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W. J. TRJITZINSKY

Singular integral equations of the first kind and those related to permutability and iteration

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Singular integral equations of the first kind and those related to permutability and iteration;

By W. J. TRJITZINSKY.

INTRODUCTION. — In this work we study the following related problems.

I. *Integral equations of the first kind*

$$\int_0^1 K(x, t)\varphi(t)\,dt = f(x) \qquad [0 \leqq x \leqq 1;\ f(x) \subset L_2],$$

where $K(x, y)$ is possibly non symmetric, is measurable and is such that there exist correspondig linear functionals L_x, R_x as stated in section 2.

II. *The permutability problem*

$$\int_0^1 p(x, t)\,q(t, y)\,dt = \int_0^1 q(x, t)\,p(t, y)\,dt,$$

where $p(x, y)$ is given L_2 in x, L_2 in y, and $q(t, y)$ is to be found.

III. *Inversion of Schmidt kernels;* that is given a symmetric $f(x, y)$, L_2 in x, in y, to find a possibly non symmetric $q(x, y)$ so that

$$f(x, y) = \int_0^1 q(x, t)\,q(y, t)\,dt.$$

IV. *The interation problem* of finding a symmetric $q(x, y)$ whose n-th iterant is equal to an assigned symmetric $f(x, y)$ (L_2 in x, L_2 in y).

The term *singular* in our title is justified by the fact that in III, IV, $f(x,y)$ is a given function merely L_2 in x and in y, but not necessarily L_2 in (x, y), that in II $p(x, y)$ is L_2 in x, L_2 in y, but not necessarily L_2 in (x, y) and that in I, $K(x, y)$ is stilless restricted. The *regular* cases adequately treated in earlier literature are those in which $f(x,y)$, $K(x, y)$, $p(x, y)$ are L_2 in (x, y). The present author has not seen the *regular* cases of III, treated anywhere.

The transition from the *regular* to the *singular* cases involves two distinct methods.

A. *Regularization of a given function* $f(x, y)$, L_2 in x, L_2 in y. This consists in finding functions $a(x)$, $b(y)$ $(\geqq 1)$ so that

$$\int_0^1 \int_0^1 \frac{f^2(x, y)}{a^2(x)\, b^2(y)}\, dx\, dy < +\infty.$$

Developments are then based on use of the characteristic values and functions of the regularized functions.

B. *Spectral theory*. — The background with respect to the method B is given by T. Carleman's [1] work, especially in the field of integral equations, *in the sequel referred to as* C. Most of the results in C are valid for symmetric kernels $K(x, y)$ more general than originally postulated in C; in fact, they hold for $K(x, y)$, L_2 in x (in y); this circumstance follows by another work of Carleman [2]. Whenever we make reference to a result in C, it will be understood that the result in question has been adapted to kernels which are L_2, separately in each of the variables, ore are more general as in sections 3, 4.

In sections 1, 2 *we adapt some of the spectral theory of* C *to non symmetric kernels.*

In section 3 *problem* I *is treated on the basis of method* B (*and of*

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(1) T. Carleman, *Sur les équations intégrales singulières à noyau réel et symétrique*, Uppsala, 1923, p. 1-228.

(2) T. Carleman, *La théorie des équations intégrales singulières et les applications* (*Annales de l'Institut H. Poincaré*, 1931, p. 401-430).

sections 1, 2). In Theorem **3**.10 existence of solutions is established when the sequence ρ_n (**3**.1*b*) is bounded. In section 4 the sense is indicated in which Problem I can be solved when the sequence ρ_n is unbounded; for this purpose use is made of section 2. The regular case of Problem I is well known; it has been studied by G. Lauricella, É. Picard; in this connection the reader is referred to a book by V. Volterra and J. Pérès ([1]), in the sequel referred to as (VP). Certain developments relating to the singular Problem I can be found in a previous work by the present author ([2]). The regular cases of Problems I, II are presented also in a work of J. Soula ([3]); *this work will be referred to as* (S_0).

Problem II *is treated in sections* 5, 6 *on the basis of method* A. — Theorem **5**.9 presents a very simple solution of an equation [(**5**.7), (**5**.8)], related to the permutability equation, without any use of characteristic values and functions. Theorem **6**.4, on the other hand, gives a completely general (but more complicated) solution, on the basis of four sequences of characteristic function. The regular case of Problem II has been solved by Lauricella [reference may be found in (S_0)].

Problem III *is solved in section* 7 (*Theorem* **7**.15) *on the basis of method* A. Under certain conditions the solution of this problem satisfies a second order iteration problem.

Problem IV, *with* $n = 2$, *is treated in Theorem* **8**.19 *on the basis of method* B. *Problem* IV, *with, any odd* n, *is solved in Theorem* **9**.14 *with the aid of method* B. *The combination of the two theorems enables one to treat the case when* n *is even.* It appears inconvenient to apply method A to iteration problems. The regular problem has been solved by Lauricella [cf. (S_0)].

([1]) V. Volterra et J. Pérès, *Théorie générale des fonctionnelles*, Paris, 1936, p. 308-310.

([2]) W. J. Trjitzinsky, *Singular Lebesgue-Stieltjes integral equations* (*Acta Mathematica*, vol. 74, 3-4, 1942, p. 197-310).

([3]) J. Soula, *L'équation intégrale de première espèce à limites fixes et les fonctions permutables à limites fixes* (*Mémorial des Sciences Mathématiques*, Paris, 1936).

1. Non symmetric kernels. — In this section $K(x, y)$ *is* L_2 in x, L_2 in y. We define $K_n(x, y)$ by the relations

$$(1.1)\qquad \begin{cases} K_n(x, y) = K(x, y) & [\text{wherever } |K(x, y)| \leqq n], \\ K_n(x, y) = \pm n & [\text{wherever } \pm K(x, y) > n]. \end{cases}$$

Let the $u_{nk}(x)$, $v_{nk}(x)$, $k = 1, 2, \ldots$, be the characteristic functions associated with $K_n(x, y)$; thus

$$(1.2)\qquad \begin{cases} u_{nk}(x) = \lambda_{nk} \displaystyle\int_0^1 K_n(x, s)\, v_{nk}(s)\, ds, \\ v_{nk}(x) = \lambda_{nk} \displaystyle\int_0^1 u_{nk}(s)\, K_n(s, x)\, ds \end{cases}$$

and

$$(1.3)\qquad \begin{cases} u_{nk}(x) = \lambda_{nk}^2 \displaystyle\int_0^1 \overline{K}_n(x, s)\, u_{nk}(s) ds, \\ v_{nk}(x) = \lambda_{nk}^2 \displaystyle\int_0^1 \underline{K}_n(x, s)\, v_{nk}(s)\, ds, \\ \overline{K}_n(x, s) = \displaystyle\int_0^1 K_n(x, t)\, K_n(s, t)\, dt, \\ \underline{K}_n(x, s) = \displaystyle\int_0^1 K_n(t, x)\, K_n(t, s)\, dt. \end{cases}$$

In accordance with a remark in (VP; 306) *the* λ_{nk} *will be considered positive*. In fact, if λ is a characteristic value and $u(x)$, $v(x)$ are corresponding characteristic functions we shall have $\pm u(x)$, $\mp v(x)$ as characteristic functions for $-\lambda$. If we admitted both positive and negative characteristic values, the set of all $u(v)$ functions could not be arranged as an orthogonal sequence. *Each sequence* (u_{nk}), (v_{nk}) *is arranged as an orthonormal sequence*. The λ_{nk}^2, $u_{nk}(x)$ are the characteristic values and functions of $\overline{K}_n(x, y)$ and the λ_{nk}^2, $v_{nk}(x)$ are those of $\underline{K}_n(x, y)$.

In accordance with a device of *Pérès* (VP) form the symmetric kernel.

$$\begin{aligned} H(x, y) &= 0 && (0 < x, y < 1;\ 1 < x, y < 2), \\ H(x, y) &= K(x, y - 1) && (0 < x, y - 1 < 1), \\ H(x, y) &= K(y, x - 1) && (0 < x - 1, y < 1). \end{aligned}$$

We define $H_n(x, y)$, as above, with K_n in place of K; $H_n(x, y)$ will be symmetric. *Let the* γ_{nk}, $w_{nk}(x)$ *be the characteristic values and functions of* $H_n(x, y)$, *with the sequence* $[w_{nk}(x)]$ *orthogonal on* $(0, 2)$, *chosen so that*

$$\text{(1.4)} \qquad \int_0^2 w_{nk}^2(x)\,dx = 2.$$

It is observed that the γ_{nk} consist précisely of the numbers

$$\lambda_{n1}, \quad \lambda_{n2}, \quad \ldots; \qquad -\lambda_{n1}, \quad -\lambda_{n2}, \quad \ldots;$$

to fix ideas we shall put

$$\text{(1.4}a\text{)} \qquad \gamma_{n,2k} = -\lambda_{nk}, \qquad \gamma_{n,2k-1} = \lambda_{nk} \qquad (k = 1, 2, \ldots);$$

furthermore, it is noted that

$$\text{(1.4}b\text{)} \qquad w_{n,2k-1}(x) = \begin{cases} u_{nk}(x) & (0 < x < 1), \\ v_{nk}(x-1) & (1 < x < 2) \end{cases}$$

and

$$\text{(1.4}c\text{)} \qquad w_{n,2k}(x) = \begin{cases} u_{nk}(x) & (0 < x < 1), \\ -v_{nk}(x-1) & (1 < x < 2); \end{cases}$$

$w_{n,2k-1}(x)$ corresponds to λ_{nk} (>0) and $w_{n,2k}(x)$ corresponds to $-\lambda_{nk}$; clearly these two functions are orthogonal on $(0, 2)$.

While in general $H(x, y)$ is not L_2 in (x, y), the integral

$$\int_0^2 H^2(x, y)\,dx$$

exists [almost everywhere on $(0, 2)$]. Accordingly the theory developed in C applies to $H(x, y)$. However, note must be taken that the second member in (1.4) is not unity

Form

$$\text{(1.5)} \qquad \theta_n^*(x, y \mid \lambda) = \begin{cases} \displaystyle\sum_{0<\gamma_{n\nu}<\lambda} \psi_{n\nu}(x)\,\psi_{n\nu}(y) & (\text{for } \lambda > 0), \\ \displaystyle -\sum_{\lambda \leq \gamma_{n\nu} < 0} \psi_{n\nu}(x)\,\psi_{n\nu}(y) & (\text{for } \lambda < 0), \end{cases}$$

$\theta_n^*(x, y \mid 0) = 0$, where

(1.5a) $$\psi_{n\nu}(x) = 2^{-\frac{1}{2}} w_{n\nu}(x).$$

As a consequence of C and of Carleman's developments (*cf* p. 284) there exists a sequence n_j (independent of x, y, λ) so that the limit

(1.6) $$\lim_{n_j} \theta_{n_j}^*(x, y \mid \lambda) = \theta^*(x, y \mid \lambda)$$

exists, for all (x, y) in $(0 \leq x, y \leq 2)$ except on a specific set E_0 (independent of λ) of plane measure zero; convergence takes place at all points of the diagonal $y = x$ (within the square) except, perhaps, on a set of linear measure zero. *At points of convergence the limit* (1.6) *defines " a spectral function " of* $H(x, y)$. *A particular spectral function* $\theta^*(x, y \mid \lambda)$ *is thus uniquely defined except on* E_0; in particular, $\theta^*(x, x \mid \lambda)$ *is uniquely defined for almost all* x *on* $(0, 2)$.

We write

(1.7) $$\begin{cases} \theta_n^{u,u}(x, y \mid \lambda) = \theta_n^*(x, y \mid \lambda), & \theta_n^{v,v}(x, y \mid \lambda) = \theta_n^*(x+1, y+1 \mid \lambda), \\ \theta_n^{u,v}(x, y \mid \lambda) = \theta_n^*(x, y+1 \mid \lambda), & \theta_n^{v,u}(x, y \mid \lambda) = \theta_n^*(x+1, y \mid \lambda), \end{cases}$$

on $(0 < x, y < 1)$. On taking note of (1.4a), of the fact that $\lambda_{nk} > 0$ and of (1.5), (1.5a), one obtains

$$\theta_n^*(x, y \mid \lambda) = \frac{1}{2} \sum_{\gamma_{n,2k-1} < \lambda} w_{n,2k-1}(x)\, w_{n,2k-1}(y) \qquad (\text{for } \lambda > 0),$$

$$\theta_n^*(x, y \mid \lambda) = -\frac{1}{2} \sum_{\lambda \leq \gamma_{n,2k}} w_{n,2k}(x)\, w_{n,2k}(y) \qquad (\text{for } \lambda < 0).$$

Thus by (1.7), (1.4b, c)

(1.7a) $$\theta_n^{u,u}(x, y \mid \lambda) = \begin{cases} \dfrac{1}{2} \sum\limits_{\lambda_{nk} < \lambda} u_{nk}(x)\, u_{nk}(y) & (\lambda > 0), \\ -\dfrac{1}{2} \sum\limits_{\lambda \leq -\lambda_{nk}} u_{nk}(x)\, u_{nk}(y) & (\lambda < 0), \end{cases}$$

(1.7b) $$\theta_n^{v,v}(x, y \mid \lambda) = \begin{cases} \dfrac{1}{2} \sum\limits_{\lambda_{nk} < \lambda} v_{nk}(x)\, v_{nk}(y) & (\lambda > 0), \\ -\dfrac{1}{2} \sum\limits_{\lambda \leq -\lambda_{nk}} v_{nk}(x)\, v_{nk}(y) & (\lambda < 0), \end{cases}$$

and

$$(1.7c)\qquad \theta_n^{u,v}(x, y \mid \lambda) = \begin{cases} \frac{1}{2} \sum\limits_{\lambda_{nk}<\lambda} u_{nk}(x)\, v_{nk}(y) & (\lambda > 0), \\ -\frac{1}{2} \sum\limits_{\lambda \leq -\lambda_{nk}} -u_{nk}(x)\, v_{nk}(y) & (\lambda < 0), \end{cases}$$

$$(1.7d)\qquad \theta_n^{v,u}(x, y \mid \lambda) = \begin{cases} \frac{1}{2} \sum\limits_{\lambda_{nk}<\lambda} v_{nk}(x)\, u_{nk}(y) & (\lambda > 0), \\ -\frac{1}{2} \sum\limits_{\lambda \leq -\lambda_{nk}} -v_{nk}(x)\, u_{nk}(y) & (\lambda < 0). \end{cases}$$

It is observed that $\theta_n^{v,u}$ is $\theta_n^{u,v}$ with the u_{nk} and the v_{nv} interchanged.

On the basis of (16) we infer existence of *four spectral functions of* K(x, y)

$$(1.8)\qquad \begin{cases} \theta^{u,u}(x, y \mid \lambda) = \theta^*(x, y \mid \lambda) & = \lim \theta_{n_j}^{u,u}(x, y \mid \lambda), \\ \theta^{v,v}(x, y \mid \lambda) = \theta^*(x+1, y+1 \mid \lambda) = \lim \theta_{n_j}^{v,v}(x, y \mid \lambda) \end{cases}$$

and

$$(1.8a)\qquad \begin{cases} \theta^{u,v}(x, y \mid \lambda) = \theta^*(x, y+1 \mid \lambda) = \lim \theta_{n_j}^{u,v}(x, y \mid \lambda), \\ \theta^{v,u}(x, y \mid \lambda) = \theta^*(x+1, y \mid \lambda) = \lim \theta_{n_j}^{v,u}(x, y \mid \lambda) \end{cases}$$

for $0 < x, y < 1$. Furthermore

$$(1.8b)\qquad \theta_n^{u,v}(x, y \mid \lambda) = \theta_n^{v,u}(y, x \mid \lambda), \qquad \theta^{u,v}(x, y \mid \lambda) = \theta^{v,u}(y, x \mid \lambda).$$

In accordance with (C; 40)

$$\int_0^2 \int_0^2 \mathrm{H}(x, y)\, g(x)\, h(y)\, dx\, dy$$
$$= \int_{-\infty}^{\infty} \frac{1}{\lambda}\, d\lambda \int_0^2 \int_0^2 \theta^*(x, y \mid \lambda)\, g(x)\, h(y)\, dx\, dy,$$

whenever $g, h \subset \mathrm{L}_2$ [on (0, 2)] and

$$(1.9)\qquad \int_0^2 \mathrm{H}(x) \mid g(x) \mid dx < +\infty \qquad \left[\mathrm{H}^2(x) = \int_0^2 \mathrm{H}^2(x, y)\, dy\right].$$

If in the above we put

$$g(x) = 0 \quad (1 < x < 2), \qquad h(y) = 0 \quad (0 < y < 1),$$

it is inferred that

$$\int_0^1\int_0^1 K(x, t)\, g(x)\, h(t+1)\, dx\, dt$$
$$=\int_{-\infty}^{\infty}\frac{1}{\lambda}\, d\lambda \int_0^1\int_0^1 \theta^*(x, t+1 \mid \lambda)\, g(x)\, h(t+1)\, dx\, dt.$$

As a consequence of (1.8*a*) *one finds that*

(1.10)
$$\int_0^1\int_0^1 K(x, y)\, g(x)\, h(y)\, dx\, dy$$
$$=\int_{-\infty}^{\infty}\frac{1}{\lambda}\, d\lambda \int_0^1\int_0^1 \theta^{u,v}(x, y \mid \lambda)\, g(x)\, h(y)\, dx\, dy$$

whenever $g,\ h \subset L_2$ *on* $(0,\ 1)$, *while* [by (1.9)]

(1.10*a*)
$$\int_0^1 K'(x)\, |g(x)|\, dx < \infty \qquad \left[K'^2(x)=\int_0^1 K^2(x, t)\, dt\right].$$

On letting

$$g(x)=0 \quad (0<x<1), \qquad h(y)=0 \quad (1<y<2),$$

we deduce

$$\int_0^1\int_0^1 K(y, t)\, g(t+1)\, h(y)\, dt\, dy$$
$$=\int_{-\infty}^{\infty}\frac{1}{\lambda}\, d\lambda \int_0^1\int_0^1 \theta^*(t+1, y \mid \lambda)\, g(t+1)\, h(y)\, dt\, dy.$$

Thus, in view of (1.8*a*), (1.9), *one has*

(1.11)
$$\int_0^1\int_0^1 K(y, x)\, g(x)\, h(y)\, dx\, dy$$
$$=\int_{-\infty}^{\infty}\frac{1}{\lambda}\, d\lambda \int_0^1\int_0^1 \theta^{v,u}(x, y \mid \lambda)\, g(x)\, h(y)\, dx\, dy$$

whenever $g,\ h \subset L_2$ *on* $(0, 1)$, *while* [by (1.9)]

(1.11*a*)
$$\int_0^1 K''(x)\, |g(x)|\, dx < +\infty \qquad \left[K''^2(x)=\int_0^1 K^2(x, t)\, dt\right].$$

Similar to the theorems in (C; 43) are the following résults previously indicated by the present author [1] and really a consequence of (1.10), (1.11 a).

The relation

$$(1.12) \qquad \int_0^1 K(x, t)\, g(t)\, dt = \int_{-\infty}^{\infty} \frac{1}{\lambda}\, d_\lambda \int_0^1 \theta^{u,v}(x, t \mid \lambda)\, g(t)\, dt$$

holds, whenever $g(t) \subset L_2$. Similarly

$$(1.13) \qquad \int_0^1 K(t, x)\, g(t)\, dt = \int_{-\infty}^{\infty} \frac{1}{\lambda}\, d_\lambda \int_0^1 \theta^{v,u}(x, t \mid \lambda)\, g(t)\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{\lambda}\, d_\lambda \int_0^1 \theta^{u,v}(t, x \mid \lambda)\, g(t)\, dt$$

for all $g(t) \subset L_2$. It is to be noted that (1.12), (1.13) *are limits of the same formulas with* K_n, θ_n *in place of* K, θ.

It is known [*cf.* (C; 47, 19)] that the solutions, $\subset L_2$, of the equation

$$(1.14) \qquad \int_0^2 H(x, y)\, \varphi(y)\, dy = 0$$

form a " linear closed set "; thus there exists a " base "

$$(1.14a) \qquad \varphi_1(x), \quad \varphi_2(x), \quad \ldots,$$

which may be chosen as a sequence orthonormal on (0, 2), so that for every solution φ of (1.14) there exist numbers $c_1, c_2, \ldots$ for which

$$(1.14b) \qquad \varphi(x) \sim \sum_1^{\infty} c_\nu \varphi_\nu(x), \qquad \sum c_\nu^2 < +\infty,$$

with $\sim$ *denoting convergence in the mean square.* In accordance with (C; 48) every $h(x) \subset L_2$ [on (0, 2)], is expressible in the form

$$(\alpha) \qquad h(x) \sim \sum_{\nu=1}^{N} c_\nu \varphi_\nu(x) + \int_{-l}^{l} d_\lambda \int_0^2 \theta^*(x, y \mid \lambda)\, h(y)\, dy \qquad [\text{as } N,\ l \to +\infty].$$

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(1) W. J. TRJITZINSKY, *Singular non linear integral equations* (*Duke Math. Journal*, val. 11, n° 3, 1944, p. 517-564); *cf.* in particular, p. 518-521.

Also, with $h, f \subset L_2$ [on (0, 2)], on has

$$(\beta) \quad \int_0^2 fh\,dx = \sum_p f_p h_p - \sum_{p,q} H_{p,q} f_p h_q + \int_{-\infty}^{\infty} d_\lambda \int_0^2 \int_0^2 \theta^*(x, y \mid \lambda) f(x) h(y)\,dx\,dy,$$

where

$$f = \int_0^2 f(x)\varphi_p(x)\,dx, \qquad h_q = \int_0^2 h(x)\varphi_q(x)\,dx,$$

$$H_{p,q} = \int_{-\infty}^{\infty} d_\lambda \int_0^2 \int_0^2 \theta^*(x, y \mid \lambda)\varphi_p(x)\varphi_q(y)\,dx\,dy.$$

We observe that, if φ is a solution of (1.14), the functions

$$(1.15) \qquad u(x) = \varphi(x), \qquad v(x) = \varphi(x+1) \qquad [\text{on } (0, 1)]$$

satisfy the equations

$$(1.15a) \qquad \int_0^1 u(x) K(x, y)\,dx = 0, \qquad \int_0^1 K(x, y) v(y)\,dy = 0,$$

respectively. Conversely, if u, v are solutions of (1.15a) the function $\varphi(x)$ defined on (0, 2) by the relations (1.15), will constitute a solution of (1.14). In view of these considerations it is seen that *the kernel* $H(x, y)$ *is closed* [that is, all the $\varphi_v(x)$ are zero] *if and only if* $K(x, y)$ *is closed on the left and on the right.* In this connection *closure of* $K(x, y)$ *on the left* (*right*) *signifies that every solution* $u[v]$, $\subset L_2$, *of the equation*

$$\int_0^1 u(x) K(x, y)\,dx = 0 \qquad \left[\int_0^1 K(x, y) v(y)\,dy = 0\right]$$

is necessarily zero.

