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A new definition of a fluid.
II. *The Maxwellian Fluid;*

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Dedicated to the Memory of M. Brillouin.

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(1) While this second Memoir is self-contained, a reader who has not previously glanced at the preceding, in which the same method is more fully explained in a simpler application, may find the presentation less easy to follow.

18. Historical introduction. — The classical theory of isotropic viscous fluids rests upon the Newton ([2])-Cauchy-Poisson law

$$t^i{}_j = -p\,\delta^i{}_j + \lambda\, d^k{}_k\,\delta^i{}_j + 2\mu\,\delta^i{}_j, \tag{18.1}$$

where $t^i{}_j$ is the stress tensor, p is the pressure ([3]), λ and μ are the coefficients of viscosity, and $d^i{}_j$ is the rate of deformation tensor. The general theory is due to Navier ([4]), who by calculations based upon a molecular hypothesis derived dynamical equations equivalent to those resulting from (18.1) for an incompressible fluid of constant viscosity. It is not certain that Cauchy intended his continuum presentation of (18.1) to apply to fluids, for he spoke of " corps solides entièrement dépourvus d'élasticité " ([5]), " un corps solide non élastique " ([6]), and his equations ([7]) lack the term $-p\,\delta^i{}_j$. The fully general expression (18.1) was first given by Poisson ([8]), who derived it from a molecular theory.

With the added restriction $3\lambda + 2\mu = 0$, known as the " Stokes relation ", the law (18.1) was proposed by Saint Venant ([9]) on the basis of continuum arguments, and to derive this same result Stokes ([10]) employed both continuum and rough molecular considerations. The resulting dynamical equations are called the Navier-Stokes equations. The classical theory was proposed as a first approximation, and there have been many attempts to formulate more general equations.

In a rectilinear shearing flow ([11]), according to (18.1) the shearing stress $t^x{}_y$, and hence also the resistance, is proportional to the rate of

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([2]) According to Newton [1687, **1**, lib. II, sect. IX], " *Resistentiam, quae oritur ex defectu lubricitatis partium fluidi, cæteris paribus, proportionalem esse velocitati, qua partes fluidi separantur ab invicem* ".

([3]) *See* the Appendix to the preceding Memoir for the definition of pressure.

([4]) [1821, **1**], [1822, **1**], [1825, **1**], [1827, **1**].

([5]) [1823, **1**].

([6]) [1828, **1**, § III]. Also, " l'élasticité disparaît entièrement ".

([7]) [1828, **1**, équat. (95), (96)].

([8]) [1831, **1**, § **60-63**].

([9]) [1843, **1**].

([10]) [1845, **1**, § **1-5**]. For Stokes's principle, *see* § **2** of the preceding Memoir.

([11]) *Cf.* § **12**.

shearing $d^x{}_y$. Since in many physical liquids this relation does not appear to be substantiated by experiment, a considerable engineering literature ([12]) of " non-Newtonian fluids ", in which $t^x{}_y$ is represented as a polynomial or power series in $d^x{}_y$, has accumulated. These one-dimensional treatments can neither reveal nor reflect the characteristic phenomena of non-linear continuum mechanics, and hence will not be discussed in the present Memoir.

We must take account, however, of the remarkable earlier result of O. E. Meyer ([12a]), who by an argument in the kinetic theory of gases for the special case of a rectilinear shearing flow obtained

$$t^x{}_y = \mu\left[d^x{}_y + \frac{3\lambda^2}{5}\frac{\partial^3 \dot{x}}{\partial y^3} + \dots\right], \tag{18.1 a}$$

where λ is the mean free path. Here for the first time appears a series expansion for a stress component in a fluid, and here also is the first suggestion that the stress should depend not only on $d^i{}_j$ but also on the higher velocity gradients.

Kleitz ([13]) proposed $v^1{}_1 = f(d^1{}_1)$, $v^1{}_2 = f(d^1{}_2)$, ..., a type of relation tensorially admissible only when f is a linear function.

