**6**.11) $(Q)J'(\mathfrak{J})$ may be replaced by $J'(\mathfrak{J})$.

Theorems **6**.4, **6**.10 may be extended somewhat if in place of Théodoresco's Theorem (**6**.3), we make use of Moisil's [1] extension of the latter result.

The representation in Theorems **6**.4, **6**.10 *are unique* in the sence that if $f(z)$ is representable in the form

$$(\mathbf{6}.13)\qquad f(z) = -\frac{1}{2\pi i}\iint_G \frac{\varphi(\mathfrak{J})\,d\mathfrak{J}_1\,d\mathfrak{J}_2}{\mathfrak{J} - z} + b(z)$$

[$b(z)$ analytic], where $|\varphi(\mathfrak{J})|$ is bounded almost everywhere in G, one necessarily has $\varphi(\mathfrak{J}) = J'(\mathfrak{J})$ (in case of Theorem **6**.4) and $\varphi(\mathfrak{J}) = (Q)J'(\mathfrak{J})$ (in case of theorem **6**.10) almost everywhere.

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(1) G. C. Moisil, *Sur un système d'équations fonctionnelles* (*C. R. Acad. Sc.*, t. **192**, 1931, pp. 1344-1346).

It will suffice to establish this for the case of Theorem **6**.4. Subtract (**6**.13) from (**6**.6); one obtains

$$-\frac{1}{2\pi i}\iint_G \frac{\psi(\mathfrak{z})\,d\mathfrak{z}_1\,d\mathfrak{z}_2}{\mathfrak{z}-z}=c(z) \qquad [c(z) \text{ analytic}]$$

and

$$\int_{(I)}-\frac{1}{2\pi i}\iint_G \frac{\psi(\mathfrak{z})\,d\mathfrak{z}_1\,d\mathfrak{z}_2}{\mathfrak{z}-z}\,dz=0 \qquad (\psi=\mathrm{J}'-\varphi).$$

Since $|\mathrm{J}'-\varphi|$ is bounded almost everywhere, from (**6**.2), (**6**.3) it is inferred that

$$\iint_I \psi(\mathfrak{z})\,d\mathfrak{z}_1\,d\mathfrak{z}_2=0$$

for all I in G. Thus $\psi(\mathfrak{z})=0$ and $\varphi=\mathrm{J}'$ almost everywhere.

Thus under conditions of Theorems **6**.4 or **6**.10 (whichever is the case) *the equation*

$$(6.14)\qquad \frac{1}{2\pi i}\iint_G \frac{\varphi(\mathfrak{z})\,d\mathfrak{z}_1\,d\mathfrak{z}_2}{\mathfrak{z}-z}+f(z)=a(z)$$

[$a(z)$ analytic, not assigned beforehand: $\varphi(\mathfrak{z})$ the unknown] *has a unique inversion, bounded almost everywhere,*

$$(6.14a)\qquad \varphi(\mathfrak{z})=\mathrm{J}'(\mathfrak{z}) \qquad [\text{or } \varphi(\mathfrak{z})=(\mathrm{Q})\mathrm{J}'(\mathfrak{z}')].$$

6'. Representation of functions of a complex variable (continued). —In this section we shall obtain a representation of $f(z)=u(x,y)+iv(x,y)$ without recourse to Théodoresco's theorem. In this the result to be established will be similar to a number of representations the author had obtained in a previous work [1], *in the sequel referred to as* (T).

We assume $f(z)$ continuous in G, *such that*

$$(6'.1)\qquad \overline{\lim}\,\frac{1}{|\mathrm{I}|}\left|\int_{(\mathrm{I})} f(z)\,dz\right|<+\infty \qquad (\textit{intervals } \mathrm{I}\supset z_0)$$

for all z_0 in G, *except on a denumerable infinity (at most) of segments parallel to the axes; moreover, the partial derivatives*

$$(6'.2)\qquad u_x,\quad u_y,\quad v_x,\quad v_y$$

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[1] W. J. Trjitzinsky, *Problems of representation and uniqueness for functions of a complex variable* (*Acta Mathematica*, t. 78, pp. 97-192.)

are assumed to exist almost everywhere in G, *say in* $G - G_0$, *with* $mG_0 = 0$.

Differentiability of u, v, as postulated above, implies the following. Let F_n be the set of points such that

$$(6'.3) \quad \begin{cases} \left|\frac{1}{h}(u(x+h, y) - u(x, y))\right|, & \left|\frac{1}{h}(u(x, y+h) - u(x, y))\right|, \\ \left|\frac{1}{h}(v(x+h, y) - v(x, y))\right|, & \left|\frac{1}{h}(v(x, y+h) - v(x, y))\right| \leq n, \end{cases}$$

whenever $|h| \leq \frac{1}{n}$ (the only points, considered, are of course in G). Clearly $F_1, F_2, \ldots$ are closed sets and

$$F_1 \subset F_2 \subset \ldots; \qquad G \supset \lim_n F_n \supset G - G_0.$$

We write

$$(6'.4) \qquad \rho(n) = m(G - F_n)$$

and note that

$$(6'.5) \qquad \lim_n \rho(n) = 0.$$

We establish that if $\rho(n)$ tends to zero sufficiently fast with $\frac{1}{n}$ then $f(z)$ has a " Cauchy double integral " representation.

THEOREM 6'.6. — *Suppose* $f(z)$ *is a function as described in connection with* (6'.1), (6'.2). *If a sequence of integers* $0 < j_1 < j_2 < \ldots$ *may be found so that the two series*

$$(6'.7) \qquad \Gamma_1 = \sum_\nu j_\nu^2 \frac{\log \nu}{\nu^2}, \qquad \Gamma_2 = \sum_\nu \log \nu \sum_{s > j_\nu} s\rho(s)$$

converge, then

$$(6'.8) \qquad f(z) = -\frac{1}{2\pi i} \iint_G \frac{\mathfrak{f}'(\mathfrak{z})\, d\mathfrak{z}_1\, d\mathfrak{z}_2}{\mathfrak{z} - z} + a(z),$$

with $a(z)$ *analytic in* G.

NOTE. — (6'.7) *can be satisfied if*

$$(6'.9) \qquad \rho(s) \leq cs^{-\alpha} \quad (\alpha > 4).$$

We shall use the following result due to D. Menchoff (1). Let

(1) D. MENCHOFF, *Les conditions de monogénéité*, Paris, 1936, See p. 10.