With the aid of (β) and (1.8) we conclude that *for* $f, h, \subset L_2$ *on* (0, 1), *one has*

$$(1.16) \quad \int_0^1 fh\,dx = \sum_p f_p h_p - \sum_{p,q} H_{p,q} f_p h_q + \int_{-\infty}^{\infty} d_\lambda \int_0^1 \int_0^1 \theta^{u,u}(x, y \mid \lambda) f(x) h(y)\,dx\,dy,$$

where

$$(1.16a)\quad \left\{\begin{aligned} & f_p=\int_0^1 f(x)\,u_p(x)\,dx, \qquad h_q=\int_0^1 h(x)\,u_q(x)\,dx, \\ & \int_0^1 u_p(x)\,\mathrm{K}(x,y)\,dx = 0 \\ & [\mathrm{H}_{p,q} \text{ as in } (\beta);\ u_p(x)=\varphi_p(x) \text{ on } (0,1)]; \end{aligned}\right.$$

moreover,

$$(1.17)\quad \int_0^1 fh\,dx=\sum_p h_p f_p-\sum_{p,q}\mathrm{H}_{p,q} f_p h_q + \int_{-\infty}^{\infty} d\lambda \int_0^1\int_0^1 \theta^{v,v}(x,y\,|\,\lambda)\,f(x)\,h(y)\,dx\,dy,$$

where

$$(1.17\,a)\quad \left\{\begin{aligned} & f_p=\int_0^1 f(x)\,v_p(x)\,dx, \qquad h_q=\int_0^1 h(x)\,v_q(x)\,dx, \\ & \int_0^1 \mathrm{K}(x,y)\,v_p(y)\,dy = 0. \end{aligned}\right.$$

In α we let $h(x)=0$ on $(1,2)$, obtaining

$$\lim_{N,l}\int_0^2\left|h(x)-\sum_1^N c_\nu\varphi_\nu(x)-\int_{-l}^{l} d\lambda\int_0^1 \theta^{*}(x,y\,|\,\lambda)\,h(y)\,dy\right|^2 dx$$

$$=\lim_{N,l}\left\{\int_0^1\left|h(x)-\sum_1^N c_\nu u_\nu(x)-\int_{-l}^{l} d\lambda\int_0^1 \theta^{u,u}(x,y\,|\,\lambda)\,h(y)\,dy\right|^2 dx + \int_0^1\left|-\sum_1^N c_\nu v_\nu(x)-\int_{-l}^{l} d\lambda\int_0^1 \theta^{v,u}(x,y\,|\,\lambda)\,h(y)\,dy\right|^2 dx\right\}=0;$$

necessarily each of the terms in $\{\ldots\}$, above, being positive tends to zero; *accordingly*

$$(\alpha^*)\qquad h(x)\sim\sum_1^N c_\nu u_\nu(x)+\int_{-l}^{l} d\lambda\int_0^1 \theta^{u,u}(x,y\,|\,\lambda)\,h(y)\,dy$$

and, similarly,

$$(\alpha^{**})\qquad h(x)\sim\sum_1^N c_\nu v_\nu(x)+\int_{-l}^{l} d\lambda\int_0^1 \theta^{v,v}(x,y\,|\,\lambda)\,h(y)\,dy,$$

the two formulas being valid (for some c_ν for wich $c_1^2+c_2^2+\ldots$ converges) *whenever* $h(x)\subset L_2$ *on* (0, 1).

It follows, further, that

$$(\mathbf{1}.18)\qquad \int_0^1 fh\,dx=\int_{-\infty}^{\infty} d_\lambda \int_0^1\int_0^1 \theta^{u,u}(x,y\mid\lambda)f(x)h(y)\,dx\,dy,$$

whenever $K(x, y)$ *is closed on the left* [*cf.* (**1**.16*a*)]. *Also*

$$(\mathbf{1}.19)\qquad \int_0^1 fh\,dx=\int_{-\infty}^{\infty} d_\lambda \int_0^1\int_0^1 \theta^{v,v}(x,y\mid\lambda)f(x)h(y)\,dx\,dy,$$

if $K(x, y)$ *is closed on the right* [*cf.* (**1**.17*a*)].

The above, incidentally, implies that $\theta^{u,u}(x, y\mid\lambda)$ *is a closed spectral function* (that is, leads to a relation of Parseval type) *if* $K(x, y)$ *is closed on the left*; $\theta^{v,v}(x, y\mid\lambda)$ *is a closed spectral function if* $K(x, y)$ *is closed on the right.*

2. Non symmetric kernels (continued). — *In this section* $K(x, y)$ *is measurable, possibly non symmetric, such that there exist linear functionals* (of the type previously used by Carleman)

$$L_x(\xi\mid\ldots),\qquad R_x(\eta\mid\ldots),$$

where ξ, η *are parameters of any kind*, with $K_n(x, y)$ from (**1**.1), *one has the following*

$$(\mathbf{2}.1_0)\qquad L_x\big(\xi\mid K(x,y)\big)\subset L_2 \quad \text{in } y;$$

$$(\mathbf{2}.2_0)\qquad \big|L_x\big(\xi\mid K_n(x,y)\big)\big|<\gamma(\xi\mid y),$$

where $\gamma(\xi\mid y)$ (independent of n) is L_2 in y

$$(\mathbf{2}.3_0)\qquad \lim_n L_x\big(\xi\mid K_n(x,y)\big)=L_x\big(\xi\mid K(x,y)\big);$$

$$(\mathbf{2}.4_0)\qquad \lim_n L_x\big(\xi\mid f_n(x)\big)=L_x\big(\xi\mid f(x)\big),$$

when $f_n(x)\underset{(w)}{\rightarrow} f(x)$ (weak convergence in L_2 on (0, 1)

$$(\mathbf{2}.5_0)\qquad \int_0^1 L_x\big(\xi\mid K_n(x,y)\big)\varphi(y)\,dy=L_x\left(\xi\,\middle|\int_0^1 K_n(x,y)\varphi(y)\,dy\right)$$

for all $\varphi \subset L$,

$$(\mathbf{2}.1^0) \qquad R_y(\eta \mid K(x, y)) \subset L_2 \quad \text{in } x;$$

$$(\mathbf{2}.2^0) \qquad \left| R_y(\eta \mid K_n(x, y)) \right| < \delta(\eta \mid x),$$

where $\delta(\eta \mid x)$ is L_2 in x;

$$(\mathbf{2}.3^0) \qquad \lim_n R_y(\eta \mid K_n(x, y)) = R_y(\eta \mid K(x, y)),$$

$$(\mathbf{2}.4^0) \qquad \lim_n R_y(\eta \mid f_n(y)) = R_y(\eta \mid f(y)),$$

when $f_n \underset{(\text{II}^0)}{\rightarrow} f$ on $(0, 1)$;

$$(\mathbf{2}.5^0) \quad \int_0^1 \psi(x) R_y(\eta \mid K_n(x, y))\, dx = R_y\left(\eta \,\middle|\, \int_0^1 \psi(x) K_n(x, y)\, dx\right)$$

for all $\psi \subset L_2$.

When $K(x, y)$ is symmetric one may take

$$R(\eta \mid \ldots) = L(\xi \mid \ldots).$$

$L_x(\xi \mid K(x, y))$, $R_y(\eta \mid K(x, y))$ *may be non measurable in the parameters* ξ, η.

DEFINITION 2.1. — *We define a functional* $T_x(\zeta \mid h(x))$, *where* $h(x)$ *is defined for* $0 \leqq x \leqq 2$ *as follows*

$$(\mathbf{2}.1a) \qquad T_x(\zeta \mid h(x)) = L_x(\xi \mid h(x)) + R_x(\eta \mid h(1 + x)),$$

where ζ *stands for* (ξ, η); *the above is stated only for such* $h(x)$ $(0 \leqq x \leqq 2)$ *for which the two terms in the second member exist.*

We observe that T_x is a linear functional. As a consequence of the definition of $H(x, y)$ (as given in section 1)

$$(\mathbf{2}.2) \qquad T_x(\zeta \mid H(x, y)) = R_x(\eta \mid K(y, x)) \qquad (y \text{ on } (0, 1)),$$

$$T_x(\zeta \mid H(x, y)) = L_x(\xi \mid K(x, y - 1)) \qquad (y \text{ on } (1, 2));$$

$(\mathbf{2}.2)$ also holds with $H_n(x, y)$, $K_n(x, y)$ in place of $H(x, y)$, $K(x, y)$.

By $(\mathbf{2}.2)$, $(\mathbf{2}.1_0)$-$(\mathbf{2}.5_0)$, $(\mathbf{2}.1^0)$-$(\mathbf{2}.5^0)$ one has for x, y on $(0, 2)$

$$(\mathbf{2}.\text{I}) \qquad T_x(\zeta \mid H(x, y)) \subset L_2 \quad \text{in } y;$$

$$(\mathbf{2}.\text{II}) \qquad \left| T_x(\zeta \mid H_n(x, y)) \right| < \alpha(\zeta \mid y),$$

where

$$\alpha(\zeta \mid y) = \delta(\eta \mid y) \qquad (y \text{ on } (0, 1)),$$
$$\alpha(\zeta \mid y) = \gamma(\xi \mid y - 1) \qquad (y \text{ on } (1, 2))$$

and $\alpha(\zeta \mid y)$ is L_2 in y.

(**2**.III) $$\lim_n T_x(\zeta \mid H_n(x, y)) = T_x(\zeta \mid H(x, y));$$

(**2**.IV) $$\lim_n T_x(\zeta \mid f_n(x)) = T_x(\zeta \mid f(x))$$

when $f_n(x) \underset{(m)}{\rightarrow} f(x)$ on $(0, 2)$

(**2**.V) $$\int_0^2 T_x(\zeta \mid H_n(x, y)) \psi(y)\, dy = T_x\left(\zeta \,\middle|\, \int_0^2 H_n(x, y) \psi(y)\, dy\right)$$

for all $\psi(y) \subset L_2$.

Corresponding to the equation (**1**.14) we have

(**2**.3) $$\int_0^2 T_x(\zeta \mid H(x, y)) \varphi(y)\, dy = 0.$$

The kernel

$$T(\zeta, y) = T_x(\zeta \mid H(x, y)).$$

has a "base" consisting of a sequence of functions $\{\varphi_\nu(x)\}$ $(\nu = 1, 2, \ldots)$, orthonormal on $(0, 2)$; that is,

(**2**.3 *a*) $$\int_0^2 T(\zeta, y) \varphi_\nu(y)\, dy = 0,$$

while every solution $\varphi(y)$, L_2 on $(0, 2)$, of (**2**.3) is representable in the mean square as

(**2**.3 *b*) $$\varphi(y) \sim \Sigma c_\nu \varphi_\nu(y) \qquad (c_1^2 + c_2^2 + \ldots < \infty).$$

With e'_x, e'_y denoting measurable sets on the intervals

$$0 \leq x \leq 2, \qquad 0 \leq y \leq 2,$$

respectively, we form

(**2**.4) $$\Omega_n(e'_x, e'_y \mid \lambda) = \int_{e'_x} \int_{e'_y} \theta_n^*(x, y \mid \lambda)\, dx\, dy.$$

where θ_n^* is from (**1**.5). One has

$$\Omega_n(e'_x, e'_y \mid \lambda) = \sum_{0<\gamma_{n\nu}<\lambda} \int_{e'_x} \psi_{n\nu}(x)\,dx \int_{e'_y} \psi_{n\nu}(y)\,dy \qquad (\lambda > 0),$$

$$\Omega_n(e'_x, e'_y \mid \lambda) = -\sum_{\lambda \leq \gamma_{n\nu}<0} \int_{e'_x} \psi_{n\nu}(x)\,dx \int_{e'_y} \psi_{n\nu}(y)\,dy \qquad (\lambda < 0).$$

Thus

$$\Omega_n^2(e'_x, e'_y \mid \lambda) \leq \sum_\nu \left(\int_{e'_x} \psi_{n\nu}(x)\,dx\right)^2 \sum_\nu \left(\int_{e'_y} \psi_{n\nu}(y)\,dy\right)^2.$$

Accordingly

$$\Omega_n^2(e'_x, e'_y \mid \lambda) \leq m(e'_x)\,m(e'_y) \qquad (n = 1, 2, \ldots);$$

here $m(e)$ = measure of e. The above inequalities signify that the absolute continuity of the additive function Ω_n of two sets (e'_x, e'_y) is uniform with respect to n. We infer existence of a sequence (n_j) $(n_j \to \infty$ with $j)$ so that the limit

$$\text{(2.5)} \qquad \lim_{n_j} \Omega_{n_j}(e'_x, e'_y \mid \lambda) = \Omega(e'_x, e'_y \mid \lambda)$$

exists (n_j independent of the sets and of λ); Ω is additive and absolutely continuous in the sets e'_x, e'_y; one has

$$|\Omega(e'_x, e'_y \mid \lambda)| \leq [m(e'_x)\,m(e'_y)]^{\frac{1}{2}}.$$

Let e_x be a measurable set in the interval $0 \leq x \leq 1$; designate by e_{x+1} the set of points $x+1$ such that x is in e_x; e_{x+1} will be a set on the interval $(1, 2)$. Similarly we define sets e_y, e_{y+1}. We write

$$\text{(2.6)} \quad \begin{cases} \Omega_n^{uu}(e_x, e_y \mid \lambda) = \Omega_n(e_x, e_y \mid \lambda), & \Omega_n^{vv}(e_x, e_y \mid \lambda) = \Omega_n(e_{x+1}, e_{y+1} \mid \lambda), \\ \Omega_n^{uv}(e_x, e_y \mid \lambda) = \Omega_n(e_x, e_{y+1} \mid \lambda), & \Omega_n^{vu}(e_x, e_y \mid \lambda) = \Omega_n(e_{x+1}, e_y \mid \lambda), \end{cases}$$

and by (**2**.5), in the limit,

$$\text{(2.6a)} \quad \begin{cases} \Omega^{uu}(e_x, e_y \mid \lambda) = \Omega(e_x, e_y \mid \lambda), & \Omega^{vv}(e_x, e_y \mid \lambda) = \Omega(e_{x+1}, e_{y+1} \mid \lambda), \\ \Omega^{uv}(e_x, e_y \mid \lambda) = \Omega(e_x, e_{y+1} \mid \lambda), & \Omega^{vu}(e_x, e_y \mid \lambda) = \Omega(e_{x+1}, e_y \mid \lambda). \end{cases}$$

By (**2**.4), (**1**.5a), (**1**.4b), (**1**.4c) it is inferred that

$$\text{(2.7)} \qquad \Omega_n^{uu}(e_x, e_y \mid \lambda) = \int_{e_x}\int_{e_y} \theta_n^{u,u}(x, y \mid \lambda)\,dx\,dy \qquad [\textit{cf.}\ (\mathbf{1}.7a)].$$

With suitable adaptations most developments in (C; 130-146) will hold for Ω; in this connection the symbols

$$\frac{d}{dx}, \quad \frac{\partial}{\partial y}, \quad \ldots,$$

when applied to an additive function of sets, are to denote "derivation" in set-functional sense.

We shall now state without proof a number of formulas [*cf.* (**2**.8)-(**2**.8*e*), below], closely analogous to certain resulte in (C; 130-143)

$$(\mathbf{2}.8) \qquad \int_{\lambda_1}^{\lambda_2} \alpha(\lambda)\, d_\lambda \int_0^2 g(x) \left[\frac{d}{dx}\int_0^2 \frac{\partial}{\partial y}\, \Omega(e_x, e_y \mid \lambda) \quad h(y)\, dy\right] dx$$
$$= \int_0^2 g(x) \frac{d}{dx}\left[\int_{\lambda_1}^{\lambda_2} \alpha(\lambda)\, d_\lambda \int_0^2 \frac{\partial}{\partial y}\, \Omega(e_x, e_y \mid \lambda) \quad h(y)\, dy\right] dx$$
$$= \int_0^2 g(x) \left\{\frac{d}{dx}\int_0^2 \frac{\partial}{\partial y}\left[\int_{\lambda_1}^{\lambda_2} \alpha(\lambda)\, d_\lambda \Omega(e_x, e_y \mid \lambda)\right] h(y)\, dy\right\} dx$$

[$g, h \subset L_2$ on $(0, 2)$; $-\infty < \lambda_1 < \lambda_2 < +\infty$; $\alpha(\lambda)$ of bounded variation for $\lambda_1 \leqq \lambda \leqq \lambda_2$];

$$(\mathbf{2}.8\,a) \qquad \int_{-\infty}^{\infty} d_\lambda \int_0^2 h(x) \left[\frac{d}{dx}\int_0^2 \frac{\partial}{\partial y}\Omega(e_x, e_y \mid \lambda)\, h(y)\, dy\right] dx \leqq \int_0^2 h^2\, dx$$

for $h \subset L_2$; Ω is defined to be *closed* if (**2**.8 *a*) holds with the equality sign for all $h \subset L_2$; if Ω is closed one has

$$(\mathbf{2}.8\,b) \qquad \int_{-\infty}^{\infty} d_\lambda \int_0^2 g(x) \left[\frac{d}{dx}\int_0^2 \frac{\partial}{\partial y}\Omega(e_x, e_y \mid \lambda)\, h(y)\, dy\right] dx = \int_0^2 gh\, dx$$

for all $g, h \subset L_2$ on $(0, 2)$; if Ω is closed one has

$$h(x) = \frac{d}{dx}\int_{-\infty}^{\infty} d_\lambda \int_0^2 \frac{\partial}{\partial y}\Omega(e_x, e_y \mid \lambda)\, h(y)\, dy$$

(for almost all x) for all $h \subset L_2$;

there exists a function $F(x)$, $\subset L_2$, so that

$$(\mathbf{2}.8\,c) \qquad F_l(x) \equiv f(x) - \frac{d}{dx}\int_{-l}^{l} d_\lambda \int_0^2 \frac{\partial}{\partial y}\Omega(e_x, e_y \mid \lambda) f(y)\, dy,$$
$$\underset{(w)}{\to} F(x) \qquad (\text{as } l \to \infty;\ \text{all } f \subset L_2);$$

the function $F(x)$, involved above, satisfies

$$(\mathbf{2}.8d)\qquad \int_0^2 T_x\big(\zeta \mid H(x, y)\big)\, F(y)\, dy = 0;$$

if

$$(\mathbf{2}.8e)\qquad \int_0^2 T_x\big(\zeta \mid H(x, y)\big)\, \varphi(y)\, dy = 0$$

has no solutions, $\subset L_2$ on $(0, 2)$ with $\int_0^2 \varphi^2\, dx \neq 0$, then Ω is closed.

We shall now prove the following analogue of (C; Theorem IV, p. 48).

Theorem 2.9. — *With* $\{\varphi_\nu\}$ *designating the base introduced subsequent* (**2**.3) *given* $h(x) \subset L_2$ *on* $(0,2)$, *there exist* c_ν *so that on writing*

$$(\mathbf{2}.9a)\qquad q_{N,l}(x) = \sum_{\nu=1}^{N} c_\nu \varphi_\nu(x) + \psi_l(x),$$

$$(\mathbf{2}.9b)\qquad \psi_l(x) = \frac{d}{dx}\int_{-l}^{l} d_\lambda \left[\int_0^2 \frac{\partial}{\partial y}\Omega(e_x, e_y \mid \lambda)\, h(y)\, dy\right],$$

we have

$$(\mathbf{2}.9c)\qquad q_{N,l}(x) \sim h(x) \qquad \big(\text{on } (0, 2);\ \text{as } N, l \to \infty\big).$$

If f *is also* L_2 *on* $(0, 2)$ *one has*

$$(\mathbf{2}.9d)\qquad \int_0^2 f(x)\, h(x)\, dx = \sum_p f_p h_p - \sum_{p,q} H_{pq} f_p h_q + \int_{-\infty}^{\infty} d_\lambda \int_0 f(x)$$
$$\dot{\times}\left[\frac{d}{dx}\int_0^2 \frac{\partial}{\partial y}\Omega(e_x, e_y \mid \lambda)\, h(y)\, dy\right] dx,$$

with

$$(\mathbf{2}.9e)\qquad f_p = \int_0^2 f(x)\, \varphi_p(x)\, dx, \qquad h_p = \int_0^2 h(x)\, \varphi_p(x)\, dx,$$

$$(\mathbf{2}.9f)\qquad H_{pq} = \int_{-\infty}^{\infty} d_\lambda \int_0^2 \varphi_p(x) \left[\frac{d}{dx}\int_0^2 \frac{\partial}{\partial y}\Omega(e_x, e_y \mid \lambda)\, \varphi_q(y)\, dy\right] dx.$$

By (**2**.8c), (**2**.8d) for some $H(x)$, $\subset L_2$,

$$h(x) - \psi_l(x) \underset{(w)}{\rightarrow} H(x) \qquad \big(\text{as } l \to \infty;\ x \text{ on } (0, 2)\big),$$

where

$$\int_0^2 T_x\big(\zeta \mid H(x, y)\big)\, H(y)\, dy = 0.$$

Hence by (**2**.3*b*)

$$H(x) \sim \sum_\nu c_\nu\, \varphi_\nu(x) \qquad \big(\text{on } (0, 2)\big)$$

for some c_ν such that $c_1^2 + c_2^2 + \ldots$ converges. This establishes (**2**.9*c*).