Boussinesq ([14]) was the first to show that the Newton-Cauchy-Poisson law (18.1) is merely the linear term in an infinite series for the viscous stresses. His concept of fluidity was somewhat more general than that embodied in Stokes's principle (§ 3), for while expressly stating that in a rigid rotation the viscous stresses must vanish, he nevertheless permitted the vorticity $\omega^i{}_j$ to modify their magnitude in general. Thus the mathematical form of *Boussinesq's principle* is

$$\left\{\begin{aligned} &v^i{}_j = f(d^k{}_l, \omega^m{}_n), \\ &f(0, \omega^m{}_n) = 0, \end{aligned}\right. \tag{18.2}$$

([12]) The earliest reference I have found is [1869, **1**], where Saint Venant attributes such a series to Dupuit. That $t^x{}_y$ must be an odd function of $d^x{}_y$ is observed by Reiner [1929, **1**, p. 17], [1943, 1, Sect. X, p. 139, 150].

([12a]) [1765, **1**, § **8**].

([13]) [1866, **1**], [1872, **1**]. Kleitz's equations necessarily reduce to the classical form (**18.1**); he stated that the coefficients were to be variable, but the manner of their variation is not specified in the published abstract of his paper.

([14]) [1868, **1**, Note I].

where $\mathfrak{v}^i{}_j \equiv t^i{}_j + p\,\delta^i{}_j$. It was Boussinesq also who first applied Cauchy's method of isotropic tensors to the determination of the admissible form for a non-linear hypothesis. By developing (18.2) he obtained expressions which in the present notation are

$$(18.3)\qquad \begin{aligned} t^i{}_j = &-p\,\delta^i{}_j + \lambda\, d^k{}_k\,\delta^i{}_j + 2\mu\, d^i{}_j + 2(\mathrm{A}-\mathrm{C})(d^i{}_k\omega^k{}_j + d_j{}^k\omega_k{}^i) \\ &+ \mathrm{B}\, d^k{}_k\, d^i{}_j + 4\mathrm{C}\, d^i{}_k\, d^k{}_j + \mathrm{D}(d^k{}_k)^2\,\delta^i{}_j - 4\mathrm{E}\,\mathrm{II}_{\mathrm{d}}\,\delta^i{}_j + \ldots, \end{aligned}$$

where A, B, C, D and E are second order coefficients of viscosity, not specified in form.

M. Levy ([15]), reasoning from Navier's molecular notions, proposed a theory in which higher derivatives of the velocity occur; while his resulting stresses for isotropic fluids, derived by the method of isotropic tensors, consist only of linear combinations of generalized Laplacians $d^i{}_j{}^{,l_1}{}_{,l_1}{}^{,\ldots,l_k}{}_{,l_k}$, there is no reason for limiting oneself to such expressions, and we shall prefer to state *Levy's principle* in the form

$$(18.4)\qquad \mathfrak{v}^i{}_j = f(\dot{x}^k{}_{,l},\ \dot{x}^m{}_{,np},\ \ldots,\ \dot{x}^q{}_{,rs\ldots t},\ \ldots),$$

where $\dot{x}^i$ is the velocity vector.

By an application of the notions of Maxwell's kinetic theory of gases ([16]) Butcher ([17]) obtained the equations

$$(18.5)\qquad \frac{1}{l+\dfrac{\partial}{\partial t}}\left\{\left[kl+\left(k+\frac{1}{3}\nu\right)\frac{\partial}{\partial t}\right]\dot{x}^j{}_{,ji} - \nu\,\dot{x}_i{}^{,j}{}_{,j}\right\} + \rho f_i = \rho\ddot{x}_i,$$

which for steady motions or for $l=\infty$ reduce to the Navier-Stokes equations.