$w(x, y)$ be real valued and such that

$$(6.10)\quad \begin{cases} |w(x_2, y_1) - w(x_1, y_1)| \leqq n\,|x_2 - x_1|, \\ |w(x_1, y_2) - w(x_1, y_1)| \leqq n\,|y_2 - y_1| \end{cases}$$

whenever (x_1, y_2). (x_2, y_1) are points in a square Q, while (x_1, y_1) is in FQ, where F is closed. If

$$I' = (a_1 \leqq x \leqq b_1;\ a_2 \leqq y \leqq b_2)$$

is the least interval containing FQ, one has

$$(6'.11)\quad \begin{cases} \left| \int_{a_1}^{b_1} [w(x, b_2) - w(x, a_2)]\,dx - \iint_{QF} w_y(x, y)\,dx\,dy \right| \leqq 5n\,|Q - FQ|, \\ \left| \int_{a_1}^{b_1} [w(b_1, y) - w(a_1, y)]\,dy - \iint_{QF} w_x(x, y)\,dx\,dy \right| \leqq 5n\,|Q - FQ|. \end{cases}$$

As remarked by Menchoff, $(6'.10)$ implies that w_x, w_y exist almost everywhere in QF.

As before, we write

$$J(I) = \int_{(I)} f(z)\,dz = J_1(I) + iJ_2(I).$$

Let Q be a square, the length of whose side l does not exceed $\frac{1}{n}$, such that $F_nQ \neq 0$. It is then observed that if (x, y) is in F_nQ and (x_2, y), (x, y_2) are any points of Q we have

$$|x_2 - x|, \qquad |y_2 - y| \leqq \frac{1}{n}.$$

Thus, by $(6'.3)$ the inequalities $(6'.10)$ will hold for u, v, $F = F_n$. Let I', as designated previously be the least interval containing F_nQ. By $(6'11)$ we obtain

$$J_1(I') = -\iint_{QF_n} (u_x + v_y)\,dx\,dy) - \xi_{u,2} - \xi_{v,1}; \qquad |\xi_{u,2}|, \quad |\xi_{v,1}| \leqq 5n\,|Q - QF_n|.$$

Now, in view of $(6'.3)$

$$('6'.12)\qquad |u_x|, \quad |u_y|, \quad |v_x|, \quad |v_y| \leqq n \qquad (\text{in } F_n).$$

Thus

$$|J_1(I')| \leqq 2n\,|QF_n| + 10n\,|Q - QF_n| \leqq 10n\,|Q|.$$

Similarly $|J_2(I')|$ does not exceed $10n|Q|$. Hence

(6'.13) $$|J(I')| \leqq 20n|Q|.$$

for any square Q (*in* G) *the length of whose side is* $\leqq \frac{1}{n}$ *while* $QF_n \neq 0$ *and* I' *is the least interval* $\supset QF_n$.

As a consequence of (6'.1) the lower and upper strong derivates of $J_1(I)$, $J_2(I)$ are finite almost everywhere in G. Hence the strong and ordinary derivatives of J(I) exist and are equal,

(6'.14) $$J'_s(\mathfrak{J}) = J'(\mathfrak{J}),$$

for $\mathfrak{J}$ in $G - H_0 = 0$, where $mH_0 = 0$. Let H_n be the set of points of F_n which are not points of density of F_n; necessarily $mH_n = 0$; moreover,

(6'.15) $$\frac{|IF_n|}{|I|} \to 1 \qquad (\mathfrak{J} \text{ in } F_n - H_n)$$

as interval I, containing $\mathfrak{J}$, tends to $\mathfrak{J}$. We observe that

(6'.16) $$\lim_n F_n = \sum_{n=1}^{\infty} (F_n - F_{n-1}), \qquad F_0 = 0,$$

Since $m(G - \lim F_n) = 0$ and

$$mH_0 = 0, \qquad m\sum_1^{\infty} H_n = 0,$$

by virtue of (6'.14), (6'.15) it is inferred that

(6'.17) $$G = E + G^0, \qquad mG^0 = 0,$$

where E consists of points $\mathfrak{J}$, such that

1° $$\mathfrak{J} \subset \lim F_n \qquad [cf.\ (6'.16)],$$

2° $$J'_s(\mathfrak{J}) = J'(\mathfrak{J}),$$

3° $$\frac{|IF_n|}{|I|} \to 1 \qquad (\text{as interval } I \to \mathfrak{J}),$$

where n is any integer for which $\mathfrak{J}$ belongs to F_n [as a consequence of (1°)]. Corresponding to (6'.16) we have

(6'.18) $$E = \sum_{n=1}^{\infty} (F_n - F_{n-1})E,$$

Lemma 6'. 19. — *For $\mathfrak{J}$ in* $(F_n - F_{n-1})E$ *one has*

$$|\mathfrak{J}'_s(\mathfrak{J})| = |J'(\mathfrak{J})| \leqq 20n. \tag{6'.20}$$

In fact, by (2°) the limit

$$J'_s(\mathfrak{J}) = J'(\mathfrak{J}) = \lim_{\nu} \frac{J(I_\nu)}{|I_\nu|} \tag{α}$$

exists (and is unique) for *any* sequence of intervals I_ν, containing $\mathfrak{J}$ and tending to $\mathfrak{J}$. Let Q_ν be a square with $\mathfrak{J}$ for center and having the length of its sides equal to $\frac{1}{\nu}$. With $\mathfrak{J}$ in $(F_n - F_{n-1})E$, $\mathfrak{J}$ is a point of F_n. Hence $Q_\nu F_n \neq 0$ $(\nu = 1, 2, \ldots)$. By (6.13)

$$J(I'_\nu) \leqq 20n\,|Q_\nu| \qquad (\nu \geqq n), \tag{β}$$

where I'_ν is the least interval containing $Q_\nu F_n$; clearly I'_ν contains $\mathfrak{J}$ and the diameter of I'_ν tends to zero with $\frac{1}{\nu}$. Thus by (α)

$$\lim_{\nu} \frac{J(I'_\nu)}{|I'_\nu|} = J'_s(\mathfrak{J}) = J'(\mathfrak{J}). \tag{γ}$$

Now, by (β) and since $|I'_\nu| \geqq |Q_\nu F_n|$,

$$\frac{|J(I'_\nu)|}{|I'_\nu|} \leqq 20n \frac{|Q_\nu|}{|I'_\nu|} \leqq 20n \frac{|Q_\nu|}{|Q_\nu F_n|} \qquad (\nu \geqq n),$$

In view of (3°) (with $I = Q_\nu$ and $\nu \to \infty$) the last member above tends to $20n$ when $\nu \to \infty$. Thus, as a consequence of (γ)

$$|J'_s(\mathfrak{J})| = |J'(\mathfrak{J})| \leqq 20n,$$

which establishes the Lemma.