We shall now prove that there exists a function $t_\nu(x)$, $\subset L_2$, so that

$$(\mathbf{2}.10) \qquad \frac{d}{dx}\int_{-\infty}^{\infty} d_\lambda \frac{\partial}{\partial y}\Omega(e_x, e_y \mid \lambda)\, \varphi_\nu(y)\, dy \sim t_\nu(x),$$

$$\sum_p H_{\nu p}\, \varphi_p(x) \sim t_\nu(x) \qquad \big(\text{on } (0, 2)\big).$$

As a consequence of (**2**.9*c*), applied to $h(y) = \varphi_p(y)$, we obtain

$$(\mathbf{2}.11) \qquad \sum_{\nu=1}^{N} c_{p\nu}\, \varphi_\nu(x) + \psi_l^p(x) \sim \varphi_p(x) \qquad \big(\text{as } N,\ l \to \infty;\ \text{on } (0, 2)\big),$$

(some $c_{p\nu}$), where

$$\psi_l^p(x) = \frac{d}{dx}\int_{-l}^{l} d_\lambda \left[\int_0^2 \frac{\partial}{\partial y}\Omega(e_x, e_y \mid \lambda)\, \varphi_p(y)\, dy\right];$$

by (**2**.8*c*)

$$\psi_l^p(x) \underset{(w)}{\rightarrow} \psi^p(x),$$

where $\psi^p(x)$ is some function $\subset L_2$; convergence here is, in fact, in the mean square. In view of (**2**.11)

$$\begin{aligned} c_{p\nu} &= \int_0^2 \big(\varphi_p(x) - \psi^p(x)\big)\varphi_\nu(x)\, dx \\ &= \delta_{\nu p} - \lim_l \int_0^2 \psi_l^p(x)\, \varphi_\nu(x)\, dx \\ &= \delta_{\nu p} - \lim_l \int_0^2 \varphi_\nu(x)\left[\frac{d}{dx}\int_{-l}^{l} d_\lambda \int_0^2 \frac{\partial}{\partial y}\Omega(e_x, e_y \mid \lambda)\, \varphi_p(y)\, dy\right] dx \\ &= \delta_{\nu p} - \lim_l \int_{-l}^{l} d_\lambda \int_0^2 \varphi_\nu(x)\left[\frac{d}{dx}\int_0^2 \frac{\partial}{\partial y}\Omega(e_x, e_y \mid \lambda)\, \varphi_p(y)\, dy\right] dx, \end{aligned}$$

where $\delta_{\nu\nu}=1$, $\delta_{\nu p}=0(\nu\neq p)$. Thus by (**2**.9$f$)

$$c_{p\nu}=\delta_{\nu p}-H_{\nu p}.$$

Hence (**2**.11) becomes

$$-\sum_{\nu=1}^{N} H_{\nu p}\,\varphi_\nu(x)+\psi_l^p(x)\sim 0 \qquad (\text{as } N,\ l\to\infty);$$

(**2**.10) will follow.

We observe that c_ν in (**2**.9a) is the ν-th Fourier coefficient of the function $H(x)$ introduced subsequent (**2**.9f). One has

$$c_\nu=\int_0^2 H(x)\,\varphi_\nu(x)\,dx=\lim_l\int_0^2[h(x)-\psi_l(x)]\,\varphi_\nu(x)\,dx$$
$$=h_\nu-\lim_l\int_0^2\psi_l(x)\,\varphi_\nu(x)\,dx=h_\nu-\lim\alpha_l(x),$$

where

$$\alpha_l(x)=\int_0^2\varphi_\nu(x)\left\{\frac{d}{dx}\int_{-l}^{l}d\lambda\left[\int_0^2\frac{\partial}{\partial y}\Omega(e_x,e_y\,|\,\lambda)\,h(y)\,dy\right]\right\}dx$$
$$=\int_0^2 h(y)\left[\frac{d}{dy}\int_{-l}^{l}d\lambda\int_0^2\frac{\partial}{\partial x}\Omega(e_x,e_y\,|\,\lambda)\,\varphi_\nu(x)\,dx\right]dy,$$

Now, by (**2**.8c), (**2**.8d)

$$\frac{d}{dy}\int_{-l}^{l}d\lambda\int_0^2\frac{\partial}{\partial x}\Omega(e_x,e_y\,|\,\lambda)\,\varphi_\nu(x)\,dx$$

converges weakly (as $l\to\infty$) to the function

$$\lambda_\nu(y)=\frac{d}{dy}\int_{-\infty}^{\infty}d\lambda\int_0^2\frac{\partial}{\partial x}\Omega(e_x,e_y\,|\,\lambda)\,\varphi_\nu(x)\,dx.$$

Therefore

$$c_\nu=h_\nu-\int_0^2 h(y)\,\lambda_\nu(y)\,dy.$$

In view of (**2**.10)

$$(\mathbf{2}.12)\qquad c_\nu=h_\nu-\lim_N\int_0^2 h(y)\sum_{p=1}^{N}H_{\nu p}\,\varphi_p(x)\,dx=h_\nu-\sum_p H_{\nu p}h_p.$$

With $q_{n,n}(x)$ defined by (**2**.9a) ($N=n$, $l=n$) we form the integral

$$\int_0^2 f(x)\, q_{n,n}(x)\, dx = \sum_{\nu=1}^{N} c_\nu f_\nu + \int_0^2 f(x) \times \left\{ \frac{d}{dx} \int_{-n}^{n} d_\lambda \left[\int_0^2 \frac{\partial}{\partial y} \Omega(e_x, e_y \mid \lambda)\, h(y)\, dy \right] \right\} dx.$$

By (**2**.8) and (**2**.12)

$$\int_0^2 f(x)\, q_{n,n}(x)\, dx = \sum_{\nu=1} \left(h_\nu - \sum_p^n \mathrm{H}_{\nu p} h_p \right) f_\nu + \int_{-n}^{n} d_\lambda \int_0^2 f(x) \times \left[\frac{d}{dx} \int_0^2 \frac{\partial}{\partial y} \Omega(e_x, e_y \mid \lambda)\, h(y)\, dy \right] dx.$$

Since, in view of (**2**.9c), $q_{n,n} \sim h$, in the limit we obtain (**2**.9d). This establishes the theorem.

If φ, $\subset L_2$ on (0,2), is a solution of

$$\int_0^2 \mathrm{T}_x\big(\zeta \mid \mathrm{H}(x, y)\big)\, \varphi(y)\, dy = 0,$$

then the functions

$$(\mathbf{2}.13) \qquad u(x) = \varphi(x), \qquad v(x) = \varphi(x+1) \qquad \big(x \text{ on } (0, 1)\big)$$

satisfy the equation

$$(\mathbf{2}.14) \qquad \int_0^1 \Big[\mathrm{R}_x\big(\eta \mid \mathrm{K}(y, x)\big)\, u(y) + \mathrm{L}_x\big(\xi \mid \mathrm{K}(x, y)\big)\, v(y) \Big]\, dy = 0.$$

From the above we conclude that if $\mathrm{R}_x\big(\eta \mid \mathrm{K}(y, x)\big)$ or $\mathrm{L}_x\big(\xi \mid \mathrm{K}(x, y)\big)$ is not closed then $\mathrm{T}_x\big(\zeta \mid \mathrm{H}(x, y)\big)$ is not closed.

In (**2**.9d) we put

$$f(x) = h(x) \quad \big(\text{on } (0, 1)\big), \qquad f = h = 0 \quad \big(\text{on } (1, 2)\big);$$

with the aid of (**2**.6a) it is then inferred that

$$(\mathbf{2}.15) \qquad \left\{ \begin{aligned} & \int_0^1 f^2(x)\, dx = \sum_p f_p^2 - \sum_{p,q} \mathrm{H}_{pq} f_p f_q + \int_{-\infty}^{\infty} d_\lambda \int_0^1 f(x) \\ & \qquad \times \left[\frac{d}{dx} \int_0^1 \frac{\partial}{\partial y} \Omega^{uu}(e_x, e_y \mid \lambda) f(y)\, dy \right] dx, \\ & f_p = \int_0^1 f \varphi_p\, dx. \end{aligned} \right.$$

By virtue of (**2**.8a)

$$(\mathbf{2}.16)\qquad \int_{-\infty}^{\infty} d_{\lambda}\int_0^1 f(x)\left[\frac{d}{dy}\int_0^1 \frac{\partial}{\partial y}\Omega^{uu}(e_x, e_y \mid \lambda)\, h(y)\, dy\right]dx \leqq \int_0^1 f^2(x)\, dx$$

for all $f \subset L_2$ on (0, 1). When Ω is closed [that is, when $T_x(\zeta \mid H(x, y))$ is closed in L_2] then (**2**.16) will hold with the equality sign.

3. THE FIRST KIND EQUATION. — Suppose $f(x)$ subject to

HYPOTHESIS **3**.1, — *On writing*

$$(\mathbf{3}.1a)\qquad f_{nk} = \int_0^1 f(x)\, u_{nk}(x)\, dx,$$

we have

$$(\mathbf{3}.1b)\qquad \rho_n^2 = \sum \lambda_{nk}^2 f_{nn}^2 < +\infty \qquad (n = 1, 2, \ldots).$$

By the Riesz Fisher theorem we infer that there exists a function $h_n(x)$ such that as a consequence of (**1**.2)

$$(\mathbf{3}.2)\qquad h_n(x) \sim \sum_i \lambda_{ni} f_{ni}\, v_{ni}(x), \qquad \int_0^1 h_n(x)\, v_{ni}(x)\, dx = \lambda_{ni} f_{ni}.$$

The function

$$(\mathbf{3}.2a)\qquad F_n(x) \equiv \int_0^1 K_n(x, s)\, h_n(s)\, ds - f(x)$$

has the property

$$(\mathbf{3}.2b)\qquad \int_0^1 F_n(x)\, u_{ni}(x)\, dx = 0 \qquad (i = 1, 2, \ldots);$$

we also have

$$(\mathbf{3}.2c)\qquad \int_0^1 h_n^2(x)\, dx = \rho_n^2.$$

Now by (**2**.5$_0$)

$$L_x(\xi \mid F_n(x)) = \int_0^1 L_x(\xi \mid K_n(x, s))\, h_n(s)\, ds - L_x(\xi \mid f(x)).$$

Hence, as a consequence of (**2**.2$_0$), (**3**.2c),

$$(\mathbf{3}.3)\qquad \begin{aligned} \left| L_x(\xi \mid F_n(x)) \right| &\leqq \left| L_x(\xi \mid f(x)) \right| + \int_0^1 \gamma(\xi \mid s)\, |h_n(s)|\, ds \\ &\leqq \left| L_x(\xi \mid f(x)) \right| + \gamma'(\xi)\rho_n, \end{aligned}$$

where

$$(\mathbf{3}.3\,a) \qquad \gamma'^2(\xi) = \int_0^1 \gamma^2(\xi \mid s)\, ds.$$

If the (ρ_n) are bounded,

$$\rho_n \leqq \rho < +\infty \qquad (n = n_1, n_2, \ldots),$$

then by $(\mathbf{3}.2c)$ one can choose a subsequence of $(h_n(x))$, say

$$h_{n^1}(x), \qquad h_{n^2}(x), \ldots,$$

converging *weakly* (in the space L_2) to a function $h(x)$,

$$(\mathbf{3}.4) \qquad h_{n^j}(x) \underset{(w)}{\rightarrow} h(x) \qquad (\text{as } n^j \rightarrow \infty).$$

In view of $(\mathbf{2}.3_0)$, $(\mathbf{2}.2_0)$ and $(\mathbf{3}.4)$; on making use of a theorem of Carleman (C; p. 20) it is inferred that the limit

$$(\mathbf{3}.5) \qquad \lim_n \int_0^1 L_x\big(\xi \mid K_n(x, s)\big)\, h_n(s)\, ds = \int_0^1 L_x\big(\xi \mid K(x, s)\big)\, h(s)\, ds$$

(as $n = n^j \rightarrow \infty$). Hence by the formula preceding $(\mathbf{3}.3)$

$$(\mathbf{3}.6) \qquad \lim L_x\big(\xi \mid F_n(x)\big) = F^*(\xi) \equiv \int_0^1 L_x\big(\xi \mid K(x, s)\big)\, h(s)\, ds - L_x\big(\xi \mid f(x)\big);$$

moreover, by $(\mathbf{3}.3)$

$$\mid F^*(\xi) \mid \leqq \mid L_x\big(\xi \mid f(x)\big) \mid + \gamma'(\xi)\rho \qquad [cf.\ (\mathbf{3}.3\,a)].$$

We write

$$(\mathbf{3}.7) \qquad F_{n\nu}(x) = \int_0^1 K_n(x, s)\, h_{n,\nu}(s)\, ds - f(x), \qquad h_{n,\nu}(s) = \sum_{i=1}^{\nu} \lambda_{ni} f_{ni} \varphi_{ni}(s).$$

By $(\mathbf{3}.2)$

$$h_{r,\nu}(s) \sim h_n(s) \qquad (\text{as } \nu \rightarrow \infty);$$

hence

$$\lim_\nu F_{n\nu}(x) = \int_0^1 K_n(x, s)\, h_n(s)\, ds - f(x);$$

in view of $(\mathbf{2}.2a)$

$$(\mathbf{3}.7\,a) \qquad F_n(x) = \lim_\nu F_{n\nu}(x).$$

By (**3**.7) and the first relation (**1**.2)

$$F_{n\nu}(x) = f^*_{n,\nu}(x) - f(x), \qquad f^*_{n,\nu}(x) = \sum_{i=1}^{\nu} f_{ni} u_{ni}(x);$$

the limit

$$\lim_{\nu} f^*_{n,\nu}(x) = f^*_n = \sum_{i=1}^{\infty} f_{ni} u_{ni}(x) \tag{3.7b}$$

exists in the sense of ordinary convergence; moreover,

$$F_n(x) = f^*_n(x) - f(x). \tag{3.7c}$$

In view of (**3**.7*b*), even though the sequence $(u_{ni}(x))$ $(i = 1, 2, \ldots)$ may be not complete, one has

$$\int_0^1 f^{*2}_n(x)\,dx = \sum_i f^2_{ni}.$$

Hence by Bessel's inequality one has

$$\int_0^1 f^{*2}_n(x)\,dx \leq \int_0^1 f^2(x)\,dx. \tag{3.8}$$

As a consequence of (**3**.7*c*)

$$\int_0^1 F^2_n(x)\,dx = \int_0^1 f^*_n(x)\,F_n(x)\,dx - \int_0^1 f(x)\,F_n(x)\,dx$$

and

$$\int_0^1 F^2_n(x)\,dx \leq \left\{ \int_0^1 f^{*2}_n(x)\,dx \int_0^1 F^{*2}_n(x)\,dx \right\}^{\frac{1}{2}} + \left\{ \int_0^1 f^2\,(x)\,dx \int_0^1 F^{*2}_n(x)\,dx \right\}^{\frac{1}{2}}.$$

Thus by (**3**.8)

$$\left[\int_0^1 F^2_n(x)\,dx \right]^{\frac{1}{2}} \leq 2 \left[\int_0^1 f^2(x)\,dx \right]^{\frac{1}{2}} < \infty,$$

where the second member is independent of *n*. Hence there exists a function $F(x)$, $\subset L_2$, so that for a sequence n_j

$$F_{n_j}(x) \underset{(w)}{\rightarrow} F(x) \qquad (\text{as } n_j \rightarrow \infty). \tag{3.9}$$

We choose (n_j) as a subsequence of (n^j) [from (**3**.5)]. Thus by (**2**.4_0) and (**3**.6) one has

$$L_x(\xi \mid F(x)) = \int_0^1 L_x(\xi \mid K(x, s))\, h(s)\, ds - L_x(\xi \mid f(x)).$$

We form the expression

$$I_n^{\alpha,\beta} = \int_\alpha^\beta \frac{1}{\lambda}\, d\lambda \int_0^1 F_n(x)\, \theta_n^{u,v}(x, y \mid \lambda)\, dx,$$

where $\alpha < \beta$. Now, in view of (**1**.7c)

$$\theta_n^{u,v}(x, y \mid \lambda + \Delta) - \theta_n^{u,v}(x, y \mid \lambda) = \frac{1}{2} \sum_{\lambda \leq \lambda_{nk} < \lambda + \Delta} u_{nk}(x)\, v_{nk}(y),$$

for $0 \leqq \lambda < \lambda + \Delta$ and

$$\theta_n^{u,v}(x, y \mid \lambda + \Delta) - \theta_n^{u,v}(x, y \mid \lambda) = -\frac{1}{2} \sum_{\lambda \leq -\lambda_{nk} < \lambda + \Delta} u_{nk}(x)\, v_{nk}(y)$$

for $\lambda < \lambda + \Delta \leqq 0$; accordingly

$$I_n^{\alpha,\beta} = \frac{1}{2} \sum_{\alpha \leq \lambda_{nk} < \beta} \frac{1}{\lambda_{nk}} \int_0^1 F_{nk}(x)\, u_{nk}(x)\, v_{nk}(y)\, dx$$

for $0 \leqq \alpha < \beta$, as well as for $\alpha < \beta \leqq 0$; inasmuch as the $\lambda_{nk} \neq 0$, it is seen that the above formula holds for all $\alpha < \beta$. Thus, in view of (**3**.2b),

$$I_n^{\alpha,\beta} = 0, \qquad I_n^{-\infty,+\infty} = 0.$$

Hence by (**1**.13), applied to the kernel $K_n(x, t)$ and the function $F_n(t)$, one has

$$\int_0^1 F_n(t)\, K_n(t, y)\, dt = I_n^{-\infty,+\infty} = 0.$$

Consequently, F_n being L_2, by (**2**.5°) one obtains

$$\int_0^1 F_n(t)\, R_y(\eta \mid K_n(t, y))\, dt = 0.$$

By virtue of (**2**.3°), (**2**.2°), (**3**.9) and the theorem in (C; p. 20), in the limit we obtain

$$\int_0^1 F(t)\, R_y(\eta \mid K(t, y))\, dt = 0.$$

We have proved

Theorem 3.10. — *Suppose $f(x)$ satisfies Hypothesis* 3.1, *with*

$$\rho_n^2 \leq \rho^2 < \infty.$$

One can then find a function

$$h(s) \subset L_2 \qquad [cf. \ (3.4), (3.2)]$$

so that

$$(3.10\,a) \qquad \int_0^1 L_x\big(\xi \mid K(x, s)\big)\, h(s)\, ds - L_x\big(\xi \mid f(x)\big) = L_x\big(\xi \mid F(x)\big),$$

where $F(x)$ *is a certain solution*, $\subset L_2$, *of the equation*

$$(3.10\,b) \qquad \int_0^1 F(t)\, R_y\big(\eta \mid K(t, y)\big)\, dt = 0.$$

Note. — *If* $R(t, \eta) = R_y\big(\eta \mid K(t, y)\big)$ [which by (2.1^0) is L_2 in t] *is closed* L_2, that is if

$$\int_0^1 \psi(t)\, R(t, \eta)\, dt = 0, \qquad \psi(t) \subset L_2,$$

implies $\psi = 0$, *the function $h(s)$ in the theorem will be a solution of*

$$(3.10\,c) \qquad \int_0^1 L_x\big(\xi \mid K(x, s)\big)\, h(s)\, ds = L_x\big(\xi \mid f(x)\big).$$

If, in addition, $L(\xi, s) = L_x\big(\xi \mid K(x, s)\big)$ [which by (2.1_0) is L_2 in s] *is right closed* L_2 (that is, if

$$\int_0^1 L(\xi, s)\, \varphi(s)\, ds = 0, \qquad \varphi(s) \subset L_2,$$

implies $\varphi = 0$), *the solution $h(s)$ of* $(3.10c)$ *will be unique in the field of solutions* L_2.

4. The case of unbounded (ρ_n). — We shall establish the following

Theorem 4.1. — *Suppose $f(x)$ is such that* $\rho_n^2 < +\infty$ [$n = 1, 2, \ldots$; *cf.* (3.1 *b*), (3.1 *a*)], *the sequence* (ρ_n^2) *not being necessarily bounded. Suppose* $T(\zeta, y) = T_x\big(\zeta \mid H(x. y)\big)$ (*Definition* 2.1) *is closed. The* *equation*

$$(4.2)\qquad I(x \mid h) \equiv f(x) = \int_0^1 K(x, s)\, h(s)\, ds = 0$$

can then be satisfield in the following sense. There exists a function $h_n(s) \subset L_2$, *say*

$$(4.2\,a)\qquad h_n(s) \sim \sum_k \lambda_{nk} f_{nk} v_{nk}(s),$$

so that the integral

$$(4.2\,b)\qquad \int_0^1 I_n^2(x \mid h_n)\, dx \qquad \left[I_n(x \mid h_n) \equiv f(x) - \int_0^1 K_n(x, s)\, h_n(s)\, ds \right]$$

is arbitrarily small for n suitably great.