([15]) [1869, 1]. In an earlier paper [1867, 1, p. 240-241] he proposed $\mathrm{F}\left(\dot{x}, \dfrac{\partial\dot{x}}{\partial y}\right)$ for the shearing stress in a rectilinear shearing flow, which he reduced to $\mathrm{F}(\dot{x})\left(\dfrac{\partial\dot{x}}{\partial y}\right)$.

([16]) In the succeeding review, reference is made only to those papers where stresses or dynamical equations containing new terms not included in the classical theory based upon (13.1) are proposed. No mention is made of the extensive literature concerned with the values of μ and λ and various other coefficients, either in gases or liquids, as derived from the kinetic theory.

([17]) [1876, 1, p. 103-111]. *Cf.* [1882, 1, p. 79-80].

The experimental discovery of Reynolds ([18]) that a temperature gradient in a fluid at rest is of itself sufficient to produce a non-equilibrated state of stress and resultant motion shows that the concept of fluidity as embodied in Stokes's, Boussinesq's, or even Levy's principle is too narrow. Reynolds called this phenomenon *thermal transpiration*, and gave a theoretical explanation ([19]) of it, founded upon kinetic theory considerations of a rather heuristic nature.

The subject was taken up at once by Maxwell; employing his celebrated hypothesis of molecular forces varying inversely as the fifth power of the distance, he calculated the following equations for the stress in a rarefied gas ([20]):

$$(\mathbf{18}.6) \qquad t^i{}_j = -p\,\delta^i{}_j - \frac{2}{3}\mu\, d^k{}_k\,\delta^i{}_j + 2\mu\, d^i{}_j - 3\frac{\mu^2}{\rho\theta}\theta^{,i}{}_{,j} - \frac{3}{2}\frac{\mu^2}{\rho\theta}\theta^{,k}{}_{,k}\,\delta^i{}_j.$$

The last two terms indicate a stress of the type required for thermal transpiration. Equally important is the occurrence of μ^2 in the coefficients of these terms, suggesting that in the general case the stress may be a power series in μ.

At a later time (1883) Reynolds ([21]) claimed that the ordinary Navier-Stokes equations are inconsistent with the condition of adherence at a solid boundary at the commencement of the motion, and by some obscure arguments of mixed molecular and phenomenological nature proposed equations which (if I have correctly understood his unexplained symbols) contain the term $-l^2\dot{x}^{i,j}{}_{,j}$ added to the acceleration, l being a constant length. His result cannot be derived from any symmetric stress tensor.

([18]) [1879, **2**, § **2**]. Since his similarity laws for this and related phenomena involve both the density of the gas and the dimensions of the boundaries, Reynolds concluded [§ **4-6**] that they cannot result from a continuum theory, and thus afford proof of the molecular nature of gases. The conclusion is incorrect. In the one dimensional case considered by Reynolds, the boundary conditions and the dimensions of the boundaries determine the characteristic pressure which accompanies a prescribed thermal gradient and prescribed temperature, and which occurs in the similarity parameter (**28**.1) of the continuum theory.

([19]) [1879, **1**, Sects. VI-VII].

([20]) [1879, **1**, § **14**].

([21]) [1901, **1**]. *Cf.* [1932, **1**, Part II, § **1**.7].

The systematic use of the kinetic theory to deduce expressions for the stresses in rarefied gases by approximate integration of the Maxwell-Boltzmann equation is the contribution of M. Brillouin ([22]). His method, however, is really almost that of a continuum theory. He supposed that the ratio of the distribution function F to the Maxwellian distribution may be written as an isotropic linear function ([23]) of $\dot{x}^i{}_{,j}$, $\ddot{x}^i$, $\dot{\rho}$, and $\dot{\theta}$, where the dot notation indicates material differentiation. The Maxwell-Boltzmann equation contains both derivatives and integrals of F; the terms in F containing first derivatives thus gives rise to certain terms containing second derivatives, so that corresponding second derivative terms must now be added to F in order to yield a solution; these terms in their turn give rise to third derivatives, etc. From any approximation for F may be determined a corresponding approximation to the stress $t^i{}_j$ and heat flux q_i. Brillouin wrote down terms containing first and second derivatives, intentionally neglecting all products of the various vectors and tensors, which he expected to be of importance only in a higher approximation. The formulae he obtained are ([24])