Convergence of the series

$$\Gamma_3 = \sum_{1}^{\infty} s\rho(s),$$

which is implied by (6'.7), signifies that $J'(\mathfrak{J})$ *is summable over* G. In fact, by Lemma (6'.19), (6'.18) and (6'.17) it is inferred that

$$\iint_G |J'(\mathfrak{J})|\, d\mathfrak{J}_1\, d\mathfrak{J}_2 = \iint_E |J'(\mathfrak{J})|\, d\mathfrak{J}_1\, d\mathfrak{J}_2 = \sum_{n=1}^{\infty} \iint_{E_n} |J'(\mathfrak{J})|\, d\mathfrak{J}_1\, d\mathfrak{J}_2, \tag{6'.21}$$

where

$$\mathfrak{J} = \mathfrak{J}_1 + i\mathfrak{J}_2, \qquad E_n = (F_n - F_{n-1})E,$$

and

$$\iint_G |J'(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2 \leqq 20 \sum_{n=1}^{\infty} n\, |E_n|;$$

moreover, E_n is a subset of $G - F_{n-1}$ so that

$$(6'.22) \qquad |E_n| \leqq \rho(n-1) \leqq \rho(n);$$

thus

$$\iint_G |J'(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2 \leqq 20\, \Gamma_3.$$

As a consequence of (6'.1) and of the established summability (over G) of $J'(\mathfrak{J})$ an application of Theorem **2**.15 will yield

$$(6'.23) \qquad J(I) = \iint_I J'(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2$$

for all intervals I in G. *The additive function*

$$(6'.24) \qquad J(e) = \iint_e J'(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2$$

of measurable sets e (in G*) is obviously absolutely continuous; futhermore, this function coincides with* J(I) *on intervals;* J(I) is absolutely continuous as a function of intervals.

With e denoting a measurable set in G, in place of (6'.21) we have

$$\iint_e |J'(\mathfrak{J})|\, d\mathfrak{J}_1\, d\mathfrak{J}_2 = \iint_{eE} |J'(\mathfrak{J})|\, d\mathfrak{J}_1\, d\mathfrak{J}_2 = \sum_{n=1}^{\infty} \iint_{eE_n} |J'(\mathfrak{J})|\, d\mathfrak{J}_1\, d\mathfrak{J}_2.$$

Thus, in view of (6'.20),

$$\iint_e |J'(\mathfrak{J})|\, d\mathfrak{J}_1\, d\mathfrak{J}_2 \leqq \sum_{n=1}^{\infty} 20\, n\, |eE_n|.$$

Let j denote a positive, as yet undefined, integer. On making use of the inequalities

$$|eE_n| \leqq |e| \quad (n \leqq j), \qquad |eE_n| \leqq |E_n| \leqq \rho(n) \quad (n > j)$$

[*cf.* (6'.22)] it is accordingly inferred that

$$\iint_e |J'(\mathfrak{J})|\, d\mathfrak{J}_1\, d\mathfrak{J}_2 \leqq \sum_{n=1}^{j} 20\, n\, |e| + \sum_{n>j} 20\, n\, \rho(n).$$

Thus by virtue of (6'.24)

(6'.25) $$|J(e)| \leqq cj^2|e| + 20\sum_{n > j} n\rho(n) = w(j, |e|),$$

where c is positive and independent of j, e.

Let l be a segment in G, parallel to one of the axes-say, the axis of reals, its end points being $p+i\eta$, $h+i\eta$ $(p>h)$. Let I_ν be the interval containing l and having sides in the lines $y=\eta\pm\frac{r_0}{\nu}$, $x=p-\frac{r_0}{\nu}$, $x=h+\frac{r_0}{\nu}$ (suitable $r_0>0$). It has been established in (T) that

$$\gamma(\mathfrak{J}) = \int_l \frac{|dz|}{|z-\mathfrak{J}|} = \int_p^h \frac{dx}{|z-\mathfrak{J}|} \leqq c' \log\nu \qquad (\mathfrak{J} \text{ in } I_\nu - I_{\nu+1})$$

for $\geqq 3$. By (6'.25)

(6'.26) $$|J_1(e)| \leqq w(j, |e|) \qquad |J_2(e)| \leqq w(j, |e|).$$

Designating by

$$V^+, \quad V^-, \quad V = V^+ - V^-$$

the upper, lower and total variations, we have

$$V^+J_1(e) = \text{u. b. } J_1(e_0), \quad V^-J_1(e) = \text{l. b. } J_1(e_0) \qquad (\text{for } e_0 \subset e).$$

In view of (6'.26)

$$0 \leqq V^+J_1(e), \qquad -V^-J_1(e) \leqq w(j, |e|), \qquad VJ_1(e) \leqq 2w(j, |e|).$$

There are similar inequalities for $J_2(e)$. We define *the total variation of* $J = J_1 + iJ_2$ as

$$V^*J(e) = VJ_1(e) + VJ_2(e);$$

by the preceding

(6.27) $$V^*J(e) \leqq 4\,w(j, |e|).$$

We want to secure

(6'.28) $$\int_{(I)} \left[\iint_G \frac{J'(\mathfrak{J})\,d\mathfrak{J}_1\,d\mathfrak{J}_2}{\mathfrak{J}-z}\right] dz = \iint_G \left[\int_{(I)} \frac{dz}{\mathfrak{J}-z}\right] J'(\mathfrak{J})\,d\mathfrak{J}_1\,d\mathfrak{J}_2$$

for all intervals I in G. Thus, together with our other considerations, would enable us to dispense with Theodoresco's theorem.

Inasmuch as by (**6**'.24)

$$\iint_G \frac{J'(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2}{\mathfrak{J} - z} = \iint_G \frac{dJ(e_{\mathfrak{J}})}{\mathfrak{J} - z} \qquad (\mathfrak{J} \text{ variable point of integration}),$$

it is observed that (**6**'.28) is secured provided the integral

$$\iint_G \gamma(\mathfrak{J})\, dV^*J(e_{\mathfrak{J}})$$

exists. It is sufficient to prove existence of

$$T = \iint_{I_{\nu_0}} \gamma(\mathfrak{J})\, dV^*J(e_{\mathfrak{J}}) \qquad (\text{suitable } \nu_0).$$

Now in view of the inequality preceding (**6**'.26)

$$T = \sum_{q \geqq \nu_0} \iint_{I_q - I_{q+1}} \cdots \leqq c' \sum_{q \geqq \nu_0} \log q\, V^*J(I_q - I_{q+1}).$$

Utilizing (**6**'.27), with $e = I_q - I_{q+1}$ and $j = j_q$, we obtain

$$T \leqq 4\, c' \sum_{q \geqq \nu_0} \log q\, w(j_q, |I_q - I_{q+1}|).$$

With the aid of the definition of $w(j, |e|)$ in (**6**'.25) and since

$$|I_q - I_{q+1}| \leqq c_0 \frac{1}{q^2},$$

it is inferred that

$$T \leqq 4\, c_0 c c' \sum_{q \geqq \nu_0} j_q^2 \frac{\log q}{q^2} + 80\, c' \sum_{q \geqq \nu_0} \log q \sum_{\nu > j_q} n\, \rho(n).$$

Here the integers $j_1 < j_2 < \ldots$ are at our disposal. If the j_2 can be chosen so that the séries involved above converge, then T exists and (**6**'.28) will hold. Accordingly we observe that the condition (**6**'.7) implies that a relation of the type involved in (**6**.2), (**6**.3) hold in the present case; *thus*

$$(6'.29) \qquad \int_{(I)} \left\{ -\frac{1}{2\pi i} \iint_G \frac{J'(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2}{\mathfrak{J} - z} \right\} dz = \iint_I J'(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2$$

for all intervals I *in* G.