On writing

$$h_{n:\nu}(x) = \sum_{k=1}^{\nu} \lambda_{nk} f_{nk} v_{nk}(x),$$

we obtain

$$h_{n:\nu}(x) \sim h_n(x) \qquad (\text{as } \nu \to \infty).$$

Hence

$$(4.3)\qquad \lim_{\nu} \int_0^1 K_n(x, s)\, h_{n:\nu}(s)\, ds = \int_0^1 K_n(x, s)\, h_n(s)\, ds.$$

Now

$$\int_0^1 K_n(x, s)\, h_{n:\nu}(s)\, ds = \sum_{k=1}^{\nu} \lambda_{nk} f_{nk} \int_0^1 K_n(x, s)\, v_{nk}(s)\, ds$$

and, by (1.2),

$$\int_0^1 K_n(x, s)\, h_{n:\nu}(s)\, ds = \sum_{k=1}^{\nu} f_{nk} u_{nk}(x);$$

in view of (4.3)

$$(4.3\,a)\qquad \int_0^1 K_n(x, s)\, h_n(s)\, ds = \sum_k f_{nk} u_{nk}(x).$$

It has been noted before that the series last displayed converges. We observe that

$$\left| \sum_k f_{nk} u_{nk}(x) \right|^2 = \left| \sum_k \lambda_{nk} f_{nk} \frac{u_{nk}(x)}{\lambda_{nk}} \right|^2 \geqq \rho_n^2 \sum_k \lambda_{nk}^{-2} u_{nk}^2(x)$$

and, by virtue of (1.2) and Bessel's inequality,

$$\left|\sum_k f_{nk}\, u_{nk}(x)\right|^2 \leqq \rho_n^2 \sum_k \left[\int_0^1 K_n(x, s)\, v_{nk}(s)\, ds\right]^2 \leqq \rho_n^2 n^2.$$

By (4.3a)

$$I_n(x \mid h_n) = f(x) - \sum_k f_{nk} u_{nk}(x).$$

Hence

$$\int_0^1 I_n^2(x \mid h_n)\, dx = \int_0^1 f^2\, dx - \sum_k f_{nk}^2.$$

Accordingly, the theorem is proved if it is established that for some $n_1 < n_2 < \ldots$ one has

$$\lim_{n_j} \sum_k f_{n_j,k}^2 = \int_0^1 f^2\, dx. \tag{4.4}$$

Now by (1.7a)

$$\theta_n^{u,u}(x, y \mid \lambda + \Delta) - \theta_n^{u,u}(x, y \mid \lambda) = \frac{1}{2} \sum_{\lambda \leq \lambda_{nk} < \lambda + \Delta} u_{nk}(x)\, u_{nk}(y),$$

for $0 \leqq \lambda < \lambda + \Delta$, and

$$\theta_n^{u,u}(x, y \mid \lambda + \Delta) - \theta_n^{u,u}(x, y \mid \lambda) = \frac{1}{2} \sum_{\lambda \leq -\lambda_{nk} < \lambda + \Delta} u_{nk}(x)\, u_{nk}(y)$$

when $\lambda < \lambda + \Delta \leqq 0$; hence

$$\begin{aligned}
&\int_0^1 \int_0^1 \theta_n^{u,u}(x, y \mid \lambda + \Delta) f(x) f(y)\, dx\, dy \\
&- \int_0^1 \int_0^1 \theta_n^{u,u}(x, y \mid \lambda) f(x) f(y)\, dx\, dy \\
&= \frac{1}{2} \sum_{\lambda \leq \lambda_{nk} < \lambda + \Delta} f_{nk}^2 \qquad (0 \leqq \lambda < \lambda + \Delta), \\
&= \frac{1}{2} \sum_{\lambda \leq -\lambda_{nk} < \lambda + \Delta} f_{nk}^2 \qquad (\lambda < \lambda < \Delta \leqq 0).
\end{aligned}$$

Accordingly

$$\begin{aligned}
&\int_0^\infty d\lambda \int_0^1 \int_0^1 \theta_n^{u,u}(x, y \mid \lambda) f(x) f(y)\, dx\, dy \\
&= \int_{-\infty}^0 d\lambda \int_0^1 \int_0^1 \theta_n^{u,u}(x, y \mid \lambda) f(x) f(y)\, dx\, dy = \frac{1}{2} \sum_k j_{nk}^2.
\end{aligned}$$

Therefore as a consequence of (2.7)

$$(4.5)\quad \sum_k f_{nk}^2 = \int_{-\infty}^{\infty} d\lambda \int_0^1 \int_0^1 \theta_n^{uu}(x, y \mid \lambda) f(x) f(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} d\lambda \int_0^1 f(x) \left[\frac{d}{dx} \int_0^1 \frac{\partial}{\partial y} \Omega_n^{uu}(e_x, e_y \mid \lambda) f(y)\, dy \right] dx = b_{n,\infty}.$$

In view of the concluding statement of section 2 and of the assumed closure of $T(\zeta, y)$

$$(4.5\,a)\quad \int_0^1 f^2\, dx = \int_{-\infty}^{\infty} d\lambda \int_0^1 f(x) \left[\frac{d}{dx} \int_0^1 \frac{\partial}{\partial y} \Omega^{uu}(e_x, e_y \mid \lambda) f(y)\, dy \right] dx = b_\infty.$$

By virtue of (4.5), (4.5a) and Bessel's inequality we have

$$b_{n,\infty} \leqq b_\infty.$$

Hence the sequence (n_j), involved in the definition of Ω^{uu}, can be chosen so that

$$\lim_{n_j} b_{n_j,\infty} = \nu, \qquad \nu \leqq b_\infty.$$

Il $\nu = b_\infty$ the desired relation (4.4) holds. 'Assume now the contrary,

$$(4.6)\qquad \nu = b_\infty - 2\varepsilon \qquad (\text{some } \varepsilon > 0).$$

Since the integral in (4.5a) over $(-\infty, +\infty)$ converges, we have

$$(4.6\,a)\quad \begin{cases} 0 \leqq b_\infty - b_l < \varepsilon, \\ b_l = \int_{-l}^{l} d\lambda \int_0^1 f(x) \left[\frac{d}{dx} \int_0^1 \frac{\partial}{\partial y} \Omega^{uu}(e_x, e_y \mid \lambda) f(y)\, dy \right] dx \end{cases}$$

for l sufficiently great. We recall *Helly's* theorem, according to which

$$\lim \int_a^b c(\lambda)\, d\psi_n(\lambda) = \int_a^b c(\lambda)\, d\psi(\lambda)$$

[finite interval (a, b)], provided $c(\lambda)$ is continuous for $a \leqq \lambda \leqq b$, $\psi_n(\lambda) \to \psi(\lambda)$ and Var. $\psi_n(\lambda) \leqq A < +\infty$. Thus

$$(4.6\,b)\quad \lim_{n_j} \int_{-l}^{l} d\lambda \int_0^1 f(x) \left[\frac{d}{dx} \int_0^1 \frac{\partial}{\partial y} \Omega_{n_j}^{uu}(e_x, e_y \mid \lambda) f(y)\, dy \right] dx = b_l,$$

We write

$$(4.6c)\qquad r_{n,l}=\left(\int_{-\infty}^{-l}+\int_{l}^{\infty}\right)d_\lambda\int_0^1 f(x)\times\left[\frac{d}{dx}\int_0^1\frac{\partial}{\partial y}\Omega_n^{uu}(e_x,\, e_y\,|\,\lambda)f(y)\,dy\right]dx.$$

Now

$$b_{n,\infty}=\int_{-l}^{l}d_\lambda\int_0^1 f(x)\left[\frac{d}{dx}\int_0^1\frac{\partial}{\partial y}\Omega_n^{uu}(e_x,\, e_y\,|\,\lambda)f(y)\,dy\right]dx+r_{n,l};$$

thus, in view of (**4**.6b) and the existence of the $\lim b_{n_j,\infty}$, it follows that the limit

$$\lim_{n_j} r_{n_j,l}=r_l\geqq 0$$

exists; by (**4**.6) one has

$$\lim_{n_j} b_{n,\infty}=b_l+r_l=b_\infty-2\varepsilon.$$

Since $r_l\geqq 0$ it is inferred that

$$b_l\leqq b_\infty-2\varepsilon$$

and

$$2\varepsilon\leqq b_\infty-b_l,$$

which contradicts (**4**.6a). Thus (**4**.6) is impossible and the Theorem is proved.

COROLLARY **4**.7. — *The conclusion of Theorem* **4**.1 *still holds when* $T_x(\zeta\,|\,H(x,\, y))$ *is not closed, provided* $f(x)$ *is orthogonal on* (0,1) *to every function* $\varphi_p(x)\,(0\leqq x\leqq 1)$ $[\varphi_p(x)$ *from the "base" of* $T_x(\zeta\,|\,H(x,\, y))$; *cf. the text after* (**2**.3)$]$.

We repeat the developments up to (**4**.5). In place of (**4**.5a), as a consequence of (**2**.15) it is inferred that

$$(4.8)\qquad \int_0^1 f^2\,dx=\sum_p f_p^2-\sum_{p,q}H_{pq}f_pf_q+\int_{-\infty}^{\infty}d_\lambda\int_0^1 f(x)\left[\frac{d}{dx}\int_0^1\frac{\partial}{\partial y}\Omega^{uu}(e_x,\, e_y\,|\,\lambda)f(y)\,dy\right]dx,$$

where

$$f_p = \int_0^1 f(x)\,\varphi_p(x)\,dx. \tag{4.8 a}$$

By the hypothesis imposed on f we accordingly obtain

$$f_p = 0 \qquad (p = 1, 2, \ldots).$$

Thus (**4**.5 a) again holds and the reasoning given subsequent to (**4**.5 a) continues to be valid; this demonstrates the Corollary.

THEOREM **4**.9. — *Let $f(x)$ be such that $\rho_n^2 < \infty\,(n = 1, 2, \ldots)$, the sequence (ρ_n^2) not being necessarily bounded. We do not assume closure of $T(\zeta, y)$. There exists then a functional $T[f]$, such that*

$$0 \leqq T[f] \leqq Q[f], \qquad Q[f] = \sum_p f_p^2 - \sum_{p,q} H_{pq} f_p f_q \tag{4.9 a}$$

*(f_p, H_{pq} from (**4**.8 a), (**2**.9 f), so that the difference*

$$\int_0^1 I_n^2(x \mid h_n)\,dx - T[f], \tag{4.9 b}$$

*with $h_n(x)$, $I_n(x \mid \ldots)$ from Theorem **4**.1, is arbitrarily small for n suitably great.*

We note first that $Q[f] \geqq 0$, inasmuch as in (**4**.8) the integral displayed in the second member cannot exceed the first member, as a consequence of « a generalized Bessel's inequality ». As in the proof of Theorem **4**.1, it is inferred that

$$\int_4^1 I_n^2(x \mid h_n)\,dx = \int_0^1 f^2\,dx - \sum_k f_{nk}^2 = \int_0^1 f^2\,dx - b_{n,\infty} (\geqq 0) \tag{4.10}$$

[$b_{\alpha,\infty}$ from (**4**.5)]. Now by Bessel's inequality and (**4**.8)

$$b_{n,\infty} \leqq \int_0^1 f^2\,dx = Q[f] + b_\infty, \tag{4.10 a}$$

where b_∞ is the second member in (**4**.5 a). Thus for some sequence (n_j), for which $\lim \Omega_{nj} = \Omega$, the limit

$$\lim_{n_j} b_{n_j,\infty} = \nu \tag{4.11}$$

exists; one has

$$\nu \leqq Q[f] + b_\infty.$$

Since $b_{n,\infty} = r_{n,l} + b_{n,l}$, in view of (4.6*b*) we conclude that the limit

$$\lim_{n_j} r_{n_j,l} = r_l$$

exists. Inasmuch as $r_l \geqq 0$,

$$0 \leqq b_l \leqq b_l + r_l = \nu \leqq Q[f] + b_\infty.$$

On letting $l \to \infty$ it is deduced that

(4.12) $$0 \leqq b_\infty \leqq \nu \leqq Q[f] + b_\infty.$$

By (4.10) and (4.11)

$$\lim_{n_j} \int_0^1 I_n^2(x \mid h_n)\, dx = \int_0^1 f^2\, dx - \nu = T[f];$$

here, as a consequence of (4.12) and (4.10*a*),

(4.12*a*) $$0 \leqq T[f] = Q[f] + b_\infty - \nu \leqq Q[f].$$

The theorem is accordingly established.

Corollary 4.13. — *For the functional* T[*f*] *of Theorem* 4.9 *one has*

(4.13*a*) $$T[f] = Q[f],$$

provided

(4.13*b*) $$\tau_n^2 = \sum_k \lambda_{nk}^{2\eta} f_{nk}^2 \leqq \tau^2 < +\infty \qquad (\textit{some } \eta > 0).$$

Note. — The condition (4.13*b*), with $\eta < 1$, is less stringent than that involved in the requirement that $\rho_n^2 \leqq \rho^2 < +\infty$.

We have

$$r_{n,l} = \sum_k{}' f_{nk}^2 \qquad [cf.\ (4.6c)],$$

where the prime over the summation symbol indicates that the sum is taken over all those values of k for which the λ_{nk} are on the interval $(l, +\infty)$. One obtains

$$r_{n,l} = \sum_k{}' \lambda_{nk}^\eta f_{nk} \frac{f_{nk}}{\lambda_{nk}^\eta} \leqq \left[\sum_k{}' \lambda_{nk}^{2\eta} f_{nk}^2 \right]^{\frac{1}{2}} \left[\sum_k{}' \frac{f_{nk}^2}{\lambda_{nk}^{2\eta}} \right]^{\frac{1}{2}}.$$

In view of (4.13*b*) and since $\lambda_{nk} \geqq l$, it follows that

$$r_{n,l} \leqq \tau l^{-\eta} \left[\sum_k f_{nk}^2 \right]^{\frac{1}{2}} \leqq \tau l^{-\eta} \left[\sum_k f_{nk}^2 \right]^{\frac{1}{2}} \leqq \tau l^{-\eta} \left[\int_0^1 f^2 \, dx \right]^{\frac{1}{2}}.$$

Thus $r_{n,l}$ tends to zero, with $\frac{1}{l}$, uniformly with respect to n. Together with (4.6*b*) this implies

$$\lim b_{n_j, \infty} = b_\infty;$$

that is, ν of (4.11) is b_∞. The conclusion of the Corollary ensues by (4.12*a*).

5. The permutability problem. — We shall now investigate the permutability problem referred to in the introduction. We thus consider the equation

$$(5.1) \qquad \int_0^1 p(x, t)\, q(t, y)\, dt = \int_0^1 q(x, t)\, p(t, y)\, dt,$$

where $p(x, y)$ is a known kernel such that

$$(5.1\,a) \qquad p(x, y) \subset L_2 \ \ (\text{in } x), \qquad p(x, y) \subset L_2 \ \ (\text{in } y),$$

while the unknown $q(x, y)$ is to be found subject to the properties

$$(5.1\,b) \qquad q(x, y) \subset L_2 \ \ (\text{in } x), \qquad q(x, y) \subset L_2 \ \ (\text{in } y).$$

Use will be made of

Definition 5.2. — *It will be said that a function $h(x, y)$ is regular $[a, b]$ is*

$$(5.2\,a) \qquad \int_0^1 \int_0^1 \frac{h^2(x, y)}{a^2(x)}\, dx\, dy < +\infty, \qquad \int_0^1 \int_0^1 \frac{h^2(x, y)}{b^2(y)}\, dx\, dy < +\infty.$$

In the sequel we shall always choose $a(x)$, $b(y)$ so that

$$(5.2\,b) \qquad a(x) \geqq 1, \qquad b(x) \geqq 1.$$

A function $h(x, y) \subset L_2$ (in (x, y)) is regular $[1, 1]$.

Every function $h(x, y)$, such that

$$h(x, y) \subset L_2 \ \ (\text{in } x), \qquad h(x, y) \subset L_2 \ \ (\text{in } y),$$

is regular $[a, b]$ for a suitable choice of $a(x)$, $b(y)$; in fact, on writing

$$\overline{h}(x)=\left[\int_0^1 h^2(x, y)\,dy\right]^{\frac{1}{2}}, \qquad \underline{h}(y)=\left[\int_0^1 h^2(x, y)\,dx\right]^{\frac{1}{2}},$$

it is observed that

$$\int_0^1\int_0^1 \frac{h^2(x, y)}{a^2(x)}\,dx\,dy=\int_0^1 \overline{h}^2(x)\frac{dx}{a^2(x)},$$

$$\int_0^1\int_0^1 \frac{h^2(x, y)}{b^2(x)}\,dx\,dy=\int_0^1 \underline{h}^2(y)\frac{dy}{b^2(y)},$$

so that one may choose for example

$$a(x)=\begin{cases} 1 & (\text{when } \overline{h}(x)\leqq 1),\\ \overline{h}(x) & (\text{when } \overline{h}(x)>1),\end{cases}$$

$$b(y)=\begin{cases} 1 & (\text{when } \underline{h}(y)\leqq 1),\\ \underline{h}(x) & (\text{when } \underline{h}(y)>1).\end{cases}$$

We note the following. Given functions F, G of x and y, $\subset L_2$ (in x), $\subset L_2$ (in y), a pair of functions a, b can be found (the same for F, G) so that F and G are each regular $[a, b]$.

Suppose $q(x, y)$ is a solution of (5.1), subject to (5.1b). We think of this function substituted in (5.1). Let $a(x)$, $b(y)$ be functions so that the functions $p(x, y)$, $q(x, y)$ are regular $[a, b]$. Multiplying (5.1) by $a^{-1}(x)\,b^{-1}(y)\,dx\,dy$ and integrating, we obtain

$$\text{(5.3)}\qquad \int_0^1\int_0^1\int_0^1 \frac{p(x, t)}{a(x)}\,\frac{q(t, y)}{b(y)}\,dx\,dy\,dt$$
$$=\int_0^1\int_0^1\int_0^1 \frac{q(x, t)}{a(x)}\,\frac{p(t, y)}{b(y)}\,dx\,dy\,dt.$$

One has

$$\mathrm{I}=\int_0^1\int_0^1\int_0^1 \left|\frac{p(x, t)}{a(x)}\right|\left|\frac{q(t, y)}{b(y)}\right|dx\,dy\,dt$$
$$\leqq\int_0^1\int_0^1\left[\int_0^1\left|\frac{p(x, t)}{a(x)}\right|\left|\frac{q(t, y)}{b(y)}\right|dt\right]dx\,dy$$
$$\leqq\int_0^1\int_0^1\left[\int_0^1 \frac{p^2(x, t)}{a^2(x)}\,dt\right]^{\frac{1}{2}}\left[\int_0^1 \frac{q^2(t, y)}{b^2(y)}\,dt\right]^{\frac{1}{2}}dx\,dy$$
$$=\int_0^1\left[\int_0^1 \frac{p^2(x, t)}{a^2(x)}\,dt\right]^{\frac{1}{2}}dx\int_0^1\left[\int_0^1 \frac{q^2(t, y)}{b^2(y)}\,dt\right]^{\frac{1}{2}}dy;$$

applying the Schwartz's inequality once more we obtain

$$1 \leqq \left[\int_0^1 \int_0^1 \frac{p^2(x,t)}{a^2(x)}\,dx\,dt\right]^{\frac{1}{2}} \left[\int_0^1 \int_0^1 \frac{q^2(t,y)}{b^2(y)}\,dt\,dy\right]^{\frac{1}{2}};$$

since $p(x, y)$, $q(x, y)$ are regular $[a, b]$ the integrals in the second member above exist. Hence the order of integration in the first member of (5.3) is immaterial. A similar property is established for the second member of (5.3), by merely interchanging the roles of $p(x, y)$ and $q(x, y)$. Accordingly, (5.3) may be rewritten in the form

$$\int_0^1 \int_0^1 q(t,y)\left[\frac{1}{b(y)}\int_0^1 \frac{p(x,t)}{a(x)}\,dx\right]dt\,dy$$
$$=\int_0^1 \int_0^1 q(x,t)\left[\frac{1}{a(x)}\int_0^1 \frac{p(t,y)}{b(y)}\,dy\right]dx\,dt.$$

On replacing x, y, t by t, x, y, respectively, in the last member, the latter becomes

$$\int_0^1 \int_0^1 q(t,y)\left[\frac{1}{a(t)}\int_0^1 \frac{p(y,x)}{b(x)}\,dx\right]dt\,dy.$$

Consequently

$$\int_0^1 \int_0^1 q(t,y)\,\mathrm{H}(t,y)\,dt\,dy = 0, \tag{5.4}$$

where

$$\mathrm{H}(t,y) = \frac{1}{b(y)}\int_0^1 \frac{p(x,t)}{a(x)}\,dx - \frac{1}{a(t)}\int_0^1 \frac{p(y,x)}{b(x)}\,dx. \tag{5.4 a}$$

By (5.2b) one has

$$|\mathrm{H}(t,y)| \leqq \mathrm{H}_1(t) + \mathrm{H}_2(y), \qquad \mathrm{H}_1^2(t) = \int_0^1 \frac{p^2(x,t)}{a^2(x)}\,dx,$$

$$\mathrm{H}_2^2(y) = \int_0^1 \frac{p^2(y,x)}{b^2(x)}\,dx, \qquad \int_0^1 \mathrm{H}_1^2(t)\,dt < +\infty, \qquad \int_0^1 \mathrm{H}_2^2(y)\,dy < +\infty,$$

inasmuch as $p(x, y)$ is regular $[a, b]$. Whence the integral

$$c^2 = \int_0^1 \int_0^1 \mathrm{H}^2(t,y)\,dt\,dy \tag{5.4 b}$$

exists. *If* $c = 0$, *i.e. if* $\mathrm{H}(x, y) = 0$ *almost everywhere on* $(0 \leqq x, y \leqq 1)$

then $b(x)^{-1}a(y)^{-1}$ *will be a solution of the permutability problem.* If $\pi(t, y)$ is any function regular $[a, b]$, we have

$$(5.4c) \qquad \left|\int_0^1\int_0^1 \pi(t, y)\,\mathrm{H}(t, y)\,dt\,dy\right| < +\infty.$$

To demonstrate this it is sufficient to show that

$$\Lambda = \left|\int_0^1\int_0^1 \frac{\pi(t, y)}{b(y)}\left[\int_0^1 \frac{p(x, t)}{a(x)}\,dx\right]dt\,dy\right| < +\infty;$$

now

$$\Lambda \leqq \int_0^1\left[\int_0^1 \frac{\pi^2(t, y)}{b^2(y)}\,dy\right]^{\frac{1}{2}}\left[\int_0^1 \frac{p^2(x, t)}{a^2(x)}\,dx\right]^{\frac{1}{2}} dt$$

$$\leqq \left[\int_0^1\int_0^1 \frac{\pi^2(t, y)}{b^2(y)}\,dt\,dy\right]^{\frac{1}{2}}\left[\int_0^1\int_0^1 \frac{p^2(x, t)}{a^2(x)}\,dx\,dt\right]^{\frac{1}{2}};$$

thus the assertion ensues since $p(x, y)$ is also regular $[a, b]$.

Suppose $c \neq 0$; let α be any constant and form the function

$$(5.5) \qquad \pi(t, y) = q(t, y) + \alpha c^{-1}\,\mathrm{H}(t, y).$$

Since $q(t, y)$ is regular $[a, b]$ and $\mathrm{H}(t, y)$ is regular $[1, 1]$, $\pi(t, y)$ will be regular $[a, b]$. Multiplying (5.5) by $c^{-1}\mathrm{H}(t, y)$, integrating and taking note of (5.4), (5.4b), we obtain

$$\alpha = \int_0^1\int_0^1 \pi(\zeta, \eta)c^{-1}\,\mathrm{H}(\zeta, \eta)\,d\zeta\,d\eta.$$

Thus, provided $\mathrm{H}(t, y) \not\equiv 0$, *our solution (which we assumed as existent) has the form*

$$(5.6) \quad q(t, y) = \pi(t, y) - \left[\int_0^1\int_0^1 \pi(\zeta, \eta)c^{-1}\,\mathrm{H}(\zeta, \eta)\,d\zeta\,d\eta\right]c^{-1}\,\mathrm{H}(t, y),$$

where $\pi(t, y)$ *is some function regular* $[a, b]$.