$$
\begin{aligned}
(18.7)\qquad t^i{}_j = & -p_m \delta^i{}_j + K_1 \{ d^i{}_j \} + K_2 \{ \dot{\overline{d^i{}_j}} \} \\
& + K_3 \{ \theta^{\cdot i}{}_{,j} \} + K_4 \{ \rho^{\cdot i}{}_{,j} \} + K_5 \{ \varepsilon^{ilm} \omega_{lm,j} \},
\end{aligned}
$$

$$
\begin{aligned}
(18.8)\qquad q_i = & \, C_1 \ddot{x}_i + C_2 \theta_{,i} + C_3 \rho_{,i} + C_4 \varepsilon_{ijk} \omega^{jk} + C_5 f_j d^j{}_i + C_6 \dddot{x}^i \\
& + C_7 \dot{\overline{\theta}}_{,i} + C_8 \dot{\overline{\rho}}_{,i} + C_9 f_j \dot{\overline{d^j{}_k}} + C_{10} d^k{}_{k,i} + C_{11} f_j \dot{\overline{x^j{}_{,i}}} + C_{12} f_j \theta^{\cdot j}{}_{,i} \\
& + C_{13} f_j \rho^{\cdot j}{}_{,i} + C_{14} f_i \dot{\overline{d^k{}_k}} + C_{15} f_i \theta^{\cdot k}{}_{,k} + C_{16} f_i \rho^{\cdot k}{}_{,k} \\
& + C_{17} \varepsilon_{ijk} f^j \varepsilon^{klm} \dot{\overline{\omega_{lm}}} + C_{18} \varepsilon_{ijk} f^j \dot{\overline{\theta}}^{\cdot k} + C_{19} \varepsilon_{ijk} f^j \dot{\overline{\rho}}^{\cdot k} \\
& + C_{20} \varepsilon_{ijk} f^j \ddot{x}^k + C_{21} (\dot{x}_i{}^{\cdot j}{}_{,j} - d^k{}_{k,i}) + C_{22} \varepsilon_{ijk} \dot{\overline{\omega^{jk}}} \\
& + C_{23} f_j \varepsilon^{jkl} \omega_{kl,i} + C_{24} f_j \varepsilon^{ikl} \omega_{kl}{}^{\cdot j} + C_{25} \varepsilon_{ijk} f^j (\dot{x}^k{}_{,l}{}^{\cdot l} - d^l{}_l{}^k).
\end{aligned}
$$

([22]) [1900, 1].

([23]) The notion of an isotropic function can be traced back at least to Cauchy in examples, and has been much employed in continuum mechanics. A recent example is Robertson's analysis of turbulent motions [1940, 1]. Let me emphasize the fact that the theory of fluids developed in the present Memoir depends in *no way* upon the notion of isotropy. *Cf.* § 21, Remark 5.

([24]) [1900, 1, § 31-32, 36]. In the interest of brevity I have introduced different letters for his coefficients. The term whose coefficient is C_{23} is misprinted in Brillouin's paper.

Here p_m is the mean pressure $\left(p_m \equiv -\frac{t^i_i}{3}\right)$, bars are used to indicate the extent of the dot operation, the notation $\{a^i_j\}$ is defined by

$$\{a^i_j\} \equiv \frac{1}{2}(a^i_j + a_j^i) - \frac{1}{3} a^k_k \delta^i_j, \tag{18.9}$$