Form the function

$$q(z) = f(z) + \frac{1}{2\pi i}\iint_G \frac{J'(\mathfrak{Z})\, d\mathfrak{Z}_1\, d\mathfrak{Z}_2}{\mathfrak{Z} - z}.$$

By (6'.29)

$$\int_{(I)} q(z)\, dz = J(z) - \int_I J'(\mathfrak{Z})\, d\mathfrak{Z}_1\, d\mathfrak{Z}_2.$$

In view of (6'.24) on accordingly obtains

$$\int_{(I)} q(z)\, dz = 0.$$

Hence $q(z)$ is analytic and the conclusion of the Theorem ensues.

7. Integral equations on the basis of Theorem 6.4. — *We introduce*

Definition 7.1. — *A continuous function $f(z)$ will be said to be of classe* R ($f \subset$ R) *if*

$$(7.1\,a) \qquad \overline{\lim} \frac{1}{|I|}\left|\int_{(I)} f(z)\, dz\right| < +\infty \qquad (\textit{for } I \supset z_0;\ z_0 \textit{ in } G - D_0),$$

where D_0 *is sum of a denumerable infinity (at most) of segments parallel to the axes, and if*

$$(7.1\,b) \qquad J'(\mathfrak{Z}) \leqq B \qquad (\mathfrak{Z} \textit{ in } G - G_0;\ mG_0 = 0),$$

where

$$J(I) = \int_{(I)} f\, dz.$$

We shall study integral equations

$$\iint_G \frac{k(z, \mathfrak{Z})\, \varphi(\mathfrak{Z})\, d\mathfrak{Z}_1\, d\mathfrak{Z}_2}{\mathfrak{Z} - z} + f(z) = a(z) \qquad (f \subset R),$$

where $a(z)$ is a generic designation for a function analytic and not assigned beforehand; here $\varphi(\mathfrak{Z})$ is the unknown. Put

$$K(z, \mathfrak{Z}) = \frac{k(z, \mathfrak{Z}) - k(z, z)}{\mathfrak{Z} - z}.$$

and assume $k(z, z) = 1$. Thus the equation is

$$(7.2) \qquad \iint_G \frac{\varphi(\mathfrak{Z})\, d\mathfrak{Z}_1\, d\mathfrak{Z}_2}{\mathfrak{Z} - z} + \iint_G K(z, \mathfrak{Z})\, \varphi(\mathfrak{Z})\, d\mathfrak{Z}_1\, d\mathfrak{Z}_2 + f(z) = a(z);$$

that is,

$$(7.2')\quad \begin{cases} \displaystyle\iint_G \frac{\varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2}{\mathfrak{J}-z} + \bar{f}(z) = a(z), \\ \bar{f}(z) = f(z) + g(z), \qquad g(z) = \displaystyle\iint_G K(z, \mathfrak{J})\, \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2, \end{cases}$$

In the sequel $K(z, \mathfrak{J})$ will be subject to suitable conditions. Associated with (7.2) is the integral equation

$$(7.3)\qquad 2\pi i\,\varphi(z) - \iint_G \Phi'(z, \mathfrak{J})\, \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2 = J'(z),$$

where $\Phi'(z, \mathfrak{J})$ is the ordinary derivative (supposed to exist) of the interval function

$$(7.3\,a)\qquad \Phi(I, \mathfrak{J}) = \int_{(I)} K(z, \mathfrak{J})\, dz;$$

we shall write

$$\Phi'(z, \mathfrak{J}) = D_z \Phi(I, \mathfrak{J});$$

in this section D_z (or $D_{\mathfrak{J}}$) *will designate ordinary derivation of interval functions.*

LEMMA 7.4. — *Every solution* $\varphi(\mathfrak{J})$ *of* (7.2), *bounded almost everywhere in* G, *will be a solution of the regular Fredholm integral equation* (7.3), *and conversely, provided* $f(z)$, $K(z, \mathfrak{J})$ *satisfy the conditions*

$$(7.5)\qquad f(z) \subset R \qquad (\textit{Definition } (7.1));$$

$$(7.6)\qquad g(z) = \iint_G K(z, \mathfrak{J})\, \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2 \subset R$$

for all φ *almost everywhere bounded in* G

$$(7.7)\qquad \int_{(I)} \left[\iint_G K(z, J)\, \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2 \right] dz = \iint_G \Phi(I, \mathfrak{J})\, \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2$$

[Φ *from* (7.3 *a*)] *for all* φ *almost everywhere bounded in* G;

$$(7.8)\qquad \iint_G |\varphi'(z, \mathfrak{J})|^2\, d\mathfrak{J}_1\, d\mathfrak{J}_2$$

bounded almost everywhere in G;

$$(7.9)\qquad D_z \iint_G \Phi(I, \mathfrak{J})\, \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2 = \iint_G \Phi'(z, \mathfrak{J})\, \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2$$

for all φ almost everywhere bonded in G.

The condition (7.8) may be replaced by

$$(7.8') \qquad \iint_G \iint_G |\Phi'(z, \mathfrak{J})|^2 (dx\,dy)(d\mathfrak{J}_1\,d\mathfrak{J}_2)$$

exists. In view of (7.1 *b*) the condition (7.8) implies that every (regular) solution of (7.3) is bounded almost everywhere in G.