Consider now the converse. Let $a(x)$, $b(y)$ be chosen subject to (5.2b) so that $p(x, y)$ is regular $[a, b]$. We construct the function $\mathrm{H}(t, y)$ (5.4a) and evaluate the constant c of (5.4b). Suppose $\mathrm{H} \not\equiv 0$; then $c < 0$. Let $\pi(t, y)$ be an arbitrary function ($\subset \mathrm{L}_2$, in t, $\subset \mathrm{L}_2$, in y), regular $[a, b]$. We take note of the statement in connection with (5.4c) and define $q(t, y)$ by (5.6). This function will be L_2 separately in t and in y; moreover, being a sum of functions regular $[a, b]$, $[1, 1]$, $q(t, y)$ will be regular $[a, b]$. Multiplying (5.6) by $H(t, y)c^{-1}\,dt\,dy$, integrating and taking note of (5.4b), it is observed that $q(t, y)$ satisfies (5.4). Substitution of (5.4a) in (5.4) will yield

$$\int_0^1\int_0^1 q(t, y)\left[\frac{1}{b(y)}\int_0^1 \frac{p(x, t)}{a(x)}\,dx\right] dt\,dy$$
$$=\int_0^1\int_0^1 q(t, y)\left[\frac{1}{a(t)}\int_0^1 \frac{p(y, x)}{b(x)}\,dx\right] dt\,dy.$$

Replacement of t, x, y, in the second member, by x, y, t, respectively, will result in the equality preceding (5.4). In this equality the order of integration is immaterial, in view of the developments subsequent to (5.3) (valid since p, q are regular $[a, b]$). Hence we can retrace the steps back to (5.3) and, in fact, write (5.3) in the form

$$\int_0^1\int_0^1\left[\int_0^1 p(x, t)\,q(t, y)\,dt\right]\frac{dx}{a(x)}\,\frac{dy}{b(y)}$$
$$=\int_0^1\int_0^1\left[\int_0^1 q(x, t)\,p(t, y)\,dt\right]\frac{dx}{a(x)}\,\frac{dy}{b(y)}.$$

We thus have

(5.7) $$\int_0^1\int_0^1 L(x, y\,|\,q)\,dx\,dy = 0,$$

where

(5.8) $$L(x, y\,|\,q) = \frac{1}{a(x)\,b(y)}\int_0^1 [p(x, t)\,q(t, y) - q(x, t)\,p(t, y)]\,dt.$$

Accordingly, offhand there is no assurance that every function $q(t, y)$ of the form (5.6) is a solution of the problem; however, we have just shown that, provided $c \neq 0$, every such function satisfies the related equation (5.7).

Theorem 5.9. — *If q is a solution of the permutability problem (5.1), L_2 separately in each of the variables, and if $a(x)$, $b(y)$ are chosen ≥ 1 so that p, q are regular $[a, b]$ (Definition 5.2), while $H(x, y)$ of (5.4a)*

is not identically zero, then q will be necessarily of the form (5.6), *where* $\pi(t, y)$ *is some function regular* $[a, b]$ *and* $c(>0)$ *is from* (5.4 *b*).

If $a, b(\geqq 1)$ *are such that p is regular* $[a, b]$ *and* $H(x, y)$ is $\not\equiv 0$, *then every function q, as given by* (5.6), *will be a solution of* (5.7), (5.8), *provided that c is defined by* (5.4 *b*), *while* $\pi(t, y)$ *is an arbitrary function,* L_2 *in t and* L_2 *in y, regular* $[a, b]$.

6. Permutability (continued). — We shall now obtain a complete, though more complicated solution.

With $a, b\ (\geqq 1)$ such that $p(x, y)$ is regular $[a, b]$, introduce kernels

$$(6.1)\qquad P(x, y)=\frac{p(x, y)}{a(x)}, \qquad P^*(x, y)=\frac{p(y, x)}{b(x)}.$$

We have

$$\int_0^1\int_0^1 P^2(x, y)\,dx\,dy<+\infty, \qquad \int_0^1\int_0^1 P^{*2}(x, y)\,dx\,dy<+\infty.$$

Let the u_i, v_i, λ_i be the characteristic functions and values of $P(x, y)$ and let the w_i, z_i, μ_i be the characteristic functions and values of $P^*(x, y)$; thus

$$(6.1\,a)\quad \frac{u_i(x)}{\lambda_i}=\int_0^1 P(x, t)\,v_i(t)\,dt, \qquad \frac{v_i(x)}{\lambda_i}=\int_0^1 u_i(t)\,P(t, x)\,dt,$$

$$(6.1\,b)\quad \frac{w_i(x)}{\mu_i}=\int_0^1 P^*(x, t)\,z_i(t)\,dt, \qquad \frac{z_i(x)}{\mu_i}=\int_0^1 w_i(t)\,P^*(t, x)\,dt.$$

The sequences

$$(u_i),\quad (v_i),\quad (w_i),\quad (z_i)$$

will be chosen orthonormal. We complete them by sequences

$$(u'_i),\quad (v'_i),\quad (w'_i),\quad (z'_i),$$

respectively; that is, each of the four sequences

$$[(u_i), (u'_i)],\quad [(v_i), (v'_i)],\quad [(w_i), (w'_i)],\quad [(z_i), (z'_i)]$$

are complete orthonormal. Moreover, we shall have

$$(6.1\,c)\qquad 0=\int_0^1 u'_i(t)\,P(t, x)\,dt=\int_0^1 P(x, t)\,v'_i(t)\,dt$$
$$=\int_0^1 w'_i(t)\,P^*(t, x)\,dt=\int_0^1 P^*(x, t)\,z'_i(t)\,dt.$$

Form the sequence $\eta_{ij}(x, y)\ (i, j = 1, 2, \ldots)$ *consisting of the functions*

$$(6.2)\quad \frac{v_\nu(x)}{\lambda_\nu}\frac{w_n(y)}{b(y)} - \frac{u_\nu(x)}{a(x)}\frac{z_n(y)}{\mu_n}, \qquad \frac{u'_\nu(x)}{a(x)} z_n(y), \qquad v_\nu(x)\frac{w'_n(y)}{b(y)}$$

$(\nu, n = 1, 2, \ldots)$. Since $a, b \geqq 1$, the functions (6.2) do not exceed in absolute value the functions

$$\left|\frac{v_\nu(x)\, w_n(y)}{\lambda_\nu}\right| + \left|\frac{u_\nu(x)\, z_n(y)}{\mu_n}\right|, \qquad |u'_\nu(x)\, z_n(y)|, \qquad |v_\nu(x)\, w'_n(y)|,$$

respectively. Clearly the $\eta_{ij}(x, y)$ are all L_2 in (x, y). By a familiar process *we orthonormalize the* $\eta_{ij}(x, y)$ *on* $(0 \leqq x, y \leqq 1)$, *designating the resulting sequence by*

$$(6.3)\qquad \mu_{ij}(x, y) \qquad (i, j = 1, 2, \ldots);$$

we have

$$\int_0^1\int_0^1 \mu_{ij}^2(x, y)\, dx\, dy = 1, \qquad \int_0^1\int_0^1 \mu_{ij}(x, y)\, \mu_{\alpha\beta}(x, y)\, dx\, dy = 0$$
$$[\text{for } (i, j) \neq (\alpha, \beta)].$$

The following result will be proved.

THEOREM 6.4. — *Suppose* $q(x, y)$ (L_2 *in* x, L_2 *in* y) *is a solution of* (5.1). *Choose* $a(x)$, $b(y)$ $(\geqq 1)$ *so that* p, q *are regular* $[a, b]$; *with these* a, b *construct the sequence* $\{\mu_{ij}(x, y)\}$ (6.3). *Then* q *will be representable in the form*

$$(6.4\,a)\quad q(x, y) \sim \pi(x, y) - \sum_{i,j}\left[\int_0^1\int_0^1 \pi(t, \tau)\, \mu_{ij}(t, \tau)\, dt\, d\tau\right] \mu_{ij}(x, y)$$

[$\sim$ *is symbol of convergence in the mean square over* $(0 \leqq x, y \leqq 1)$], *where* $\pi(x, y)$ *is some function* L_2 *in* x *and* L_2 *in* y, *regular* $[a, b]$ *and such that*

$$(6.4\,b)\qquad \sum_{i,j}\left[\int_0^1\int_0^1 \pi(t, \tau)\, \mu_{ij}(t, \tau)\, dt\, d\tau\right]^2 < \infty.$$

The converse. Let a, b $(\geqq 1)$ *be chosen so that* p *is regular* $[a, b]$. *Let* $\pi(x, y)$ *be any function,* L_2 *in* x *and* L_2 *in* y, *regular* $[a, b]$ *and* *such that* (**6**.4b) *holds. Then* $q(x, y)$, *as given by* (**6**.4a), *will be a solution of* (**5**.1), L_2 *in* x *and* L_2 *in* y, *regular* $[a, b]$.

Suppose q, a, b, μ_{ij} are functions as specified at the beginning of the theorem. We think of q as substituted in (**5**.1). One may thus write

(**6**.5)
$$F_1(x, y) = F_2(x, y),$$

$$F_1(x, y) = \int_0^1 \frac{p(x, t)}{a(x)} \frac{q(t, y)}{b(y)} dt, \qquad F_2(x, y) = \int_0^1 \frac{q(x, t)}{a(x)} \frac{p(t, y)}{b(y)} dt.$$

Designate by $(\bar{u}_i(x))$ the sequence consisting of the $u_i(x)$ and the $u'_i(x)$ and by $(\bar{w}_j(y))$ the sequence consisting of the $(w_j(y))$, $(w'_j(y))$. *The sequence*

(**6**.6)
$$\{\bar{u}_i(x)\bar{w}_j(y)\} \qquad (i, j = 1, 2, \ldots)$$

is complete orthonormal on $0 \leqq x$, $y \leqq 1$, inasmuch as each of the sequences $(\bar{u}_i)$, $(\bar{w}_j)$ has this property on $(0, 1)$. One has

(**6**.6a)
$$\bar{u}_{i_n} = u_n, \qquad \bar{u}_{i'_n} = u'_n; \qquad \bar{w}_{j_\nu} = w_\nu, \qquad \bar{w}_{j'_\nu} = w'_\nu.$$

It follows without difficulty that $F_1(x, y)$, $F_2(x, y)$ are L_2 in (x, y), inasmuch as the four functions

$$\frac{p(x, y)}{a(x)}, \quad \frac{q(x, y)}{b(y)}, \quad \frac{q(x, y)}{a(x)}, \quad \frac{p(x, y)}{b(y)}$$

have this property. Accordingly (**6**.5) is equivalent to the relations

(**6**.7)
$$\alpha_{ij} = \beta_{ij},$$

where

(**6**.7a)
$$\alpha_{ij} = \int_0^1 \int_0^1 \int_0^1 \frac{p(x, t)}{a(x)} \frac{q(t, y)}{b(y)} \bar{u}_i(x)\bar{w}_j(y)\, dx\, dy\, dt,$$

$$\beta_{ij} = \int_0^1 \int_0^1 \int_0^1 \frac{q(x, t)}{a(x)} \frac{p(t, y)}{b(y)} \bar{u}_i(x)\bar{w}_j(y)\, dx\, dy\, dt.$$

Now the order of integration here is immaterial. In fact, if we consider the integral for α_{ij}, for instance, it is inferred that

$$\int_0^1\int_0^1\int_0^1\left|\frac{p(x,t)}{a(x)}\frac{q(t,y)}{b(y)}\bar{u}_i(x)\bar{w}_j(y)\right|dx\,dy\,dt$$
$$=\int_0^1\left[\int_0^1\left|\frac{p(x,t)}{a(x)}\bar{u}_i(x)\right|dx\right]\left[\int_0^1\frac{q(t,y)}{b(y)}\bar{w}_j(y)\,dy\right]dt$$
$$\leq\int_0^1\left[\int_0^1\left|\frac{p^2(x,t)}{a^2(x)}dx\right]^{\frac{1}{2}}\left[\int_0^1\frac{q^2(t,y)}{b^2(y)}dy\right]^{\frac{1}{2}}dt$$
$$\leq\left[\int_0^1\int_0^1\frac{p^2(x,t)}{a^2(x)}dx\,dt\right]^{\frac{1}{2}}\left[\int_0^1\int_0^1\frac{q^2(t,y)}{b^2(y)}dy\,dt\right]^{\frac{1}{2}}<+\infty.$$

We obtain

$$\alpha_{ij}=\int_0^1\int_0^1\left[\int_0^1\bar{u}_i(x)\,\mathrm{P}(x,t)\,dx\right]q(t,y)\frac{\bar{w}_j(y)}{b(y)}dt\,dy$$

and, by (**6**.1 *a*), (**6**.1 *c*), (**6**.6 *a*),

$$(\mathbf{6}.8)\qquad \alpha_{i_n,j}=\int_0^1\int_0^1 q(t,y)\left[\frac{v_n(t)}{\lambda_n}\frac{\bar{w}_j(y)}{b(y)}\right]dt\,dy,\qquad \alpha_{i'_n,j}=0$$

similarly

$$\beta_{ij}=\int_0^1\int_0^1\left[\int_0^1\bar{w}_j(y)\,\mathrm{P}^*(y,t)\,dy\right]q(x,t)\frac{\bar{u}_i(x)}{a(x)}dx\,dt,$$

so that, in view of (**6**.1 *b*), (**6**.1 *c*), (**6**.6 *a*),

$$(\mathbf{6}.8\,a)\qquad \beta_{i,j_\nu}=\int_0^1\int_0^1 q(x,t)\left[\frac{\bar{u}_i(x)}{a(x)}\frac{z_\nu(t)}{\mu_\nu}\right]dx\,dt,\qquad \beta_{i,j'_\nu}=0.$$

Vanishing of the $\alpha_{i'_n,j}$, β_{i,j'_ν} is a consequence of (**6**.1 *c*) and is not contingent on q; however, in view of (**6**.7) the vanishing of these numbers necessitates that

$$\alpha_{i,j'_\nu}=0=\beta_{i'_n,j};$$

here one may discard, as superfluous, the values $i=i''_n$, $j=j'_\nu$, which gives as a necessary condition

$$\alpha_{i_n,j'_\nu}=0=\beta_{i'_n,j_\nu}\qquad(n,\nu=1,2,\ldots);$$

in fact, (**6**.7) is equivalent to the above and to

$$\alpha_{i_n,j_\nu}=\beta_{i_n,j_\nu}.$$

In view of the expressions given in (**6**.8), (**6**.8a) for the $\alpha_{i n, j}$, $\beta_{i, j}$, we thus conclude that (**6**.7) *is equivalent to*

(**6**.9)
$$\int_0^1 \int_0^1 q(x, y)\left[\frac{v_n(x)}{\lambda_n} \frac{w_\nu(y)}{b(y)}\right] dx\, dy = \int_0^1 \int_0^1 q(x, y)\left[\frac{u_n(x)}{a(x)} \frac{z_\nu(y)}{\mu_\nu}\right] dx\, dy,$$

(**6**.9a)
$$\int_0^1 \int_0^1 q(x, y)\left[v_n(x) \frac{w'_\nu(y)}{b(y)}\right] dx\, dy = \int_0^1 \int_0^1 q(x, y)\left[\frac{u'_n(x)}{a(x)} z_\nu(y)\right] dx\, dy;$$

(**6**.9) we rewrite in the form

(**6**.9$'$)
$$\int_0^1 \int_0^1 q(x, y)\left[\frac{v_n(x)}{\lambda_n} \frac{w_\nu(y)}{b(y)} - \frac{u_n(x)}{a(x)} \frac{z_\nu(y)}{\mu_\nu}\right] dx\, dy = 0.$$

The relations (**6**.9a), (**6**.9$'$) accordingly imply that, if q is a solution of (**5**.1) (with the stated properties), then q is orthogonal to all the functions (**6**.2); that is

$$\int_0^1 \int_0^1 q(x, y)\, \eta_{ij}(x, y)\, dx\, dy = 0.$$

As a consequence one has

(**6**.10)
$$\int_0^1 \int_0^1 q(x, y)\, \mu_{ij}(x, y)\, dx\, dy = 0 \qquad [i, j = 1, 2, \ldots;\ \text{cf. } (\mathbf{6}.3)].$$

The integrals in the left members here exist inasmuch as the integrals in (**6**.9a), (**6**.9$'$) exist, while any μ_{ij} is a linear combination with constant coefficients of a finite number of $\eta_{n\nu}$. The following is, in fact, true. *If* $s(x, y)$, L_2 *in* x, L_2 *in* y, *is regular* $[a, b]$, *then the integrals*

(**6**.11)
$$\int_0^1 \int_0^1 s(x, y)\, \mu_{ij}(x, y)\, dx\, dy$$

exist. This follows from the fact that the $s(x, y)\eta_{ij}(x, y)$ are integrable in (x, y), which ensues by virtue of inequalities, of which the following is typical

$$\int_0^1 \int_0^1 \left| s(x, y)\, v_n(x) \frac{w_\nu(y)}{b(y)} \right| dx\, dy$$
$$\leqq \left[\int_0^1 \int_0^1 \frac{s^2(x, y)}{b^2(y)}\, dx\, dy \right]^{\frac{1}{2}} \left[\int_0^1 \int_0^1 v_n(x)\, w_\nu(y)\, dx\, dy \right]^{\frac{1}{2}}.$$

Let the $\gamma_{ij}(i, j = 1, 2, \dots)$ be a set of real constants such that

$$\sum_{i,j} \gamma_{ij}^2 < +\infty.$$

There exists (by the Riesz-Fisher theorem, for example) a function $\gamma(x, y)$, such that

$$\gamma(x, y) \sim \sum_{i,j} \gamma_{ij}\, \mu_{ij}(x, y), \tag{6.12}$$

$$\int_0^1 \int_0^1 \gamma(x, y)\, \mu_{ij}(x, y)\, dx\, dy = \gamma_{ij}, \qquad \int_0^1 \int_0^1 \gamma^2(x, y)\, dx\, dy = \sum_{i,j} \gamma_{ij}^2. \tag{6.12 a}$$

We put

$$\pi(x, y) = q(x, y) + \gamma(x, y); \tag{6.13}$$

$\pi(x, y)$ is L_2 in x and in y and is regular $[a, b]$, since $q(x, y)$ has these properties and since $\gamma(x, y)$ is regular $[1, 1]$. On taking account of the italics in connection with (**6**.11) and of (**6**.10), (**6**.12a), from (**6**.13) it is inferred that

$$\int_0^1 \int_0^1 \pi(x, y)\, \mu_{ij}(x, y)\, dx\, dy = \int_0^1 \int_0^1 \gamma(x, y)\, \mu_{ij}(x, y)\, dx\, dy = \gamma_{ij}. \tag{6.13'}$$

This fact, together with (**6**.13), (**6**.12), implies that *the postulated solution $q(x, y)$ is representable as stated in the first part of the theorem.* The inequality (**6**.4b) holds by (**6**.13′), in view of the convergence of the sum of the γ_{ij}^2.

To prove the remaining part of the theorem assume that $a(x)$, $b(y)$, $\pi(x, y)$ are functions as specified in the converse part of the theorem. The numbers

$$\gamma_{ij} = \int_0^1 \int_0^1 \pi(t, \tau)\, \mu_{ij}(t, \tau)\, dt\, d\tau \tag{6.14}$$

can be defined in view of the italics in connection with (**6**.11). By (**6**.4b) the sum of the γ_{ij}^2 converges. Accordingly, (**6**.4a) can serve to define a function $q(x, y)$,

$$(6.15)\quad \begin{cases} q(x, y) = \pi(x, y) - \gamma(x, y), \qquad \gamma(x, y) \sim \sum_{i,j} \gamma_{ij}\,\mu_{ij}(x, y), \\ \int_0^1 \int_0^1 \gamma(x, y)\,\mu_{ij}(x, y)\,dx\,dy = \gamma_{ij}. \end{cases}$$

Multiplying by $\mu_{ij}(x, y)\,dx\,dy$ and integrating, as a consequence of (**6**.14) it is deduced that

$$\int_0^1 \int_0^1 q(x, y)\,\mu_{ij}(x, y)\,dx\,dy = \int_0^1 \int_0^1 \pi(x, y)\,\mu_{ij}(x, y)\,dx\,dy - \gamma_{ij} = 0;$$

that is, (**6**.10) will hold. Accordingly $q(x, y)$ is orthogonal to the $\eta_{ij}(x, y)$. In other words, the function $q(x, y)$ defined by (**6**.15) satisfies (**6**.9a), (**6**.9'). However, it has been indicated previously that the latter relations are equivalent to (**6**.7), which in turn implies (**6**.5). Whence $q(x, y)$ is a solution of (**5**.1). Now in (**6**.15)

$$\int_0^1 \int_0^1 \gamma^2(x, y)\,dx\,dy = \sum_{i,j} \gamma_{ij}^2 < +\infty;$$

thus $\gamma(x, y)$ is regular [1, 1] and $q(x, y)$ is regular $[a, b]$, as is the case with $\pi(x, y)$. This completes the proof of the theorem.