and f_i is the extraneous force vector. There are several noteworthy features of this remarkable result. Fourier's law of heat conduction ([25]) is generalized for the first time, and Brillouin's formula predicts (among other interesting new phenomena) a counterpart of thermal transpiration, in that deformation of a fluid at uniform temperature gives rise to flow of heat. In fact, the same variables occur both in t^i_j and in q_i. In the equations as actually written there is a certain redundancy, in that both $\ddot{x}^i$ and f^i occur, while by the momentum equation one or the other may be eliminated. Furthermore, the terms whose coefficients are C_4, C_{18}, C_{19}, C_{20}, C_{22}, C_{23}, C_{24}, C_{25}, are axial vectors, while q_i is a polar vector, and thus these terms are incorrect; the term whose coefficient is K_5 is an axial tensor, and thus is similarly incorrect. Not only are the coefficients unspecified in form or value, but also Brillouin's purely formal ordering of the terms by the order of the derivatives occurring is not satisfactory, since neither order of magnitude nor analytic character can be determined in this way. The generality of Brillouin's method has not been equalled by any subsequently proposed, and his brilliant work, though insufficiently appreciated in the current literature, remains to this day the greatest single advance in the theory of rarefied gases. His ideas are the starting point of the present memoir, as shall appear in paragraph 19.

Claiming that capillary films are but narrow regions where large but continuous density gradients occur, Korteweg ([26]) realized that if such were indeed the case the Navier-Stokes equations would become inadequate in these layers. For anisotropic fluids be proposed

$$t^i_j = -p\delta^i_j + A^i_j{}^k \theta_{,k} + B^i_j{}^k \rho_{,k} + C^i_j{}^k_l \rho^{,l}_{,k} + D^i_j{}^k_l \rho^{,l}_{,k} + E^i_j{}^k_l \dot{x}^l_{,k}, \tag{18.10}$$

([25]) *Cf.* § 13.

([26]) [1901, **2**, § 5-7]. *Cf.* [1932, **1**, Part IV, § 1.3].

where $A^i{}_j{}^k, \ldots, E^i{}_j{}^k{}_l$ are tensorial coefficients, an expression which for isotropic fluids he reduced to the form

$$(18.11) \qquad t^i{}_j = -p\,\delta^i{}_j + \lambda\, d^k{}_k \delta^i{}_j + 2\,\mu\, d^i{}_j - \alpha \rho_{,k} \rho^{,k} \delta^i{}_j - \beta \rho^{,i} \rho_{,j} + \gamma \rho^{,k}{}_{,k} \delta^i{}_j + \delta \rho^{,i}{}_{,j},$$

where α, β, γ, δ are coefficients of unspecified form.

Hilbert ([27]) gave a method of finding all solutions F of the Maxwell-Boltzmann equation which are of the form

$$(18.12) \qquad F = \frac{f_0}{\lambda} + f_1 + f_2 \lambda + f_3 \lambda^2 + \ldots,$$

where $f_0, f_1, \ldots,$ are also solutions, and λ is an *arbitrary* parameter, to be identified with the mean free path—truly an extraordinary type of solution for a non-linear integro-differential equation. Since as $\lambda \to 0$ the simplifying assumptions ([28]) upon which the Maxwell-Boltzmann equation is derived lose their validity, while as $\lambda \to \infty$ any polynomial approximation to (18.12) becomes less and less accurate, the results obtained by this method can have at most a narrow range of validity, and an asymptotic character.

Claiming that Hilbert's parmeter λ need not be identified with the mean free path, but may be given different significances as the collision integral in the Maxwell-Boltzmann equation is modified to describe various types of gas conditions, Bolza, Born, and von Kármán ([28a]) applied Hilbert's method to the analysis of phenomena at very low pressures, when encounters of the molecules with each oter are so uncommon as to be negligible.

Hilbert's method has not been sufficiently appreciated, and among physicists a more elaborate and less definite scheme devised by Enskog ([29]) has found acceptance. Like Hilbert, Enskog too begins

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([27]) [1912, **1**], [1912, **2**, Chap. **21**].

([28]) Only binary encounters are considered and the transfer of energy at collisions is neglected.

([28a]) [1913, **1**].