Suppose φ is a bounded solution of (7.2). — By (7.5), (7.6) $\bar{f}$ of (7.2′) belongs to R. Consider (7.2) in the form (7.2′) and apply the inversion (6.14), (6.14 *a*) (where f is replaced by $\bar{f}$); thus

$$\varphi(\mathfrak{J}) = \frac{1}{2\pi i} J^{0\prime}(\mathfrak{J}),$$

where

$$J^0(I) = \int_{(I)} f(z)\,dz = J(I) + J_0(I), \qquad J_0(I) = \int_{(I)} g(z)\,dz.$$

In view of (7.7), (7.3 *a*) one may write φ in the form

$$2\pi i\,\varphi(z) = J'(z) + D_z \iint_G \Phi(I, \mathfrak{J})\,\varphi(\mathfrak{J})\,d\mathfrak{J}_1\,d\mathfrak{J}_2;$$

hence (7.9) will yield

$$(7.10) \qquad 2\pi i\,\varphi(z) = J'(z) + \iint_G \Phi'(z, \mathfrak{J})\,\varphi(\mathfrak{J})\,d\mathfrak{J}_1\,d\mathfrak{J}_2;$$

that is, φ will be a solution of the Fredholm equation (7.3).

Conversely, *suppose φ is a bounded solution of* (7.10). — By (7.9) it is inferred that φ will satisfy the relation preceding (7.10), which as a consequence of (7.7) yields

$$2\pi i\,\varphi(z) = J'(z) + D_z \int_{(I)} \left[\iint_G K(\tau, \mathfrak{J})\,\varphi(\mathfrak{J})\,d\mathfrak{J}_1\,d\mathfrak{J}_2 \right] d\tau$$
$$= J'(z) + D_z \int_{(I)} g(\tau)\,d\tau = J'(z) + J'_0(z).$$

Integrating over I and utilizing (7.5), (7.6), with the aid of Theorem 2.15 we obtain

$$(7.11) \qquad 2\pi i \iint_I \varphi(z)\,dx\,dy = \iint_I J'(z)\,dx\,dy + \iint_I J'_0(z)\,dx\,dy$$
$$= J(I) + J(I_0) = \int_{(I)} (f+g)\,dz = \int_{(I)} \bar{f}(z)\,dz.$$

Since φ is bounded, by (6.3) it is inferred that

$$2\pi i \iint_{I} \varphi(z)\,dx\,dy = 2\pi i \int_{(I)} \left[-\frac{1}{2\pi i} \iint_{G} \frac{\varphi(\mathfrak{Z})\,d\mathfrak{Z}_1\,d\mathfrak{Z}_2}{\mathfrak{Z} - z} \right] dz.$$

Thus from (7.11) it follows that

$$\int_{(I)} \left[\iint_{G} \frac{\varphi(\mathfrak{Z})\,d\mathfrak{Z}_1\,d\mathfrak{Z}_2}{\mathfrak{Z} - z} + \bar{f}(z) \right] dz = 0$$

for all I in G; hence [...], above, is analytic; accordingly φ is a solution of (7.2′), that is of (7.2). The proof of the Lemma is complete.

THEOREM 7.12. — *The integral equations* (7.2), (7.3) *are equivalent in the sense that a solution φ of one, bounded almost everywhere, is a solution of the other, provided : $f(z) \subset$* R *(Definition* 7.1) *and* $\left[\textit{with } \Phi(I, \mathfrak{Z}) = \int_{(I)} K(z, \mathfrak{Z})\,dz \right]$

(1°) $$\left\{ \begin{array}{l} \dfrac{|\Phi(I, \mathfrak{Z})|}{|I|} \leq h(z)\,q(\mathfrak{Z}) \\ [\textit{for all } I \supset z;\ h(z) \neq \infty \textit{ in } G - D_0;\ q(\mathfrak{Z}) \textit{ integrable over } G]; \end{array} \right.$$

(2°) $$\left\{ \begin{array}{l} \dfrac{|\Phi(I, \mathfrak{Z}) - \Phi(I, \mathfrak{Z}')|}{|I|} \leq h(z)\,q(\mathfrak{Z}, \mathfrak{Z}') \\ [\textit{for all } I \supset z;\ q(\mathfrak{Z}, \mathfrak{Z}') \to 0 \quad \textit{with } |\mathfrak{Z} - \mathfrak{Z}'|]; \end{array} \right.$$

(3°) $$\left\{ \begin{array}{ll} \displaystyle\iint_{G} |K(b, \mathfrak{Z})|\,d\mathfrak{Z}_1\,d\mathfrak{Z}_2 < +\infty & (\textit{some } b \textit{ in } G); \\ \displaystyle\iint_{G} |K(z, \mathfrak{Z}) - K(z', \mathfrak{Z})|\,d\mathfrak{Z}_1\,d\mathfrak{Z}_2 \to 0 & \textit{with } |z - z'| \end{array} \right.$$

(4°) $$\iint_{G} |\Phi'(z, \mathfrak{Z})|^2\,d\mathfrak{Z}_1\,d\mathfrak{Z}_2 \quad \textit{is bounded almost everywhere.}$$

The proof of this result is based on Lemma 7.4. Let l denote a segment, which with its end points lies in G and is parallel to one of the axes. To establish (7.7) it will suffice to show that

$$\int_{l} \left[\iint_{G} K(\tau, \mathfrak{Z})\,\varphi(\mathfrak{Z})\,d\mathfrak{Z}_1\,d\mathfrak{Z}_2 \right] d\tau = \iint_{G} \left[\int_{l} K(\tau, \mathfrak{Z})\,d\tau \right] \varphi(\mathfrak{Z})\,d\mathfrak{Z}_1\,d\mathfrak{Z}_2$$

for $|\varphi| \leqq c$ (almost everywhere in G). We have

$$\int_l \iint_G |K(\tau, \mathfrak{J})|\,|\varphi(\mathfrak{J})|\,d\mathfrak{J}_1\,d\mathfrak{J}_2\,|d\tau| \leqq c \int_l \left[\iint_G |K(\tau, \mathfrak{J})|\,d\mathfrak{J}_1\,d\mathfrak{J}_2\right] |d\tau|;$$

now by (3°)

$$\iint_G |K(\tau, \mathfrak{J})|\,d\mathfrak{J}_1\,d\mathfrak{J}_2 \leqq \iint_G |K(\tau, \mathfrak{J}) - K(b, \mathfrak{J})|\,d\mathfrak{J}_1\,d\mathfrak{J}_2 + \iint_G |K(b, \mathfrak{J})|\,d\mathfrak{J}_1\,d\mathfrak{J}_2 < +\infty$$

for τ in any closed subset of G-hence on l; thus the last member in the preceding formula is finite and (7.7) holds.