It is worth noting that, as we proceed constructing a solution $q(x, y)$ of (**5**.1), in accordance with the theorem, $\pi(x, y)$ can be always chosen not L_2 in (x, y). This is due to the fact that $\mu_{ij}(x, y)$, being a linear combination with constant coefficients of a number of $\eta_{n\nu}(x, y)$, consists of two terms

$$\mu'_{ij}(x, y), \quad \mu''_{ij}(x, y),$$

where $\mu'_{ij}(x, y)$ is a linear combination of terms

$$\frac{u_\nu(x)}{a(x)}\,z_n(y), \qquad \frac{u'_\nu(x)}{a(x)}\,z_n(y)$$

and $\mu''_{ij}(x, y)$ is a linear combination of terms

$$v_\nu(x)\,\frac{w_n(y)}{b(y)}, \qquad v_\nu(x)\,\frac{w'_n(y)}{b(y)}$$

[*cf.* (**6**.2)]. It is the presence of the factors $a^{-1}(x)$, $b^{-1}(y)$ in μ'_{ij}, μ''_{ij}, respectively, that enables us to choose $\pi(x, y)$, if desired, regular $[a, b]$, so that the inequality (**6**.4*b*) holds, while the integral

$$\int_0^1 \int_0^1 \pi^2(x, y)\, dx\, dy$$

diverges. *The corresponding solution $q(x, y)$ will then certainly be not L_2 in (x, y).*

7. Inversion of Schmidt kernels. — In the Schmidt theory of non symmetric regular kernels, given a non symmetric kernel $q(x, y)$, there is associated with it a symmetric kernel

$$(\mathbf{7}.1) \qquad f(x, y) = \int_0^1 q(x, t)\, q(y, t)\, dt.$$

In this section we shall consider the converse of this problem; that is, given a symmetric function $f(x, y)$, to find $q(x, y)$ (possibly non symmetric) so that (**7**.1) *holds.* Furthermore, this problem will be considered in the *singular* form in the sense that we merely assume

$$(\mathbf{7}.2) \qquad f(x, y) \subset L_2 \quad (\text{in } x), \qquad \subset L_2 (\text{in } y).$$

Choose $a(x) (\geqq 1)$ so that $F(x, y) = a^{-1}(x) a^{-1}(y) f(x, y)$ is L_2 in (x, y). Suppose (**7**.1) has a solution, L_2 in x and in y, such that

$$(\mathbf{7}.3) \qquad \int_0^1 \int_0^1 Q^2(x, y)\, dx\, dy < +\infty, \qquad Q(x, y) = \frac{q(x, y)}{a(x)}.$$

We think of $q(x, y)$ as substituted in (**7**.1), writing (**7**.1) in the form

$$(\mathbf{7}.1') \qquad \int_0^1 Q(x, t)\, Q(y, t)\, dt = F(x, y).$$

Since $F(x, y)$ is L_2 in (x, y) and is symmetric, $F(x, y)$ has real characteristic values and functions λ_ν, $u_\nu(x)$

$$(\mathbf{7}.4) \qquad \frac{u_\nu(x)}{\lambda_\nu} = \int_0^1 F(x, t)\, u_\nu(t)\, dt \qquad (\nu = 1, 2, \ldots).$$

Let $(u'_\nu(x))$ be the sequence complementary to $(u_\nu(x))$; thus

$$(7.4a) \qquad \int_0^1 F(x, t)\, u'_\nu(t)\, dt = 0.$$

We arrange to have $[(u_\nu(x)), (u'_\nu(x))]$ orthonormal and write

$$(7.4b) \qquad \bar{u}_{i_n}(x) = u_n(x), \qquad \bar{u}_{i'_n}(x) = u'_n(x)$$

$[(i_1, i_2, \ldots), (i'_1, i'_2, \ldots) = (1, 2, \ldots)]$. If one thinks of the first member of $(7.1')$ as a kernel, it is observed that it is positive definite. Hence, if there exists a solution, as stated, the characteristic values of $F(x, y)$ must be positive,

$$(7.5) \qquad \lambda_\nu > 0,$$

which is a necessary condition. The sequence

$$(7.6) \qquad \{\bar{u}_i(x)\, \bar{u}_j(y)\}$$

is complete orthonormal on $(0 \leqq x, y \leqq 1)$. By $(7.1')$

$$(7.7) \qquad F_{i,j} = \int_0^1 \int_0^1 F(x, y)\, \bar{u}_i(x)\, \bar{u}_j(y)\, dx\, dy$$
$$= \int_0^1 \left[\int_0^1 Q(x, t)\, \bar{u}_i(x)\, dx\right]\left[\int_0^1 Q(y, t)\, \bar{u}_j(y)\, dy\right] dt.$$

The last member here is obtained on making use of the permissible change of order of integration in

$$\int_0^1 \int_0^1 \int_0^1 Q(x, t)\, Q(y, t)\, \bar{u}_i(x)\, \bar{u}_j(y)\, dx\, dy\, dt,$$

a fact ensuing from (7.3). In view of (7.4), $(7.4b)$

$$F_{i_n, j} = \int_0^1 u_n(y)\, \frac{\bar{u}_j(y)}{\lambda_n}\, dy, \qquad F_{i'_n, j} = 0.$$

Inasmuch as $F_{ij} = F_{ji}$, one has

$$(7.7a) \qquad F_{i_n, i_n} = \frac{1}{\lambda_n}, \qquad F_{i,j} = 0 \qquad [\text{for } (i, j) \neq (i_n, i_n)],$$

Accordingly, on letting $i = j = i_n$ and then $i = j = i'_n$, from (7.7) it is inferred that

$$\int_0^1 \left[\int_0^1 Q(x, t)\, u_n(x)\, dx\right]^2 dt = \frac{1}{\lambda_n}, \qquad \int_0^1 \left[\int_0^1 Q(x, t)\, u'_n(x)\, dx\right]^2 dt = 0,$$

Thus

$$(7.8) \qquad \int_0^1 u'_n(x)\, Q(x, t)\, dt = 0;$$

furthermore, since $F_{i,j} = 0$ $(i \neq j)$, in view of (7.7) it is deduced that the sequence

$$(7.8\,a) \qquad v_n(t) = \lambda_n^{\frac{1}{2}} \int_0^1 u_n(x)\, Q(x, t)\, dx \qquad (n = 1, 2, \ldots).$$

is orthonormal. It will be now shown that

$$(7.8\,b) \qquad u_n(x) = \lambda_n^{\frac{1}{2}} \int_0^1 Q(x, t)\, v_n(t)\, dt.$$

In view of (7.8a), by (7.1′) one has

$$\lambda_n^{\frac{1}{2}} \int^1 Q(x, t)\, v_n(t)\, dt = \lambda_n \int_0^1 \left[\int_0^1 Q(x, t)\, Q(y, t)\, dt\right] u_n(y)\, dy$$

$$= \lambda_n \int_0^1 F(x, y)\, u_n(y)\, dy;$$

(7.8b) will ensue from (7.4). Thus, u_n, v_n, $\lambda_n^{\frac{1}{2}}$ are evidently the characteristic functions and values of the non symmetric kernel $Q(x, y)$ [note (7.8) and completeness of the sequence $(\bar{u}_\nu)$]. Let the v'_n complete the sequence (v_n),

$$(7.8\,c) \qquad \int_0^1 Q(x, t)\, v'_n(t)\, dt = 0.$$

We form a sequence $(\bar{v}_j)$, with

$$(7.8\,d) \qquad \bar{v}_{j_\nu} = v_\nu, \qquad \bar{v}_{j'_\nu} = v'_\nu.$$

The sequence

$$(7.9) \qquad \{\, \bar{u}_i(x)\, \bar{v}_j(y) \,\}$$

is complete orthonormal on $(0 \leq x, y \leq 1)$. Correspondingly

$$(7.10) \qquad Q(x, y) \sim \sum_{i,j} Q_{i,j} \bar{u}_i(x) \bar{v}_j(y)$$

(convergence in the mean square in (x, y)), where

$$(7.10a) \qquad Q_{i,j} = \int_0^1 \int_0^1 Q(x, y) \bar{u}_i(x) \bar{v}_j(y)\, dx\, dy.$$

Inasmuch as $Q(x, y)$ is L_2 in (x, y), the order of integration here is immaterial. By (7.4b), (7.8d) and (7.8), (7.8c)

$$Q_{i'_n, j} = 0 = Q_{i, j'_\nu} \qquad (n, \nu = 1, 2, \ldots).$$

For the remaining $Q_{i,j}$ one has

$$Q_{i_n, j_\nu} = \int_0^1 \int_0^1 Q(x, y)\, u_n(x)\, v_\nu(y)\, dx\, dy$$

and, by virtue of (7.8a),

$$Q_{i_n, j_\nu} = \int_0^1 \lambda_n^{-\frac{1}{2}} v_n(y)\, v_\nu(y)\, dy.$$

Thus

$$(7.10b) \qquad Q_{i_n, j_n} = \lambda_n^{-\frac{1}{2}}, \qquad Q_{i,j} = 0 \qquad [\text{for } (i, j) \neq (i_n, j_n)]$$

Whence (7.10) takes the form

$$(7.10') \qquad Q(x, y) \sim \sum \frac{1}{\lambda_n^{\frac{1}{2}}} u_n(x)\, v_n(y),$$

while [by (7.3)]

$$(7.11) \qquad \sum_n \frac{1}{\lambda_n} \left[= \sum_{i,j} Q^2_{i,j} = \int_0^1 \int_0^1 Q^2(x, y)\, dx\, dy \right] < +\infty.$$

This is another necessary condition for existence of the postulated solution.

We now consider the converse. *Assume* (7.5), (7.11) *and let* (v_n) *be any orthonormal sequence.* Designate by (v'_n) the set complementary to (v_n) and let $(\overline{v}_n)$ be the totality of the v_n, v'_n,

(7.12) $$\overline{v}_{j_\nu} = v_\nu, \qquad \overline{v}_{j'_\nu} = v'_\nu.$$

In view of (7.11) we can construct a function $Q(x, y)$,

(7.13) $$Q(x, y) \sim \sum_n \frac{1}{\lambda_n^{\frac{1}{2}}} u_n(x)\, v_n(y).$$

As a consequence of (7.11) $Q(x, y)$ is L_2 in (x, y). On writing

$$Q_{i,j} = \int_0^1 \int_0^1 Q(x, y)\, \overline{u_i}(x)\, \overline{v}_j(y)\, dx\, dy = \int_0^1 \left[\int_0^1 \overline{u_i}(x)\, Q(x, y)\, dx \right] \overline{v}_j(y)\, dy,$$

one has (7.10*b*). Furthermore, by (7.4*b*)

$$Q_{i'_n, j} = \int_0^1 \left[\int_0^1 u'_n(x)\, Q(x, y)\, dx \right] \overline{v}_j(y)\, dy = 0$$

$(j = 1, 2, \ldots)$; since $(\overline{v}_j)$ is a complete sequence, this implies that

$$\int_0^1 u'_n(x)\, Q(x, y)\, dy = 0 \qquad (n = 1, 2, \ldots).$$

On the other hand,

$$Q_{i_n, j} = \int_0^1 \left[\int_0^1 u_n(x)\, Q(x, y)\, dx \right] \overline{v}_j(y)\, dy = \begin{cases} 0 & (j \neq j_n), \\ \lambda_n^{-\frac{1}{2}} & (j = j_n), \end{cases}$$

$(j = 1, 2, \ldots)$; also, inasmuch as $(\overline{v}_j)$ is complete,

$$\int_0^1 u_n(x)\, Q(x, y)\, dx = \lambda_n^{-\frac{1}{2}} \overline{v}_{j_n}(y) = \lambda_n^{-\frac{1}{2}} v_n(y).$$

Accordingly, *the function* $Q(x, y)$, *given by* (7.13), *satisfies* (7.8), (7.8*a*); (7.8*a*) signifies that the sequence

(7.13 *a*) $$\lambda_n^{\frac{1}{2}} \int_0^1 u_n(x)\, Q(x, y)\, dx \qquad (n = 1, 2, \ldots)$$

is orthonormal, inasmuch as the sequence (v_n) has this property.

Consider the symmetric function

(α) $$H(x, y) = \int_0^1 Q(x, t)\, Q(y, t)\, dt.$$

Since the sequence $\{\bar{u}_i(x)\bar{u}_j(y)\}$ is complete, *equation* $(7.1')$ *will be satisfied if*

$$(\beta)\qquad \begin{cases} H_{ij} = \int_0^1\int_0^1 H(x, y)\,\bar{u}_i(x)\,\bar{u}_j(y)\,dx\,dy \\ \quad = F_{ij}\left[=\int_0^1\int_0^1 F(x, y)\,\bar{u}_i(x)\,\bar{u}_j(y)\,dx\,dy\right]. \end{cases}$$

Now the order of integration in

$$I = \int_0^1\int_0^1\int_0^1 Q(x, t)\,Q(y, t)\,\bar{u}_i(x)\,\bar{u}_j(y)\,dx\,dy\,dt$$

is immaterial, since $Q(x, y)$ is L_2 in (x, y). Thus by (α)

$$H_{ij} = I = \int_0^1\left[\int_0^1 \bar{u}_i(x)\,Q(x, t)\,dx\right]\left[\int_0^1 \bar{u}_j(y)\,Q(y, t)\,dy\right]dt.$$

Clearly $H_{ij} = H_{ji}$ and, as a consequence of (7.8), $(7.4b)$, $(7.8a)$,

$$H_{i'_n, j} = 0 = H_{i, i'_n},$$

$$H_{i_n, i_\nu} = \int_0^1\left[\int_0^1 u_n(x)\,Q(x, t)\,dx\right]\left[\int_0^1 u_\nu(y)\,Q(y, t)\,dy\right]dt$$

$$= \int_0^1 \frac{\varphi_n(t)}{\lambda_n^{\frac{1}{2}}}\,\frac{\varphi_\nu(t)}{\lambda_\nu^{\frac{1}{2}}}\,dt = \begin{cases} 0 & (n \neq \nu), \\ \dfrac{1}{\lambda_n} & (n = \nu); \end{cases}$$

that is,

$$H_{i_n, i_n} = \frac{1}{\lambda_n}, \qquad H_{ij} = 0 \qquad [\text{for } (i, j) \neq (i_n, i_n)].$$

Hence by $(7.7a)$ $H_{ij} = F_{ij}$ $(\text{all } (i, j))$. The function (7.13) therefore satisfies $(7.1')$. Multiplying by $a(x)\,a(y)$, one obtains

$$\int_0^1 q(x, t)\,q(y, t)\,dt = f(x, y),$$

where

$$(7.14)\qquad q(x, t) = a(x)\,Q(x, t);$$

$q(x, y)$ will constitute a solution of (7.1). We thus obtained the following result.

THEOREM 7.15. — *Consider the problem* (7.1), *where* $f(x, y)$ *is symmetric*, L_2 *in* x, L_2 *in* y. *We look for solutions* $q(x, y)$ *not necessarily symmetric.*

If for some $a(x)(\geqq 1)$, *such that*

$$F(x, y) = \frac{f(x, y)}{a(x)\,a(y)} \subset L_2 \qquad [\text{in } (x, y)],$$

the characteristic values λ_i *of* F *satisfy*

$$(7.15\,a) \qquad \lambda_i > 0, \qquad \sum_i \frac{1}{\lambda_i} < +\infty,$$

while the problem has a solution $q(x, y)$ (L_2 *in* x *and* L_2 *in* y) *such that* $Q(x, y) = a^{-1}(x)q(x, y)$ *is* L_2 *in* (x, y), *then necessarily* $q(x, y)$ *must be of the form*

$$(7.15\,b) \qquad \frac{q(x, y)}{a(x)} \sim \sum_n \frac{1}{\lambda_n^{\frac{1}{2}}} u_n(x)\, v_n(y),$$

where the $u_n(x)$ *are the characteristic functions of* F *and* (v_n) *constitutes a certain orthonormal sequence* [*cf.* 7.8*a*)].

The converse. If $a(x)$ *is such that* $F(x, y)$ (*cf. above*) *is* L_2 *in* (x, y), *while* (7.15*a*) *holds, then functions* $q(x, y)$ *of the form* (7.15*b*), *where* (v_n) *is any orthonormal sequence, will satisfy the problem; furthemore,* $a^{-1}(x)q(x, y)$ *will be* L_2 *in* (x, y).

We observe that *a solutiou* $q(x, y)$ *of* (7.1) *satisfies the iteration problem*

$$(7.16) \qquad \int_0^1 q(x, t)\,q(t, y)\,dt = f(x, y),$$

if $q(x, y)$ *is symmetric.* It is of interest to note that *for symmetry of* $q(x, y)$ *it is necessary and sufficient that*

$$\frac{q(x, y)}{a(x)\,a(y)} [= Q(x, y)\,a^{-1}(y) = T(x, y)]$$

be orthogonal to the functions of the sequence

$$(7.16\,a) \qquad u_i(x)\,u_j'(y) \qquad (i, j = 1, 2, \ldots),$$
$$2^{-\frac{1}{2}}[u_n(x)\,u_\nu(y) - u_\nu(x)\,u_n(y)] \qquad (n < \nu),$$

orthonormal on $(0 \leqq x, y \leqq 1)$.

In fact, $T(x, y)$ is L_2 in (x, y). Now

$$T(x, y) \sim \sum_{i,j} T_{ij}\, \bar{u}_i(x)\, \bar{u}_j(y), \qquad T(y, x) \sim \sum_{i,j} T_{ij}\, \bar{u}_i(y)\, \bar{u}_j(x).$$

Interchange of i, j in the latter relation yields

$$T(y, x) \sim \sum_{ij} T_{ji}\, \bar{u}_i(x)\, \bar{u}_j(y).$$

Hence symmetry of $q(x, y)$ is equivalent to the relations $T_{ij} = T_{ji}$; that is, to

$$\int_0^1 \int_0^1 \frac{Q(x, y)}{a(y)}\, \bar{u}_i(x)\, \bar{u}_j(y)\, dx\, dy = \int_0^1 \int_0^1 \frac{Q(x, y)}{a(y)}\, \bar{u}_j(x)\, \bar{u}_i(y)\, dx\, dy.$$

The order of integration here being immaterial, one may write the above in the form

$$\int_0^1 \left[\int_0^1 \bar{u}_i(x)\, Q(x, y)\, dx \right] \frac{\bar{u}_j(y)}{a(y)}\, dy = \int_0^1 \left[\int_0^1 \bar{u}_j(x)\, Q(x, y)\, dx \right] \frac{\bar{u}_i(y)}{a(y)}\, dy.$$

In view of (7.8), (7.4b) the above is equivalent to

$$\int_0^1 \left[\int_0^1 u_n(x) \frac{Q(x, y)}{a(y)}\, dx \right] u'_\nu(y)\, dy = 0,$$

$$\int_0^1 \left[\int_0^1 u_n(x)\, Q(x, y)\, dx \right] \frac{u_\nu(y)}{a(y)}\, dy = \int_0^1 \left[\int_0^1 u_\nu(x)\, Q(x, y)\, dx \right] \frac{u_n(y)}{a(y)}\, dy.$$

The conclusion (7.16), (7.16a) ensues.

Let $(w_i(x, y))$ $(i = 1, 2, \ldots)$ *be a sequence completing the sequence* (7.16a) *on* $(0 \leqq x, y \leqq 1)$ in such a way that the $w_i(x, y)$ and the functions (7.16a) together form an orthonormal sequence. *If* $q(x, y)$ *is a symmetric solution (with stated properties) of the second order iteration problem* (7.16), *we necessarily have*

$$\text{(7.17)} \qquad \frac{q(x, y)}{a(x)\, a(y)} \sim \sum_i \gamma_i w_i(x, y)$$

in the sense of mean square convergence on $(0 \leqq x, y \leqq 1)$. This fact ensues as a consequence of the orthogonality of $T(x, y)$, to the functions (7.16a).

8. THE ITERATION PROBLEM ($n=2$). — We shall now investigate the second order iteration problem

$$(\mathbf{8}.1)\qquad [q^{[2]}(x, y)\equiv]\int_0^1 q(x, t)\,q(t, y)\,dt = f(x, y),$$

where $f(x, y)$ is given symmetric, L_2 in x, L_2 in y, and the solution $q(x, y)$ is to be symmetric, L_2 in x (in y); the equation is to be satisfied for almost all (x, y) in the square $(0\leq x, y\leq 1)$.

We observe that for existence of a solution (or solutions) of ($\mathbf{8}.1$) it is necessary that $f(x, y)$ be positive definite, *this we henceforth assume*. One can represent $f(x, y)$, in infinitely many ways, in the form

$$(\mathbf{8}.2)\qquad f(x, y)=\lim_m f_m(x, y),$$

where $f_m(x, y)$ is L_2 in (x, y) and $f_m(x, y)$ is symmetric, positive definite. Furthermore, *the $f_m(x, y)$ may be chosen so that*

$$(\mathbf{8}.2')\qquad \left[\int_0^1 f_m^2(x, t)\,dt\right]^{\frac{1}{2}}\leq f^*(x)<+\infty$$

almost everywhere, with $f^*(x)$ independent of m. An example of f_m, satisfying the above conditions, is given by

$$f_m(x, y)=f(x, y)\qquad (\text{wherever } |f|\leq m),$$
$$f_m(x, y)=\pm m\qquad (\text{wherever } \pm f> m).$$

Consider any particular representation ($\mathbf{8}.2$) (that is, assume a particular sequence $(f_m(x, y))$. Designate by $\lambda_{m\nu}$, $u_{m\nu}$ the characteristic values and functions of $f_m(x, y)$,

$$(\mathbf{8}.2\,a)\qquad u_{m\nu}(x)=\lambda_{m\nu}\int_0^1 f_m(x, t)\,u_{m\nu}(t)\qquad (\nu=1, 2, \ldots);$$

here $\lambda_{m\nu}>0$. Corresponding to a fixed m we break up the sequence

$$(\nu)=(1, 2, \ldots)$$

into two sequences

$$(\mathbf{8}.2\,b)\qquad (p_j),\quad (n_j)\qquad (j=1, 2, \ldots)$$

and, correspondingly, write

$$(\mathbf{8}.2\,c)\qquad \Gamma_m^{[2]'}(x, y \mid \lambda) = \sum_{0<\lambda_{m,p_j}<\lambda} u_{m,p_j}(x)\, u_{m,p_j}(y) \qquad (\lambda > 0),$$

$$\Gamma_m^{[2]'}(x, y \mid \lambda) = 0 \qquad (\lambda \leqq 0),$$

$\Gamma_m^{[2]''}(x, y \mid \lambda) = 0 (\lambda \leqq 0)$, and

$$(\mathbf{8}.2\,d)\qquad \Gamma_m^{[2]''}(x, y \mid \lambda) = \sum_{0<\lambda_{m,n_j}<\lambda} u_{m,n_j}(x)\, u_{m,n_j}(y) \qquad (\lambda > 0),$$

$\Gamma_m^{[2]''}(x, y \mid \lambda) = 0 (\lambda \leqq 0)$. The spectral function of $f_m(x, y)$ is

$$(\mathbf{8}.3)\qquad \Gamma_m^{[2]}(x, y \mid \lambda) = \sum_{0<\lambda_{m\nu}<\lambda} u_{m\nu}(x)\, u_{m\nu}(y) \qquad (\lambda > 0),$$

$$\Gamma_m^{[2]}(x, y \mid \lambda) = 0 \qquad (\lambda \leqq 0).$$

One has

$$(\mathbf{8}.4)\qquad \Gamma_m^{[2]}(x, y \mid \lambda) = \Gamma_m^{[2]'}(x, y \mid \lambda) + \Gamma_m^{[2]''}(x, y \mid \lambda).$$

For some (m_j) $(\lim m_j = +\infty)$ the limit

$$(\mathbf{8}.4')\qquad \lim \Gamma_{m_j}^{[2]}(x, y \mid \lambda) = \Gamma^{[2]}(x, y \mid \lambda)$$

exists and represents a spectral function of $f(x, y)$ [in the sense of (C)]. We shall show that (m_j) *may be chosen so that the limits*

$$(\mathbf{8}.5)\quad \lim \Gamma_{m_j}^{[2]'}(x, y \mid \lambda) = \Gamma^{[2]'}(x, y \mid \lambda), \qquad \lim \Gamma_{m_j}^{[2]''}(x, y \mid \lambda) = \Gamma^{[2]''}(x, y \mid \lambda)$$

exist also and represent functions of bounded variation in λ (on every finite real interval).