([29]) [1917, **1**], [1939, **1**, App. A]. Some of Enskog's results were derived also by Chapman [1916, **1**], [1917, **2**], who considered the equations of transfer

with a " λ-series " (18.12), but while in Hilbert's method the successive approximations are uniquely determined at each stage, in Enskog's a considerable arbitrariness is introduced, to be eliminated later by a formal procedure whose mathematical and physical meaning is obscure. Enskog's second approximation ([30]) for the stresses includes terms containing products such as $\left(\frac{p_{,i}}{\rho}\right)_{,j}$ and $\theta_{,i}\theta_{,j}$.

Enskog's results were extended by Lennard-Jones ([31]), who discovered new terms in the stress equations.

Jaffé ([32]) proposed to find a series solution of the Maxwell-Boltzmann equation in the form $F = f_0 + f_1\lambda^{-1} + f_2\lambda^{-2} + \ldots$, and hence appropriate to very rarefied gases; the first approximation is " free molecule flow " rather than Eulerian hydrodynamics.

A theory of viscous fluids based upon Stokes's principle was constructed by Girault ([33]), but following a mistaken premise he incorrectly annulled all the terms of even degree in the rate of deformation.

The most general stresses yet derived from the kinetic theory by Enskog's method are those of Burnett ([34]); they contain those of Maxwell, Enskog, and Lennard-Jones as special cases. Burnett's equations were derived in a shorter way by Chapman and Cowling ([35]),

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directly. The claim, frequent in the physical literature, that Hilbert's method fails to yield the Navier-Stokes equations is shown to be false by Grad [1950, **1**, Appendix **8**].

([30]) [1917, **1**, Ch. VI; *see* especially p. 122-127].

([31]) [1923, **1**, § 9-12]. Jones's results are discussed by Rocard [1924, **1**], [1927, **1**, § **10-16**], [1932, **2**, Chap. VIII] who notices [1927, **1**, § **12**] that they are incomplete. *Cf.* [1932, **1**, Part III, § **1**.1].

([32]) [1930, **1**, § **4**]. *Cf.* the earlier work of Bolza, Born, and von Kármán, mentioned above.

([33]) [1931, **1**, Chap. III]. *Cf.* § **12**, ([27]) of the preceding Memoir. Three applications of Girault's theory are given by Viguier [1947, **1**], [1949, **5**], [1950, **2**]. The terms incorrectly annulled by Girault do not affect the results discussed in the second and third of these papers; in the second is given a numerical estimate for one of the third order coefficients of viscosity.

([34]) [1936, **1**].

([35]) [1939, **1**, § **15**.3, **15**.41]. *Cf.* also [1950, **3**].

who present them in the following form :

$$
\begin{aligned}
(18.12)\qquad -t^i{}_j = p\,\delta^i{}_j - 2\,\mu\,\{d^i{}_j\} &+ \varpi_1 \frac{\mu^2}{p} d^k{}_k \{d^i{}_j\} \\
&+ \varpi_2 \frac{\mu^2}{p}\left\{ f^i{}_{,j} - \left(\frac{1}{\rho} p^{,i}\right)_{,j} - \dot{x}_k{}^{,i}\dot{x}_j{}^{,k} - 2\,\dot{x}^i{}_{,k}\{d^k{}_j\}\right\} \\
&+ \varpi_3 \frac{\mu^2}{\rho\theta}\{\theta^{,i}{}_{,j}\} + \varpi_4 \frac{\mu^2}{\rho p\theta}\{p^{,i}\theta_{,j}\} \\
&+ \varpi_5 \frac{\mu^2}{\rho\theta}\{\theta^{,i}\theta_{,j}\} + \varpi_6 \frac{\mu^2}{p}\{\{d^i{}_k\}\{d^k{}_j\}\},
\end{aligned}
$$

the bracket notation being defined by (18.9).