In view of (3°) $g(z)$ *of* (7.6) *is continuous.* On writing

$$J_0(I) = \int_{(I)} g(z)\,dz,$$

by virtue of (7.7) it is deduced that

$$J_0(I) = \iint_G \Phi(I, \mathfrak{J})\,\varphi(\mathfrak{J})\,d\mathfrak{J}_1\,d\mathfrak{J}_2. \tag{7.13}$$

Accordingly, by (1°), for any I containing z one obtains

$$\frac{|J_0(I)|}{|I|} \leqq \iint_G \frac{|\Phi(I, \mathfrak{J})|}{|I|}\,|\varphi(\mathfrak{J})|\,d\mathfrak{J}_1\,d\mathfrak{J}_2 \leqq cc'h(z) < +\infty$$

[here c' is the integral over G of $q(\mathfrak{J})$] for all z in $G - D_0$; hence

$$\overline{\lim}\, \frac{1}{|I|}\left|\int_{(I)} q(z)\,dz\right| < +\infty \qquad (\text{for } I \supset z_0;\ z_0 \text{ in } G - D_0). \tag{7.13a}$$

Now $q(\mathfrak{J})$ can be assumed as finite for every $\mathfrak{J}$ in G since, if necessary, the values of $q(\mathfrak{J})$ can be suitably assigned on a set of zero measure. By 1°

$$\overline{\lim}\, \frac{|\Phi(I, \mathfrak{J})|}{|I|} \leqq h(z)\,g(\mathfrak{J}) < +\infty \qquad (I \supset z)$$

for all $\mathfrak{J}$ in G and for z in $G - D_0$. Hence the lower and upper strong derivates of the real and imaginary parts of $\Phi(I, \mathfrak{J})$ are finite for z in $G - D_0$ (for all $\mathfrak{J}$ in G). Accordingly

$$\Phi'_s(z, \mathfrak{J}) = \Phi'(z, \mathfrak{J} \qquad (\text{for } z \text{ in } G - G_{\mathfrak{J}}),$$

where $G_{\mathcal{J}}$ is some set of zero measure, possibly depending on $\mathcal{J}$. Let $\{\mathcal{J}_\nu\}$ $(\nu = 1, 2, \ldots)$ denote a set of points everywhere dense in G. In view of the above the derivatives

$$(7.14) \qquad \Phi'_s(z, \mathcal{J}_\nu) = \Phi'(z, \mathcal{J}_\nu) \qquad (\nu = 1, 2, \ldots)$$

exist for z in $G - G^0$, where

$$G^0 = D_0 + \sum_1^\infty G_{\mathcal{J}_\nu}, \qquad m(G^0) = 0.$$

Let $\mathcal{J}$ denote any point in G and the $I_n (n = 1, 2, \ldots)$ denote any sequence of intervals tending to z (z in $G - G_0$). By (2°)

$$l_n = \frac{\Phi(I_n, \mathcal{J})}{|I_n|} = \frac{\Phi(I_n, \mathcal{J}_\nu)}{|I_n|} + r_{n,\nu}, \qquad |r_{n,\nu}| \leqq h(z)\, q(\mathcal{J}, \mathcal{J}_\nu).$$

In view of (7.14) all the limits l of the sequence $l_1, l_2, \ldots$ satisfy

$$|l - \Phi'_s(z, \mathcal{J}_\nu)| \leqq h(z)\, q(\mathcal{J}, \mathcal{J}_\nu).$$

The numbers l are independent of the $\mathcal{J}_\nu$. Let $\mathcal{J}_{\nu_j} (\nu_1 < \nu_2 < \ldots)$ be a subsequence of $\{\mathcal{J}_\nu\}$ converging to $\mathcal{J}$. Replacing, above, $\mathcal{J}_\nu$ by $\mathcal{J}_{\nu_j}$, in the limit we obtain

$$(1_0) \qquad \lim_{\nu_j} \Phi'_s(z, \mathcal{J}_{\nu_j}) = l.$$

Necessarily l is unique, that is the limit

$$\lim_n l_n = \lim_n \frac{\Phi(I_n, \mathcal{J})}{|I_n|} = l$$

exists and is independent of the choice of the sequence $\{\nu_j\}$. We repeat now the above argument with $\{I_n\}$ replaced by any other sequence of intervals $\{I^*_n\}$, tending to z. It is found that all the limits l^* of the sequence

$$l^*_n = \frac{\Phi(I^*_n, \mathcal{J})}{|I^*_n|} \qquad (n = 1, 2, \ldots)$$

satisfy

$$|l^* - \Phi'_s(z, \mathcal{J}_\nu)| \leqq h(z)\, q(\mathcal{J}, \mathcal{J}_\nu)$$

and

$$\lim_{\nu_j} \Phi'_s(z, \mathcal{J}_{\nu_j}) = l^* \qquad (\text{as } \mathcal{J}_{\nu_j} \to \mathcal{J}).$$

By (1°) $l^* = l$; that is, l is independent of the choice of the sequence

$\{I_n\}$. Consequently *the strong derivative*

$$(7.14\,a)\qquad \Phi'_s(z, \mathfrak{J}) = l = \lim_{\nu_j} \Phi'_s(z, \mathfrak{J}_{\nu_j}) = \Phi'(z, \mathfrak{J})$$

exists (and is finite) *for all* z *in* $G - G^0$ *and for all* $\mathfrak{J}$ *in* G. It is essential to note that G^0 *is a set of zero measure independent of* $\mathfrak{J}$.

In view of the relation preceding (7.13 *a*)

$$\overline{\lim}\, \frac{1}{|I|} \left| \iint_G \Phi(I, \mathfrak{J}) \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2) \right| < +\infty \qquad (\text{for } I \supset z_0;\ z_0 \text{ in } G - D_0).$$

Thus the strong derivates of the real and imaginary parts of

$$\iint_G \Phi(I, \mathfrak{J}) \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2$$

are finite in $G - D_0$; accordingly

$$(7.15)\qquad D_z \iint_G \Phi(I, \mathfrak{J}) \varphi(\mathfrak{J})\, d\mathfrak{J}_1,\, d\mathfrak{J}_2 = D_{sz} \iint_G \Phi(I, \mathfrak{J}) \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2$$

for almost all z in G (here $D_{sz} \ldots$ denotes the strong derivative at z).

To prove (7.9) let z denote a point at which (7.14 *a*), (7.15) hold (necessarily $h(z) \neq \infty$). It will suffice to secure

$$(7.9')\qquad D_{sz} \iint_G \Phi(I, \mathfrak{J}) \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2 = \iint_G \Phi'_s(z, \mathfrak{J}) \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2.$$

Let $\{I_n\}$ denote a sequence tending to z. We have

$$\begin{aligned} D_{sz} \iint_G \Phi(I, \mathfrak{J}) \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2 &= \lim_n \iint_G \frac{\Phi(I_n, \mathfrak{J})}{|I_n|} \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2 \\ &= \iint_G \lim_n \frac{\Phi(I_n, \mathfrak{J})}{|I_n|} \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2 \\ &= \iint_G \Phi'_s(z, \mathfrak{J}) \varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2. \end{aligned}$$

In fact, by (1°) and since $|\varphi| \leqq c$,

$$\frac{|\Phi(I_n, \mathfrak{J})|}{|I_n|} |\varphi(\mathfrak{J})| \leqq c h(z) q(\mathfrak{J}) \qquad (n = 1, 2, \ldots);$$

the last member is independent of n and is integrable in $\mathfrak{J}$ over G; this justifies transition from the second member, above, to the third.