We write

$$0 = l_0 < l_1 < \ldots < l_s = l$$

and note that, by $(\mathbf{8}.2\,a)$,

$$\Lambda^{m,s} \equiv \sum_{i=1}^{s} \left| \Gamma_m^{[2]'}(x, y \mid l_j) - \Gamma_m^{[2]'}(x, y \mid l_{j-1}) \right| \leqq \sum_{\lambda_{m,p_j}<l} \left| u_{m,p_j}(x)\, u_{m,p_j}(y) \right|$$

$$\leqq \sum_{\lambda_{m,p_j}<l} \lambda_{m,p_j}^2 \left| \int_0^1 f_m(x, t)\, u_{m,p_j}(t)\, dt \int_0^1 f_m(y, t)\, u_{m,p_j}(t)\, dt \right|$$

$$\leqq l^2 \left\{ \sum_j \left| \int_0^1 f_m(x, t)\, u_{m,p_j}(t)\, dt \right|^2 \sum_j \left| \int_0^1 f_m(y, t)\, u_{m,p_j}(t)\, dt \right|^2 \right\}^{\frac{1}{2}}$$

and, by Bessel's inequality and (**8**.2′),

$$\Lambda^{m,s} \leqq l^2 \left\{ \int_0^1 f_m^2(x, t)\, dt \int_0^1 f_m^2(y, t)\, dt \right\}^{\frac{1}{2}} \leqq l^2 f^*(x)\, f^*(y)$$

Hence

$$|\Gamma_m^{[2]'}(x, y \mid \lambda)|, \quad \mathrm{Var}|_0^l \Gamma_m^{[2]'}(x, y \mid \lambda) \leqq l^2 f^*(x) f^*(y)$$

for $|\lambda| \leqq l$ and for all $l > 0$. The latter inequalities signify that the sequence (m_j), involved in (**8**.4′), contains a subsequence, which we still term (m_j), so that

$$\lim \Gamma_{m_j}^{[2]'}(x, y \mid \lambda) = \Gamma^{[2]'}(x, y \mid \lambda)$$

exists; in view of (**8**.4), (**8**.4′) the same will be true for $\Gamma_{mj}^{[2]''}(x, y \mid \lambda)$. The italics with respect to (**8**.4a) are thus demonstrated.

The condition (**8**.2′) may be deleted, if in some way we can demonstrate that

$$\Lambda^{m,s} \leqq G(x, y, l) \qquad [G(x, y, l) < +\infty \text{ almost everywhere, when } 0 < l < +\infty]$$

where the second member is independent of m, s, and if a similar inequality can be established for $\Gamma_m^{[2]'}$.

In view of (**8**.4), (**8**.5) *there results a decomposition of the spectral function* $\Gamma^{[2]}$ *of* f,

$$\text{(D)} \qquad \Gamma^{[2]}(x, y \mid \lambda) = \Gamma^{[2]'}(x, y \mid \lambda) + \Gamma^{[2]''}(x, y \mid \lambda).$$

This decomposition has a considerable degree of arbitrariness. The formula (D) depends on the choice of the approximating functions $f_m(x, y)$ [*cf.* (**8**.2)], the choice of the sequences (p_j), (n_j) [(**8**.2b)] and on the choice of the sequence (m_j) in (**8**.4a).

Suppose $q(x, y)$ is a solution with properties as stated at the beginning of this section. Define $q_m(x, y)$ by the relations

$$(\mathbf{8}.6) \qquad \begin{cases} q_m(x, y) = q(x, y) & [\text{when } |q(x, y)| < m], \\ q_m(x, y) = \pm m & [\text{when } \pm q(x, y) > m]. \end{cases}$$

Let the $\rho_{m\nu}$, $u_{m\nu}(x)$ be the characteristic values and functions of $q_m(x, y)$

$$(\mathbf{8}.6\,a) \qquad u_{m\nu}(x) = \rho_{m\nu} \int_0^1 q_m(x, t)\, u_{m\nu}(t)\, dt.$$

Designate by $\Gamma_m(x, y \mid \rho)$ the spectral function of $q_m(x, y)$; thus

$$(8.6\,b) \quad \Gamma_m(x, y \mid \rho) = \begin{cases} \sum_{0 < \rho_{m\nu} < \rho} u_{m\nu}(x)\, u_{m\nu}(y) & (\text{for } \rho > 0), \\ -\sum_{\rho \leq \rho_{m\nu} < 0} u_{m\nu}(x)\, u_{m\nu}(y) & (\text{for } \rho < 0), \end{cases}$$

$\Gamma(x, y \mid 0) = 0$. Inasmuch as $q(x, y)$ is L_2 in x, it follows by spectral theory that for some sequence (m_j) the limit

$$(8.6\,c) \qquad \lim \Gamma_{m_j}(x, y \mid \rho) = \Gamma(x, y \mid \rho)$$

exists. The spectral function $\Gamma(x, y \mid \rho)$ of $q(x, y)$ is not necessarily unique. The kernel

$$(8.7) \qquad q_m^{[2]}(x, y) = \int_0^1 q_m(x, t)\, q_m(t, y)\, dt$$

is regular, that is $\subset L_2$ in (x, y); it is observed that *the*

$$\lambda_{m\nu} = \rho_{m\nu}^2, \qquad u_{m\nu}(x)$$

are the characteristic values and functions of $q_m^{(2)}(x, y)$. Since $q(x, y)$ has been assumed to be a solution of (8.1), with the understanding that *existence of an integral related to an iterant implies absolute integrability*, we note that the integral

$$(8.7') \qquad |q(x, y)|^{[2]} = \int_0^1 |q(x, t)\, q(t, y)|\, dt$$

exists; thus, inasmuch as

$$|q_m(x, t)\, q_m(t, y)| \leq |q(x, t)\, q(t, y)|,$$

we obtain [for almost all (x, y)]

$$(8.7\,a) \quad \lim q_{m_j}^{[2]}(x, y) = \int_0^1 q(x, t)\, q(t, y)\, dt = q^{[2]}(x, y) = f(x, y).$$

The kernel $q^{(2)}(x, y)$ is therefore L_2 in x (in y)

$$f_m(x, y) = q_m^{[2]}(x, y)$$

is a regular approximating kernel of $f(x, y)$. Obviously [*cf.* (8.7)] $f_m(x, y)$ is positive definite. For any fixed m the sequence (ν)

consists of two subsequences (p_j), (n_j) so that

$$\rho_{m,p_j}=\sqrt{\lambda_{m,p_j}}>0, \qquad \rho_{m,n_j}=-\sqrt{\lambda_{m,n_j}}<0 \qquad (j=1, 2, \ldots).$$

We note that $(8.2')$, as relating to the case at hand, may be dispensed with in accordance with the remark preceding (D); in fact, the inequalities subsequent $(8.4a)$ may be replaced by

$$\Lambda^{m,\epsilon}\leqq \sum_{\lambda_{m,p_j}<l}\left|\sqrt{\lambda_{m,p_j}}\int_0^1 q_m(x, t)\, u_{m,p_j}(t)\, dt\right|\left|\sqrt{\lambda_{m,p_j}}\int_0^1 q_m(y, t)\, u_{m,p_j}(t)\, dt\right|$$

[*cf.* $(8.6a)$] so that

$$\Lambda^{m,\epsilon}\leqq l\left\{\sum\left|\int_0^1 q_m(x, t)\, u_{m,p_j}(t)\, dt\right|^2\sum_j\left|\int_0^1 q_m(y, t)\, u_{m,p_j}(t)\, dt\right|^2\right\}^{\frac{1}{2}}$$

$$\leqq l\, q(x)\, q(y) \qquad \left[q^2(x)=\int_0^1 q^2(x, t)\, dt\right].$$

A similar inequality will hold for $\Gamma_m^{[2]''}$.

$\Gamma_m(x, y\,|\,\rho)$ of $(8.6b)$ may be expressed as

$$\Gamma_m(x, y\,|\,\rho)=\sum_{\lambda_{m,p_j}<\rho^2} u_{m,p_j}(x)\, u_{m,p_j}(y) \qquad (\text{for } \rho>0),$$

$$\Gamma_m(x, y\,|\,\rho)=-\sum_{\lambda_{m,n_j}\leq\rho^2} u_{m,n_j}(x)\, u_{m,n_j}(y) \qquad (\text{for } \rho<0).$$

In view of $(8.2c)$, $(8.2d)$

$$(8.8) \qquad \Gamma_m(x, y\,|\,\rho)=\Gamma_m^{[2]'}(x, y\,|\,\rho^2) \qquad (\rho>0)$$

and

$$(8.8a) \qquad -\Gamma_m(x, y\,|\,\rho)=\Gamma_m^{[2]''}(x, y\,|\,\rho^2)+\sigma_m(x, y\,|\,\rho) \qquad (\rho<0),$$

where

$$(8.8b) \qquad \sigma_m(x, y\,|\,\rho)=\sum_{\lambda_{m,n_j}=\rho^2}{}' u_{m,n_j}(x)\, u_{m,n_j}(y);$$

the latter sum is over values j (if any) such that $\lambda_{m,n_j}=\rho^2$; there may be several λ_{m,n_j} equal to ρ^2; one has

$$(8.8c) \qquad \sigma_m(x, y\,|\,\rho)=0 \qquad (\text{for } \rho\neq-\sqrt{\lambda_{m,n_j}}).$$

The sequence $(m)=(m_j)$ in $(8.6c)$ is chosen so that the limits in (**8**.4a) exist. Since, as $m_j \to \infty$, the limits of the first member and of the first term in the second member in (**8**.8a) exist and are of bounded variation, the limit

$$\sigma(x, y \mid \rho) = \lim \sigma_{m_j}(x, y \mid \rho)$$

exists and is of bounded variation (on every finite interval); by (**8**.8c)

(**8**.9) $$\sigma(x, y \mid \rho) \mid = 0 \qquad (\rho \neq -\sqrt{\lambda_{m_j, n_i}};\, j,\, i = 1, 2, \ldots).$$

In the limit from (**8**.8), (**8**.8a) one obtains

(**8**.9a) $$\begin{cases} \Gamma(x, y \mid \rho) = \quad \Gamma^{[2]'}(x, y \mid \rho^2) & (\rho > 0), \\ \Gamma(x, y \mid \rho) = -\Gamma^{[2]''}(x, y \mid \rho^2) - \sigma(x, y \mid \rho) & (\rho < 0). \end{cases}$$

It is to be noted that, in view of (**8**.9).

(**8**.10) $$\int_\alpha^\beta c(x)\, d_\lambda \sigma(x, y \mid \lambda) = 0,$$

whenever $c(\lambda)$ is continuous on the finite closed interval (α, β). In fact, let

$$\alpha = \rho_0 < \rho_1 < \ldots < \rho_n = \beta$$

and designate by ζ_ν a point on the interval $(\rho_{\nu-1}, \rho_\nu)$. Since the set of points δ exclusive of the $-\sqrt{\lambda_{m,n_j}}(m, j = 1, 2, \ldots)$ is everywhere dense on the axis of reals, the ρ_ν may be chosen in δ so that the maximum $|\rho_\nu - \rho_{\nu-1}|$ $(\nu = 1, \ldots, n)$ is arbitrarily small. With such a choice of the ρ_ν one has

$$\Delta_\nu \sigma(x, y \mid \lambda) = \sigma(x, y \mid \rho_\nu) - \sigma(x, y \mid \rho_{\nu-1}) = 0.$$

Whence

$$S_n = \sum_{\nu=1}^{n} c(\zeta_\nu)\, \Delta_\nu \sigma(x, y \mid \lambda) = 0;$$

now σ is of bounded variation and c is continuous; hence the integral in (**8**.10) equals $\lim S_n = 0$.

On writing $l_1 = l^{\frac{1}{2}}$, by (**8**.9a) and with the aid of (**8**.10) we obtain

(**8**.11) $$\int_{-l_1}^{l_1} \frac{1}{\rho}\, d_\rho \Gamma(x, y \mid \rho) = \int_0^{l_1} \frac{1}{\rho}\, d_\rho \Gamma^{[2]'}(x, y \mid \rho^2) - \int_0^{l_1} \frac{1}{\rho}\, d_\rho \Gamma^{[2]''}(x, y \mid \rho^2)$$
$$= \int_0^{l} \frac{1}{\sqrt{\lambda}}\, d_\lambda \Gamma^{[2]'}(x, y \mid \lambda) - \int_0^{l} \frac{1}{\sqrt{\lambda}}\, d_\lambda \Gamma^{[2]''}(x, y \mid \lambda).$$

If the integral

$$\int_{-\infty}^{\infty} \frac{1}{\rho} d_\rho \Gamma(x, y \mid \rho)$$

converges in the ordinary sense or in the mean square in x (in y), it will represent $q(x, y)$. It will be shown that

$$(\mathbf{8}.12) \qquad \begin{cases} \int_0^{\infty} \frac{1}{\sqrt{\lambda}} d_\lambda \Gamma^{[2]'}(x, y \mid \lambda) \sim q'(x, y), \\ \int_0^{\infty} \frac{1}{\sqrt{\lambda}} d_\lambda \Gamma^{[2]''}(x, y \mid \lambda) \sim q''(x, y) \end{cases}$$

in the sense of mean square convergence in x (in y) to some functions q', q''. By virtue of (**8**.11) this would signify that in the indicated sense one has

$$(\mathbf{8}.13) \qquad \int_{-\infty}^{\infty} \frac{1}{\rho} d_\rho \Gamma(x, y \mid \rho) \sim q(x, y) = q'(x, y) - q''(x, y).$$

In order to establish (**8**.12) [and hence (**8**.13)] we shall first prove that

$$(\mathbf{8}.14) \qquad \Lambda(x) = \int_0^{8} \frac{1}{\lambda} d_\lambda \Gamma^{[2]}(x, x \mid \lambda) < +\infty$$

for almost all x, *this being a condition necessary of the existence of a symmetric solution* $q(x, y)$, L_2 *in* x (*for almost all* y). We note that for such a solution necessarily $q^{[2]}(x, y)$ is L_2 in x (for almost all y). We proceed now with $q(x, y)$ having the stated properties. Applying the spectral formula (**1**.12), with

$$K(x, t) = q(x, t), \qquad g(t) = q(x, t), \qquad \theta^{u,v} = \Gamma \qquad [cf. (\mathbf{8}.6\,c)],$$

we obtain

$$(1^\circ) \qquad \int_0^1 q^2(x, t)\, dt = \int_{-\infty}^{\infty} \frac{1}{\rho} d_\rho \int_0^1 \Gamma(x, t \mid \rho)\, q(x, t)\, dt [\equiv T_0^\infty] < \infty$$

(in the sense of ordinary convergence, for almost all x). As a consequence of (C; p. 33) we have

$$(2^\circ) \qquad \Gamma(x, y \mid \rho) = \int_0^{\rho} \lambda\, d_\lambda \int_0^1 \Gamma(y, t \mid \lambda)\, q(x, t)\, dt;$$

also, we note the theorem (C; p. 11) according to which

$$(3^\circ)\qquad \int_a^b \sigma(\rho)\, d_\rho \int^\rho \omega(\lambda)\, d\alpha(\lambda) = \int_a^b \sigma(\rho)\,\omega(\rho)\, d\alpha(\rho)$$

whenever σ, ω are functions continuous on the closed interval (a, b) and α is of bounded variation on (a, b). With $0 < \delta < l < \infty$, we consider the integral

$$(4^\circ)\qquad \mathbf{S}_\delta^l = \left(\int_{-l}^{-\delta} + \int_\delta^l\right) \frac{1}{\rho^2}\, d_\rho\, \Gamma(x, x \mid \rho).$$

In (4°) we substitute $\Gamma(x, x \mid \rho)$ from (2°) and then apply the theorem (3°), with

$$\sigma(\rho) = \frac{1}{\rho^2}, \qquad \omega(\lambda) = \lambda, \qquad \alpha(\lambda) = \int_0^1 \Gamma(x, t \mid \lambda)\, q(x, t)\, dt;$$

$\alpha(\lambda)$ is of course of bounded variation, since $q(x, t)$ is L_2 in x (in t) and Γ is a spectral function of $q(x, t)$. By (3°)

$$\mathbf{S}_\delta^\rho = \left(\int_{-l}^{-\delta} + \int_\delta^l\right) \frac{1}{\rho}\, d_\rho \int_0^1 \Gamma(x, t \mid \rho)\, q(x, t)\, dt.$$

Comparing the second members here and in (1°), one obtains

$$(5^\circ)\qquad \lim \mathbf{S}_\delta^l = T_0^\infty \qquad (\text{as } \delta \to 0,\ l \to +\infty).$$

Thus by (1°) and (4°)

$$T_0^\infty = \left[\mathbf{S}_0^\infty =\right] = \int_{-\infty}^\infty \frac{1}{\rho^2}\, d_\rho\, \Gamma(x, x \mid \rho) < +\infty$$

(almost all x). As a consequence of (**8**.9), (**8**.9a) it is inferred that

$$T_0^\infty = \Lambda(x) \qquad [cf.\ (\mathbf{8}.14)].$$

Thus, in view of (5°), (**8**.14) *has been proved.* By virtue of (**8**.14) one has $\Lambda = \Lambda' + \Lambda''$, where

$$(\mathbf{8}.15)\qquad \begin{cases} \Lambda'(x) = \displaystyle\int_0^\infty \frac{1}{\lambda}\, d_\lambda\, \Gamma^{[2]'}(x, x \mid \lambda) < \infty, \\ \Lambda''(x) = \displaystyle\int_0^\infty \frac{1}{\lambda}\, d_\lambda\, \Gamma^{[2]''}(x, x \mid \lambda) < \infty, \end{cases}$$

(almost all x); $\Lambda' \geqq 0$, $\Lambda'' \geqq 0$.

We let $0 < l < l'$ and observe that by (**8**.2c) and by *Fatou's* lemma

$$Q^{l,l'} = \int_0^1 \left[\int_l^{l'} \frac{1}{\sqrt{\lambda}} d_\lambda \Gamma^{[2]'}(x, y \mid \lambda) \right]^2 dx \leq \lim_{m_j} \int_0^1 \left[\int_l^{l'} \frac{1}{\sqrt{\lambda}} d_\lambda \Gamma^{[2]'}_{m_j}(x, y \mid \lambda) \right] dx$$
$$= \lim_{m_j} \int_0^1 \left[\sum_i^{(l,l')} (\lambda_{m,p_i})^{-\frac{1}{2}} u_{m_j,p_i}(x)\, u_{m_j,p_i}(y) \right]^2 dx;$$

here the summation is over values i for which λ_{m,p_i} is on (l, l'). One has

$$Q^{l,l'} \leq \lim_{m_j} \int_0^1 \sum_i^{(l,l')} \sum_s^{(l,l')} \lambda_{m,p_i}^{-\frac{1}{2}} \lambda_{m,p_s}^{-\frac{1}{2}} u_{m,p_i}(x)\, u_{m,p_s}(x) \times u_{m,p_i}(y)\, u_{m,p^s}(y)\, dx$$
$$= \lim_{m_j} \sum_i^{(l,l')} \frac{1}{\lambda_{m,p_i}} u^2_{m,p_i}(y) = \lim_{m_j} \int_l^{l'} \frac{1}{\lambda} d_\lambda \Gamma^{[2]'}_m(y, y \mid \lambda).$$

By *Helly's* theorem on the passage to the limit under the integral sign,

$$Q^{l,l'} \leq \int_l^{l'} \frac{1}{\lambda} d_\lambda \Gamma^{[2]'}(y, y \mid \lambda).$$

Since the first integral (**8**.15) converges, we have

$$[0 \leq]\, Q^{l,l'} < \varepsilon \qquad \text{for} \quad l,\ l' \geq l(\varepsilon),$$

where $l(\varepsilon)$ is suitably great. Hence the first integral in (**8**.12) converges, as indicated, to some symmetric function $q'(x, y)$, $\subset L_2$ in x (in y). We similarly prove the other relation (**8**.12). *Whence* (**8**.13) *holds*.

We consider now the converse. It is assumed that $f(x, y)$ is positive definite and that the integral (**8**.14) is finite (almost everywhere) for some spectral function $\Gamma^{[2]}$ of f [then the integrals (**8**.15) will have the same property for any decomposition (D) of $\Gamma^{[2]}$]. Consider now a choice of $f_m(x, y)$ so that (**8**.2), (**8**.2′) hold, as well as a choice of sequences (p_j), (n_j) [*cf.* (**8**.2b)] for $m = 1, 2, \ldots$ (these sequences may depend on m). *Let* (D) *be a corresponding decomposition of* $\Gamma^{[2]}$, obtained for some suitable sequence $(m) = (m_j)$ [involved in (**8**.4a)]. We envisage (**8**.15) for this decomposition. We repeat the developments given subsequent to (**8**.13) up to the inequality $Q^{l,l'} < \varepsilon$, assigning to the symbols involved their present meanings; the latter inequality will ensue by (8.15). A similar result is obtained for $\Gamma^{[2]''}$. On the refore has

(8.16) $$\begin{cases} \int_0^\infty \frac{1}{\sqrt{\lambda}} d_\lambda \Gamma^{[2]'}(x, y \mid \lambda) \sim q'(x, y), \\ \int_0^\infty \frac{1}{\sqrt{\lambda}} d_\lambda \Gamma^{[2]''}(x, y \mid \lambda) \sim q''(x, y) \end{cases}$$

[mean square convergence in x (in y)], where q', q'' are some symmetric functions, L_2 in x (L_2 in y).

We have

(8.17) $$|q(x, y)|^{[2]} < +\infty, \qquad [q(x, y) = q'(x, y) - q''(x, y)]$$

almost everywhere. It will be shown that $q(x, y)$ is a solution of the iteration problem.