For Maxwellian molecules the coefficients have the numerical values

$$
(18.13)\qquad \begin{cases} \varpi_1 = \dfrac{4}{3}\left(\dfrac{1}{2} - \dfrac{d\log\mu}{d\log\theta}\right), \qquad \varpi_2 = 2, \qquad \varpi_3 = 3, \\[2ex] \varpi_4 = 0, \qquad \varpi_5 = 3\,\dfrac{d\log\mu}{d\log\theta}, \qquad \varpi_6 = 8. \end{cases}
$$

Chapman and Cowling ([36]) derived also a corresponding expression for the heat flux

$$
\begin{aligned}
(18.14)\qquad q_i = -\varkappa\theta_{,i} + \theta_1 \frac{\mu^2}{\rho\theta} d^k{}_k \theta_{,i} + \theta_2 \frac{\mu^2}{\rho\theta}\left[\frac{2}{3}(d^k{}_k\theta)_{,i} + 2\,\theta_{,j}\dot{x}^j{}_{,i}\right] \\
+ \theta_3 \frac{\mu^2}{\rho p} p_{,j}\{d^j{}_i\} + \theta_4 \frac{\mu^2}{\rho}\{d^k{}_i\}_{,k} + \theta_5 \frac{\mu^2}{\rho\theta}\theta_{,j}\{d^j{}_i\},
\end{aligned}
$$

where for the Maxwellian molecules the coefficients as corrected by Chang and Uhlenbeck ([37]) are

$$
(18.15)\qquad \begin{cases} \theta_1 = \dfrac{15}{4}\left(\dfrac{7}{2} - \dfrac{d\log\mu}{d\log\theta}\right), \qquad \theta_2 = -\dfrac{45}{8}, \\[2ex] \theta_3 = -3, \qquad \theta_4 = 3, \qquad \theta_5 = 3\left(\dfrac{35}{4} + \dfrac{d\log\mu}{d\log\theta}\right). \end{cases}
$$

It is these last formulae which Tsien ([38]) recommends as suitable for the description of aerodynamic phenomena at altitudes of 30 to 100^{km}.

([36]) [1939, **1**, § **15**.3-**15**.4]. *Cf.* [1927, **1**, § **39**].

([37]) [1948, **3**, § III]. Kohler [1950, **4**] obtains the relation $\varpi_3 = \theta_4$ for all types of molecules.

([38]) [1946, **1**, p. 654]. The subject is considered further by Schamberg [1947, **2**].

Several special stress formulae were considered by Sakadi. First ([38a]) he proposed an expression equivalent tho the second approximation (7.7) to Stokes's principle, but, falling back on an argument of the type used by Girault, he rejected it in favor of an unexplained and very elaborate substitute. Next ([38b]) he employed a formula equivalent to the result of adding a term in $\omega^{ik}\omega_{kj}$ to Boussinesq's expression (18.3), and another equivalent to the second approximation in Girault's theory.

Proceeding by analogy to his approximate theory of large elastic deformation, Seth ([38c]) " for the sake of simplicity " proposed

$$t^i{}_j = -p\,\delta^i{}_j - \frac{2}{3}\mu s^k{}_k \delta^i{}_j + 2\mu s^i{}_j, \tag{18.15a}$$

where $s^i{}_j$ is a formal analogue of the Eulerain strain tensor of Almansi and Hamel :

$$2s^i{}_j \equiv \dot{x}^i{}_{,j} + \dot{x}_j{}^{,i} - \dot{x}^{k,i}\dot{x}_{j,k}. \tag{18.15b}$$

Here indeed we see the absurdity which can result from pursuing " simplicity " and a false analogy to elasticity in fluid dynamics, for in forming his tensor $s^i{}_j$ Seth has abused the most élementary physical principles by adding together quantities of different physical dimensions.

Reiner ([39]) reduced the mathematical expression of Stokes's principle to an elegant and perspicuous form in the isotropic case. The resulting theory, whose formulation in completeness, simplicity, and generality offers a favorable contrast to the elaborate and yet extremely restricted approximative methods of the kinetic theory, is directed rather toward phenomena of non-linear viscosity in liquids than toward the effects encountered in rarefied gases. This theory has been discussed in paragraph 11 of the preceding Memoir.