The transition to the last member ensues by (7.14 a). Consequently (7.9) holds.

It remains to establish (7.6). By (7.13) and (7.9)

$$J'_0(z) = \iint_G \Phi'(z, \mathcal{J})\varphi(\mathcal{J})\, d\mathcal{J}_1\, d\mathcal{J}_2.$$

Thus in view (4°)

$$|J'_0(x)| \leqq c \iint_G |\Phi'(z, \mathcal{J})|\, d\mathcal{J}_1\, d\mathcal{J}_2 \leqq c_0 \left[\iint_G |\Phi'(z, \mathcal{J})|^2\, d\mathcal{J}_1 d\mathcal{J}_2 \right]^{\frac{1}{2}}$$

(c_0 some constant). Whence $|J'_0(z)|$ is bounded almost everywhere; together with (7.13 a) this implies that (7.1 a), (7.1 b) hold for $g(z)$. Since $g(z)$ is continuous, $g(z) \subset R$; (7.6) holds. Now (4°) is identical with (7.8) and $f(z)$ belongs to R by hypothesis. Thus, under the assumptions of the Theorem, the conditions (7.5)-(7.8) of the Lemma all hold. The theorem is proved.

8. Integral equations on the basis of Theorem **6**.10. — We introduce

Definition **8**.1. — *A continuous function $f(z)$ will be said to be of class B_μ provided*

(**8**.1 *a*) $$\overline{\lim_{h \to 0}} \left| \frac{f(z+h) - f(z)}{h} \right| < +\infty \qquad (in\ G - E),$$

where $E = F_1 + E_2 + \ldots$, *μ-length of* $E_j < +\infty$ $(j = 1, 2, \ldots)$, *$\mu(u)$ being admissible, and the modulus of continuity $\eta(u)$ of $f(z)$ is of the form*

(**8**.1 *b*) $$\eta(u) = \omega(u)\frac{\mu(u)}{u} \qquad [\eta(u),\ \omega(u) \to 0\ with\ u]$$

and provided

(**8**.1 *c*) $$|(Q)J'(\mathcal{J})| \leqq B \qquad (in\ G - G_0;\ mG_0 = 0),$$

where

$$J(I) = \int_{)I)} f(z)\, dz = J_1 + iJ_2.$$

The class B_μ is additive in the sense that $c_1 f_1 + c_2 f_2$ (c_1, c_2 constants) belongs to B_μ with f_1, f_2. Furthermore, in accordance with a remark subsequent to Theorem 6.10, (8.1 a) implies that

$$(8.2) \qquad -\infty < (Q)\underline{J}_\nu \leqq (Q)\overline{J}_\nu < +\infty \qquad (\text{in } G - E;\ \nu = 1, 2).$$

Also, if $f \subset B_\mu$, J(I) will satisfy the condition (μ) [Definition (4.5)]; this ensues by Lemma 5.10. Consequently by virtue of Theorem 4.25, applied to J_1, J_2, *one obtains*

$$(8.3) \qquad \iint_I (Q)J'(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2 = J(I) \qquad (\textit{all } I \textit{ in } G)$$

whenever f belongs to B_μ.

LEMMA 8.4. — *Every solution $\varphi(\mathfrak{J})$ of* (7.2), *bounded almost everywhere in* G, *will be a solution of the regular Fredholm integral equation* (7.3) [*with* (Q) *derivation*] *and conversely, provided $f(z)$ $k(z, \mathfrak{J})$ satisfy conditions* (7.5), (7.9), *where* R *is replaced by* B_μ *and ordinary derivation is replaced by* (Q) *derivation. We designate the modified conditions* (7.5)-(7.9) *by* (8.5)-(8.9), respectively.

This result is established following the lines of the proof of Lemma 7.4, on making use of (8.3) (for f and g) and of Théodoresco's theorem (6.3).

THÉOREM 8.10. — *The integral equations* (7.2), (7.3) [*with* (Q) *derivatives*] *are equivalent in the sense that a solution φ of one, bounded almost everywhere, is a solution of the other, provided $f(z) \subset B_\mu$ (Definition* (8.1) *and provided* $\left(\textit{with } \Phi(I, \mathfrak{J}) = \int_{(I)} K(z, \mathfrak{J})\, dz\right)$ *one has :*

$$(1^\circ) \qquad \frac{|\Phi(I, \mathfrak{J})|}{|I|} \leqq h(z)q(\mathfrak{J}),$$

[*for all* $I \supset z$; $h(z) \neq \infty$ *in* $G - E$; $q(\mathfrak{J})$ *integrable over* G];

$$\frac{1}{|I|}|\Phi(I, \mathfrak{J}) - \Phi(I, \mathfrak{J}')| \leqq h(z)q(\mathfrak{J}, \mathfrak{J}')$$

[*for all* $I \supset z$; $q(\mathfrak{J}, \mathfrak{J}') \to 0$ *with* $|\mathfrak{J} - \mathfrak{J}'|$]:

$$(2^\circ) \qquad \nu = \iint_G |K(z', \mathfrak{J})|\, d\mathfrak{J}_1\, d\mathfrak{J}_2 < +\infty \qquad (\textit{for some } z' \textit{ in } G);$$

$$(3^\circ) \qquad \iint_G |(Q)\Phi'(z, \mathfrak{J})|^2\, d\mathfrak{J}_1,\, d\mathfrak{J}_2 \qquad \textit{boundest almost everywhere};$$

$$(4^\circ) \qquad \iint_G |K(z_1, \mathfrak{J}) - K(z, \mathfrak{J})|\, d\mathfrak{J}_1\, d\mathfrak{J}_2 \leqq a|z_1 - z| \qquad (\text{constant } a).$$

In the above E *is a sum of a denumerable infinity (at most) of sets of finite* μ*-length.*

We write

$$J_0(I) = \int_{(I)} g(z)\, dz.$$

By (4°) and the definition of $g(z)$ in (**8**.6)

$$(\mathbf{8}.11) \qquad |g(z) - g(z_1)| \leq k' |z - z_1|;$$

hence (with z_1 denoting a point interior I)

$$|J_0(I) = | = \left| \int_{(I)} (g(z) - g(z_1))\, dz \right| \leq k' \int_{(I)} |z - z_1| \, |dz| \leq k |I|$$