Consider the functions

$$q'^{(l)}(x, y) = \int_0^l \frac{1}{\sqrt{\lambda}} d_\lambda \Gamma^{[2]'}(x, y \mid \lambda), \qquad q''^{(l)}(x, y) = \int_0^l \frac{1}{\sqrt{\lambda}} d_\lambda \Gamma^{[2]''}(x, y \mid \lambda)$$

and observe that

$$q'^{(l)}(x, y) = \int_0^{l'} \frac{1}{\rho} d_\rho \Gamma^{[2]'}(x, y \mid \rho^2), \qquad q''^{(l)}(x, y) = \int_0^{l'} \frac{1}{\rho} d_\rho \Gamma^{[2]''}(x, y \mid \rho)$$

$(l' = l^{\frac{1}{2}})$. Since by (8.16) $q'^{(l)}$, $q''^{(l)}$ converge in the mean square in x (in y), as $l \to +\infty$, to q', q'', respectively, we also have

$$\int_0^\infty \frac{1}{\rho} d_\rho \Gamma^{[2]'}(x, y \mid \rho^2) \sim q'(x, y), \qquad \int_0^\infty \frac{1}{\rho} d_\rho \Gamma^{[2]''}(x, y \mid \rho^2) \sim q''(x, y).$$

One accordingly has

$$q(x, y) \sim \int_0^\infty \frac{1}{\rho} d_\rho \Gamma^{[2]'}(x, y \mid \rho^2) - \int_0^\infty \frac{1}{\rho} d_\rho \Gamma^{[2]''}(x, y \mid \rho^2)$$

[*cf.* (8(17)]; that is,

(8.18) $$q(x, y) \sim \int_{-\infty}^\infty \frac{1}{\rho} d_\rho \theta(x, y \mid \rho),$$

where

(8.18a) $$\theta(x, y \mid \rho) = \begin{cases} \Gamma^{[2]'}(x, y \mid \rho^2) & (\rho > 0). \\ -\Gamma^{[2]''}(x, y \mid \rho^2) & (\rho < 0). \end{cases}$$

It is observed that $\theta(x, y \mid \rho)$ is a spectral function (a denumerable set of values ρ possibly excepted) of $q(x, y)$. Using this fact and observing that the lines of reasoning employed in (C; 124, 125) are now valid as a consequence of the fact that $q(x, y)$ is L_2 in x, we obtain

$$\int_{-\infty}^{\infty} \frac{1}{\rho^2} d_\rho \, \theta(x, y \mid \rho) = q^{[2]}(x, y)$$

in the sense of ordinary convergence. Thus

$$q^{[2]}(x, y) = \int_0^{\infty} \frac{1}{\rho} d_\rho \, \theta(x, y \mid \sqrt{\rho}) - \int_0^{\infty} \frac{1}{\rho} d_\rho \, \theta(x, y \mid -\sqrt{\rho})$$

and, by (**8**.18*a*),

$$q^{[2]}(x, y) = \int_0^{\infty} \frac{1}{\rho} d_\rho \, \Gamma^{[2]'}(x, y \mid \rho) + \int_0^{\infty} \frac{1}{\rho} d\rho \, \Gamma^{[2]''}(x, y \mid \rho).$$

Whence by virtue of (D)

$$q^{[2]}(x, y) = \int_0^{\infty} \frac{1}{\rho} d_\rho \, \Gamma^{[2]}(x, y \mid \rho).$$

Now $\Gamma^{[2]}(x, y \mid \rho)$ is a spectral function of f; the last member above is hence a spectral representation of f [valid for almost all $(x. y)$]. Consequently $q(x, y)$ is a solution of the iteration problem.

THEOREM **8**.19. — *Consider the iteration problem* (8.1), *where* $f(x, y)$ *is positive definite (a necessary condition).*

If $q(x, y)$ *is a solution of* (**8**.1), L_2 *in* x (*in* y), *there is on hand a corresponding decomposition* (D) *of a spectral function* $\Gamma^{[2]}$ *of* f, *say* $\Gamma^{[2]} = \Gamma^{[2]'} + \Gamma^{[2]''}$, *so that*

$$q(x, y) \sim \int_0^{\infty} \frac{1}{\sqrt{\lambda}} d_\lambda \, \Gamma^{[2]'}(x, y \mid \lambda) - \int_0^{\infty} \frac{1}{\sqrt{\lambda}} d_\lambda \, \Gamma^{[2]''}(x, y \mid \lambda)$$

[*mean square convergence in* x (*in* y)] *and so that*

$$\Lambda(x) = \int_0^{\infty} \frac{1}{\lambda} d_\lambda \Gamma^{[2]}(x, x \mid \lambda) < +\infty \qquad \text{(for almost all } x\text{)}.$$

The converse. Envisage a decomposition (D), $\Gamma^{[2]} = \Gamma^{[2]'} + \Gamma^{[2]''}$, *of a spectral function* $\Gamma^{[2]}$ *of* f, *for which* $\Lambda(x) < +\infty$ (*for almost all* x), *Then*

$$\int_0^\infty \frac{1}{\sqrt{\lambda}} d_\lambda \Gamma^{[2]'}(x, y \mid \lambda) \sim q'(x, y), \qquad \int_0^\infty \frac{1}{\sqrt{\lambda}} d_\lambda \Gamma^{[2]''}(x, y \mid \lambda) \sim q''(x, y),$$

convergence being in the mean square in x (in y) to some functions q', q''. The function

$$q(x, y) = q'(x, y) - q''(x, y)$$

will represent a solution of the iteration problem.

9. THE ITERATION PROBLEM (n ODD). — We now turn to the general iteration problem

$$q^{[2]}(x, y) = f(x, y), \tag{9.1}$$

where n is any odd integer, $f(x, y)$ is given symmetric and is L_2 in x, in y; here

$$\left\{\begin{aligned} &q^{[\nu]}(x, y) = \int_0^\infty q(x, t)\, q^{[\nu-1]}(t, y)\, dt, \\ &q^{[1]}(x, y) = q(x, y) \qquad (\nu = 2, 3, \ldots). \end{aligned}\right. \tag{9.1 a}$$

We seek symmetric solution $q(x, y)$, L_2 in x (in y).

Suppose $q(x, y)$ is a solution of required type. Define $q_m(x, y)$ by the relations

$$q_m(x, y) = q(x, y) \qquad [\text{when } |q(x, y)| < m],$$
$$q_m(x, y) = \pm m \qquad [\text{when } \pm q(x, y) > m).$$

Let the $\rho_{m\nu}$, $u_{m\nu}(x)$ be the characteristic values and functions of $q_m(x, y)$

$$u_{m\nu}(x) = \rho_{m\nu} \int_0^1 q_m(x, t)\, u_{m\nu}(t)\, dt.$$

We designate by $\Gamma_m(x, y \mid \rho)$ the spectral function of $q_m(x, y)$

$$\Gamma_m(x, y \mid \rho) = \left\{\begin{aligned} &\sum_{0 < \rho_{m\nu} < \rho} u_{m\nu}(x)\, u_{m\nu}(y) \qquad (\rho > 0), \\ &-\sum_{\rho \leq \rho_{m\nu} < 0} u_{m\nu}(x)\, u_{m\nu}(y) \qquad (\rho < 0). \end{aligned}\right. \tag{α}$$

Since $q(x, y)$ is L_2 in x, there exists a sequence (m_j) so that the limit

$$\Gamma(x, y \mid \rho) = \lim \Gamma_{m_j}(x, y \mid \rho) \tag{β}$$

exists; Γ is a spectral function of $q(x, y)$. Inasmuch as we understand integrability of iterants to imply absolute integrability, it is noted that the integrals

$$|q(x, y)|^{[i]} = \int_0^1 \cdots \int_0^1 |q(x, t_1)\, q(t_1, t_2) \ldots q(t_{i-1}, y)|\, dt_1 \ldots dt_{i-1}$$

$(i = 2, \ldots, n)$ exist. Furthermore,

$$|q_m(x, t_1)\, q_m(t_1, t_2) \ldots q_m(t_{i-1}, y)| \leqq |q(x, t_1) \ldots q(t_{i-1}, y)|.$$

Whence

$$\lim_m q_m^{[i]}(x, y) = q^{[i]}(x, y) \qquad (i = 2, \ldots, n). \tag{9.2}$$

Inasmuch as the integral

$$\int_0^1 f^2(x, t)\, dt = q^{[2n]}(x, x)$$

exists, it is concluded that the integrals

$$\int_0^1 q^{[i]^2}(x, t)\, dt = q^{[2i]}(x, x) \tag{9.3}$$

$(i = 2, \ldots, n)$ exist. Therefore $q_m^{[i]}(x, y)$ is a regular approximating kernel [that is, L_2 in (x, y)] of the singular kernel $q^{[i]}(x, y)$, the latter being of the type to which the spectral theory applies $(i \leqq n)$. The spectral function of $q_m^{[i]}(x, y)$ is

$$\left\{ \begin{aligned} \Gamma_m^{[i]}(x, y \mid \rho) &= \sum_{0 < \rho_{m\nu}^i < \rho} u_{m\nu}(x)\, u_{m\nu}(y) \qquad (\text{for } \rho > 0), \\ \Gamma_m^{[i]}(x, y \mid \rho) &= -\sum_{\rho \leq \rho_{m\nu}^i < 0} u_{m\nu}(x)\, u_{m\nu}(y) \qquad (\text{for } \rho < 0), \end{aligned} \right. \tag{9.4}$$

$\Gamma_m^{[i]}(x, y \mid 0) = 0$; this assertion is made on the basis of the fact that the $\rho_{\nu m}^i$, $u_{m\nu}$ are the characteristic values and functions of $q_m^{[i]}(x, y)$. For i even one has $\Gamma_m^{[i]}(x, y \mid \rho) = 0$ for $\rho < 0$. The sequence (m_j) can be chosen independent of i so that the limits

$$\lim_{m_j} \Gamma_{m_j}^{[i]}(x, y \mid \rho) = \Gamma_{(x, y \mid \rho)}^{[i]} \qquad (i = 1, \ldots, n) \tag{9.5}$$

all exist; the second member here is a spectral function of $q^{[i]}(x, y)$. In particular, $\Gamma^{[n]}(x, y \mid \rho)$ is a spectral function of $f(x, y)$, the approximating kernel of $f(x, y)$ being

$$f_m(x, y) = q_m^{[n]}(x, y).$$

Since n is odd, (**9**.4) gives us

$$\Gamma_m^{[n]}(x, y \mid \rho) = \sum_{0<\rho_{m\nu}<\rho^{\frac{1}{n}}} u_{m\nu}(x)\, u_{m\nu}(y) \qquad (\text{for } \rho > 0),$$

$$\Gamma_m^{[n]}(x, y \mid \rho) = -\sum_{\rho^{\frac{1}{n}}\leq\rho_{m\nu}<0} u_{m\nu}(x)\, u_{m\nu}(y) \qquad (\text{for } \rho < 0).$$

Designating the characteristic values of $q^{[n]}(x, y)$ by $\lambda_{m\nu}$ and noting that $\lambda_{m\nu} = \rho_{m\nu}^n$, we have

$$\rho_{m\nu} = \lambda_{m\nu}^{\frac{1}{n}},$$

where $\rho_{m\nu}$ has the sign of $\lambda_{m\nu}$. From the above one obtains

$$\Gamma_m^{[n]}(x, y \mid \rho) = \Gamma_m\left(x, y \mid \rho^{\frac{1}{n}}\right)$$

that is

$$\Gamma_m(x, y \mid \rho) = \Gamma_m^{[n]}(x, y \mid \rho^n);$$

in the limit

(**9**.6) $$\Gamma(x, y \mid \rho) = \Gamma^{[n]}(x, y \mid \rho^n).$$

To every spectral function $\Gamma(x, y \mid \rho)$ of a solution $q(x, y)$ there corresponds a spectral function $\Gamma^{[n]}(x, y \mid \rho)$ of $f(x, y)$ so that (**9**.6) holds. If $f(x, y)$ has just one spectral function, say $\Gamma^{[n]}$, then $q(x, y)$ will have just one spectral function Γ; Γ will satisfy (**9**.6). Whenever f has only one spectral function, there is at most one solution.

Since $q^{[2]}(x, y)$ is positive definite and some of the higher iterants exist, the spectral representation

(**9**.7) $$q^{[2]}(x, y) = \int_0^{\infty} \frac{1}{\rho}\, d_\rho \Gamma^{[2]}(x, y \mid \rho)$$

[$\Gamma^{[2]}$ from (**9**.5)] will hold in the sense of ordinary convergence. By (α) for $\rho > 0$ we have

$$\sum_{0<\rho_{m\nu}<\sqrt{\rho}} u_{m\nu}(x)\, u_{m\nu}(y) = \Gamma_m(x, y \mid \sqrt{\rho}),$$

$$\sum_{-\sqrt{\rho}<\rho_m<0} u_{m\nu}(x)\, u_{m\nu}(y) = -\Gamma_m(x, y \mid -\sqrt{\rho}) - \sigma_m(x, y \mid \rho),$$

where

$$\sigma_m(x, y \mid \rho) = \sum_{\rho_{m\nu} = -\sqrt{\rho}} u_{m\nu}(x)\, u_{m\nu}(y) \qquad [= 0 \text{ (for } \rho \neq \rho_{m\nu}^2)].$$

As a consequence of (9.4; $i = 2$) and of the above one has, when $\rho > 0$,

$$\Gamma_m^{[2]}(x, y \mid \rho) = \Gamma_m(x, y \mid \sqrt{\rho}) - \Gamma_m(x, y \mid -\sqrt{\rho}) - \sigma_m(x, y \mid \rho).$$

By (9.5; $i = 2$) and (β) the limit

$$\lim \sigma_{m_j}(x, y \mid \rho) = \sigma(x, y \mid \rho) = 0 \qquad (\rho \neq \rho_{m_j,\nu}^2;\ j, \nu = 1, 2, \ldots)$$

exists; it is of bounded variation (on every finite interval) in ρ; we have

$$(9.8) \qquad \Gamma^{[2]}(x, y \mid \rho) = \Gamma(x, y \mid \sqrt{\rho}) - \Gamma(x, y \mid -\sqrt{\rho}) - \sigma(x, y \mid \rho)$$

($\rho > 0$). In view of (9.6)

$$\Gamma^{[2]}(x, y \mid \rho) = \Gamma^{[n]}\left(x, y \,\middle|\, \rho^{\frac{n}{2}}\right) - \Gamma^{[n]}\left(x, y \,\middle|\, -\rho^{\frac{n}{2}}\right) - \sigma(x, y \mid \sigma)$$

and, by (9.7), (8.10),

$$(9.9) \qquad q^{[2]}(x, y) = \int_0^\infty \frac{1}{\rho}\, d_\rho \Gamma^{[n]}\left(x, y \,\middle|\, \rho^{\frac{n}{2}}\right) - \int_0^\infty \frac{1}{\rho}\, d_\rho \Gamma^{[n]}\left(x, y \,\middle|\, -\rho^{\frac{n}{2}}\right)$$
$$= \int_{-\infty}^{\infty} \frac{1}{\lambda^{\frac{2}{n}}}\, d_\lambda \Gamma^{[n]}(x, y \mid \lambda),$$

the integral in the last member being convergent. More generally, we establish that *the Stieltjes-integral representations in terms of* $\Gamma^{[n]}$ *of the* $q^{[j]}(x, y)$ ($j = 2, \ldots, n$) *all converge,* a similar statement being valid for the $q_m^{[j]}(x, y)$ ($j = 2, \ldots, n$) (for representations in terms of $\Gamma_m^{[n]}$). Convergence in any of the above representations is asserted almost everywhere in the square $0 \leqq x, y \leqq 1$. Since $q(x, y) \subset L_2$ in x (in y) the integral

$$\int_0^1 q^2(x, t)\, dt = q^{[2]}(x, x)$$

exists almost everywhere for $0 \leqq x \leqq 1$. By reasons as in section 8, we obtain

$$(9.10) \qquad \Lambda(x) = [q^{[2]}(x, x) =] \int_{-\infty}^{\infty} \frac{1}{\lambda^{\frac{2}{n}}}\, d_\lambda \Gamma^{[n]}(x, x \mid \lambda) < +\infty$$

almost everywhere on (0, 1), which is a necessary condition for the existence of a solution with the stated properties.

On letting $0 < l < l'$ and writing

$$\int_l^{l'} + \int_{-l'}^{-l} = \int^{l,l'},$$

by (**9**.4; $i = n$; $\rho_{m\nu}^n = \lambda_{m\nu}$) we obtain

$$Q^{l,l'} = \int_0^1 \left[\int^{l,l'} \frac{1}{\lambda^{\frac{1}{n}}} d_\lambda \Gamma^{[n]}(x, y \mid \lambda) \right]^2 dx \leq \lim_{m_j} \int_0^1 \left[\int^{l,l'} \frac{1}{\lambda^{\frac{1}{n}}} d_\lambda \Gamma_{m_j}^{[n]}(x, y \mid \lambda) \right]^2 dx$$

$$= \lim_{m_j} \int_0^1 \left[\sum_\nu^{(l,l')} \frac{1}{\lambda_{m\nu}^{\frac{1}{n}}} u_{m\nu}(x)\, u_{m\nu}(y) \right]^2 dx,$$

where the summation is over values ν for which $\lambda_{m\nu}$ is on the intervals $(-l', -l)$, (l, l'); one further has

$$Q^{l,l'} \leq \lim_{m_j} \sum_\nu^{l,l'} \lambda_{m\nu}^{-\frac{2}{n}} u_{m\nu}^2(y) = \lim_{m_j} \int^{l,l'} \frac{1}{\lambda^{\frac{2}{n}}} d_\rho \Gamma_m^{[2]}(x, y \mid \lambda)$$

$$= \left(\int_{-l'}^{-l} + \int_l^{l'} \right) \frac{1}{\lambda^{\frac{2}{n}}} d_\lambda \Gamma^{[n]}(x, x \mid \lambda).$$

Since the integral (**9**.10) converges, it is seen that

$$Q^{l,l'} < \varepsilon \qquad [\text{for } l, l' \geq l(\varepsilon)].$$

Therefore

(**9**.11) $$q(x, y) \sim \int_{-\infty}^{\infty} \frac{1}{\lambda^{\frac{1}{n}}} d_\lambda \Gamma^{[n]}(x, y \mid \lambda)$$

in the sense of mean convergence in x (in y).

Consider now the converse. — Envisage a spectral function $\Gamma^{[n]}(x, y \mid \lambda)$ of $f(x, y)$. This implies that for a sequence $f_m(x, y)$ of regular kernels converging to $f(x, y)$, and having characteristic values and functions $\lambda_{m\nu}$, $u_{m\nu}$ there exists a sequence (m_j) so that the limit

$$\lim \Gamma_{m_j}^{[n]}(x, y \mid \lambda) = \Gamma^{[n]}(x, y \mid \lambda)$$

exists; here

$$\Gamma_m^{[n]}(x, y \mid \lambda) = \sum_{0<\lambda_{m\nu}<\lambda} u_{m\nu}(x)\, u_{m\nu}(y) \qquad (\text{for } \lambda > 0),$$

$$\Gamma_m^{[n]}(x, y \mid \lambda) = -\sum_{\lambda \leq \lambda_{m\nu}<0} u_{m\nu}(x)\, u_{m\nu}(y) \qquad (\text{for } \lambda < 0).$$

Assume that for this spectral function $\Gamma^{[n]}$ (**9**. 10) *holds*. Then, repeating the developments preceding (**9**. 11), with the present meanings of the symbols, we conclude that, as $l \to +\infty$, the function

$$q^{(l)}(x, y) = \int_{-l}^{l} \frac{1}{\lambda^{\frac{1}{n}}} d_\lambda \Gamma^{[n]}(x, y \mid \lambda)$$

converges in the mean square in x (in y) to some function, which we shall denote by $q(x, y)$. It will be now proved that *if the integrals*

(**9**. 12) $$|q(x, y)|^{[\nu]} \qquad (\nu \leqq n-1)$$

are L_2 *in* x (*in* y), *necessarily* $q(x, y)$ *will satisfy* (**9**. 1).

It is noted that

$$q^{(l)}(x, y) = \int_{-l'}^{l'} \frac{1}{\rho} d_\rho \Gamma^{[n]}(x, y \mid \rho^n) \qquad \left(l' = l^{\frac{1}{n}}\right).$$

Thus

(**9**. 13) $$q(x, y) \sim \int_{-\infty}^{\infty} \frac{1}{\rho} d_\rho \Gamma^{[n]}(x, y \mid \rho^n)$$

in the sense of mean convergence in x (in y); accordingly

$$\theta(x, y \mid \rho) = \Gamma^{[n]}(x, y \mid \rho^n)$$

is a spectral function of $q(x, y)$. Hence, in view of the statement with respect to (**9**. 12) and by virtue of the developments in (C; 124, 125) we conclude that the integrals

$$\int_{-\infty}^{\infty} \frac{1}{\rho^j} d_\rho \theta(x, y \mid \rho) \qquad (j = 2, \ldots, n)$$

converge (in the ordinary sense) to the iterants of $q(x, y)$,

$$q^{[j]}(x, y) \qquad (j = 2, \ldots, n)$$

respectively. In particular, for $j = n$ one has

$$q^{[n]}(x, y) = \int_{-\infty}^{\infty} \frac{1}{\rho^n} d_\rho \theta(x, y \mid \rho) = \int_{-\infty}^{\infty} \frac{1}{\lambda} d_\lambda \theta\left(x, y \,\middle|\, \lambda^{\frac{1}{n}}\right) = \int_{-\infty}^{\infty} \frac{1}{\lambda} d_\lambda \Gamma^{[n]}(x, y \mid \lambda).$$

The last member here is a spectral representation of f for almost all (x, y). Thus

$$q^{[n]}(x, y) = f(x, y);$$

q is a solution of (**9**.1).

THEOREM **9**.14. — *Consider the n-th order iteration problem* (**9**.1), *with n odd.*

If there exists a solution $q(x, y)$, L_2 *in* x (*in* y), *then* $f(x, y)$ *has a spectral function* $\Gamma^{[n]}(x, y \mid \lambda)$ *so that*

$$\Lambda(x) = \int_{-\infty}^{\infty} \frac{1}{\lambda^{\frac{2}{n}}} d_\lambda \Gamma^{[n]}(x, x \mid \lambda) < +\infty$$

(*almost everywhere*) *and so that*

$$q(x, y) \sim \int_{-\infty}^{\infty} \frac{1}{\lambda^{\frac{1}{n}}} d_\lambda \Gamma^{[n]}(x, y \mid \lambda)$$

(*mean square convergence in* x, *in* y).

The converse. Let $\Gamma^{[n]}(x, y \mid \lambda)$ *be a spectral function of* $f(x, y)$ *for which* $\Lambda(x) < +\infty$ (*almost everywhere*). *We then have*

$$\int_{-\infty}^{\infty} \frac{1}{\rho} d_\rho \Gamma^{[n]}(x, y \mid \rho^n), \quad \text{i. e.} \int_{-\infty}^{\infty} \frac{1}{\lambda^{\frac{1}{n}}} d_\lambda \Gamma^{[n]}(x, y \mid \lambda)$$

[*cf.* (**9**.13)]

convergent in a mean square in x (*in* y) *to some function, say* $q(x, y)$. *If the integrals*

$$| q(x, y) |^{[\nu]} \qquad (\nu \leq n - 1)$$

are L_2 (*in* x, *in* y), *then* $q(x, y)$ *will be a solution of the iteration problem.*

NOTE. — The solution is unique if $f(x, y)$ has just one spectral function