(k, k' constants) for squares I. Since $(Q)J'_0$ is defined with the aid of a particular sequence of squares, the above implies that

$$|(Q)J'_0(\mathcal{J})| \leq k.$$

Thus (**8**.1 c) holds for g. Also, by (**8**.11) it is observed that (**8**.1 a) is valin for g. The modulus of continuity $\eta_0(u)$ of $g(z)$ is $k'u$; thus, with (**8**.1 b) in view, on writing

$$k' u = \omega_0(u) \frac{\mu(u)}{u}$$

and on noting that on account of (**4**.9) (where $m = 2$)

$$\omega_0(u) = k' \frac{u^2}{\mu(u)} \to 0 \qquad (\text{with } u),$$

it is inferred that g satisfies (**8**.1 b). Hence $g \subset B_\mu$ and (**8**.6) *holds.*

As a consequence of (4°), (2°)

$$\iint_G |K(z, \mathcal{J})| \, d\mathcal{J}_1 \, d\mathcal{J}_2 \leq \nu + \iint_G |K(z', \mathcal{J}) - K(z, \mathcal{J})| \, d\mathcal{J}_1 \, d\mathcal{J}_2 \leq \nu + a |z' - z|.$$

Accordingly (**8**.7) is valid,

In view of (**8**.7)

$$(\mathbf{8}.12) \qquad J_0(I) = \iint_G \Phi(I, \mathcal{J}) \varphi(\mathcal{J}) \, d\mathcal{J}_1 \, d\mathcal{J}_2.$$

By (1) and inasmuch as $|\varphi| \leqq c$ (almost everywhere), we have

$$\frac{|J_0(I)|}{|I|} \leqq c \iint_G \frac{|\Phi(I, \mathfrak{J})|}{|I|} d\mathfrak{J}_1 \, d\mathfrak{J}_2 \leqq cc' h(z) < +\infty \qquad (\text{in } G - E]$$

for all interval $I \supset z$ (c' a constant). Thus

$$\overline{\lim} \frac{|\Phi(I, \mathfrak{J})|}{|I|} < +\infty, \qquad \overline{\lim} \frac{|J_0(I)|}{|I|} < +\infty \qquad [I \supset z;\ z \text{ in } G - E).$$

Now $m(E) = 0$; hence the extreme strong derivates of the real and imaginary parts of Φ, J_0 are finite almost everywhere. By a known theorem this implies

$$\Phi'_s(z, \mathfrak{J}) = \Phi'(z, \mathfrak{J}) \qquad [\text{for } z \text{ in } G - G_{\mathfrak{J}};\ m(G_{\mathfrak{J}}) = 0].$$
$$J'_{0s}(z) = J'_0(z) \qquad (\text{almost everywhere in } G).$$

The assertion for Φ'_s *holds for* z *in* $G - G^0$, *where* G^0 *is independent of* $\mathfrak{J}$, *contains* E *and has zero measure*; this is established with the aid of the second condition (1°), following the corresponding lines of reasoning in section **7**. Clearly

$$\underline{F}_s \leqq \underline{F}' \leqq (Q)\underline{F} \leqq (Q)\overline{F} \leqq \overline{F}' \leqq \overline{F}_s$$

for any real valued interval function $F(I)$. Accordingly the above yields

$$(\mathbf{8}.13) \qquad \begin{cases} (Q)\Phi'(z, \mathfrak{J}) = \Phi'_s(z, \mathfrak{J}) & (\text{for } z \text{ in } G - G_0;\ \text{all } \mathfrak{J} \text{ in } G), \\ (Q)J'_0(z) = J'_{0s}(z) & (\text{almost everywhere in } G). \end{cases}$$

We proceed to prove (**8**.9). By (**8**.12) and (**8**.13) it will suffice to establish

$$(\mathbf{8}.9') \qquad [J'_{0s}(z) =] D_{sz} \iint_G \Phi(I, \mathfrak{J}) \varphi(\mathfrak{J}) \, d\mathfrak{J}_1 \, d\mathfrak{J}_2 = \iint_G \Phi'_s(z, \mathfrak{J}) \varphi(\mathfrak{J}) \, d\mathfrak{J}_1 \, d\mathfrak{J}_2$$

($D_{sz} \ldots =$ strong derivative at z) for almost all z. Let z be a point for which $h(z) \neq \infty$ and (**8**.13) holds. Let $\{I_n\}$ be a sequence of intervals containing z and $\to z$. We have

$$J'_{0s}(z) = \lim_n \iint_G \frac{\Phi(I_n, \mathfrak{J})}{|I_n|} \varphi(\mathfrak{J}) \, d\mathfrak{J}_1 \, d\mathfrak{J}_2 = \iint_G \lim_n \frac{\Phi(I_n, \mathfrak{J})}{|I_n|} \varphi(\mathfrak{J}) \, d\mathfrak{J}_1 \, d\mathfrak{J}_2$$
$$= \iint_G \Phi'_s(z, \mathfrak{J}) \varphi(\mathfrak{J}) \, d\mathfrak{J}_2 \, d\mathfrak{J}_2.$$

Here passage to the limit under the integral sign is justified since

$$\lim \frac{\Phi(I_n, \mathfrak{J})}{|I_n|} = \Phi'_s(z, \mathfrak{J})$$

and since, by (1°), one has

$$\frac{1}{|I_n|}|\Phi(I_n, \mathfrak{J})||\varphi(\mathfrak{J})| \leqq ch(z)q(\mathfrak{J}) \qquad [h(z) < +\infty],$$

where the last member is independent of n and is integrable in $\mathfrak{J}$ over G. *Thus* (**8**.9′), (**8**.9) *hold*. The theorem is accordingly established.

Theorem **7**.12, **8**.10 furnish the following result regarding repre sentations of functions $f(z)$ of a complex variable.

If $f(z) \subset R$ and $k(z, \mathfrak{J})$ satisfies the conditions of Theorem **7**.12, while the regular Fredholm integral equation (**7**.3) has a solution φ (necessarily bounded almost everywhere), then $f(x)$ has the representations

(**8**.14) $$f(z) = -\iint_G \frac{k(z, \mathfrak{J})\varphi(\mathfrak{J})\, d\mathfrak{J}_1\, d\mathfrak{J}_2}{\mathfrak{J} - z} + a(z);$$

$a(z)$ analytic

$$\frac{k(z, \mathfrak{J})}{\mathfrak{J} - z} = \frac{1}{\mathfrak{J} - z} + k(z, \mathfrak{J}).$$

A similar statement can be made for functions $f(z) \subset B_\mu$.

An analogous study of integral equations can be developped on the basis of section **6**′.