Leaf ([39a]) has proposed to treat viscosity in an inert mixture of

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([38a]) [1941, **1**].
([38b]) [1942, **1**].
([38c]) [1944, **1**].
([39]) [1945, **1**].
([39a]) [1946, **2**].

several fluids by assigning to each fluid Γ a partial stress $t^{\Gamma i}{}_j$ of the type

$$t^{\Gamma i}{}_j = -p^{\Gamma}\delta^i{}_j + \lambda^{\Gamma\Delta} d_{\Delta}{}^k{}_k \delta^i{}_j + 2\mu^{\Gamma\Delta} d_{\Delta}{}^i{}_j, \tag{18.15c}$$

where p^{Γ} is the partial pressure

$$2\, d_{\Delta ij} \equiv \dot{x}_{\Delta i,j} + \dot{x}_{\Delta j,i}, \tag{18.15d}$$

$\dot{x}_{\Delta i}$ is the velocity of the fluid Δ, and $\lambda^{\Gamma\Delta}$, $\mu^{\Gamma\Delta}$ are coefficients.

Garcia ([39b]) has committed the same astonishing blunder as Seth : his " formula cardinal completa y exacta ... par α deformaciones finitas " differs from (18.15 a) but in a change of sign.

Reiner's theory was developed for the case of an incompressible fluid by Rivlin ([40]), and applied to several extremely important examples. Rivlin's results have been discussed in paragraphs 11-12 of the preceding Memoir.

Chang and Uhlenbeck ([41]) have consecrated their energies to the formidable labor of extracting part of a third approximation by Enskog's method, but their results are not yet published.

In an attempt to represent thermal phenomena in gases Verschaffelt ([42]) in effect proposed

$$v^i{}_j = \left(\lambda d^k{}_k + \alpha \log \frac{\theta}{\theta_0}\right)\delta^i{}_j + 2\,\mu d^i{}_j, \tag{18.16}$$

where α is a coefficient which is a function of $\frac{\theta}{\theta_0}$ only.

([39b]) [1947, 2], [1948, 9].

([40]) [1947, 3], [1948, 7], [1949, 2]; *cf.* [1949, 3]. Rivlin obtained also a formula for one of the second order viscosity coefficients in a liquid containing long chain molecules in solution [1948, 10], [1949, 4].

([41]) [1948, 3].

([42]) Verchaffelt's actual proposal [1948, 4], [1948, 5] that *extraneous forces* of the type $f_i = \alpha \frac{\theta_{,i}}{\theta}$ be presumed is contradictory to the whole notion of stress which divides all forces acting upon the medium into extraneous forces, which are given functions of position and velocity, and unknown mutual forces, to whose effect it is postulated that the stress forces are equivalent. Thermal forces certainly belong to the latter category and hence should be specified only in terms of stresses. Formally, of course, since the extraneous forces and the stress forces enter the dynamical equations (A.5_1) in precisely the same way,

The hydrodynamical equations derived from heuristic arguments of mixed molecular and phenomenological nature by Predvoditelev ([43]) differ from the classical ones in that the ordinary acceleration is replaced by $(1-\beta)\ddot{x}^i + \beta\left(\frac{\partial \dot{x}_i}{\partial t} - \dot{x}^i \dot{x}^k{}_{,k}\right)$, where β is a coefficient.

Jaffé's kinetic theory method for very rarefied gases has been organized and applied by J. Keller ([44]).

A new method of integration of the Maxwell-Boltzmann equation discovered by H. Grad ([45]) does not really employ the notion of stress and heat flux, but instead yields gross equations of motion in terms of moments of the distribution function up to any specified order.

19. Heuristic preliminaries. Brillouin's principle. — Our purpose in the present Memoir is to give a rather general definition of a fluid as a special type of continuous medium susceptible of the principal phenomena observed in rarefied gases. A first step in this direction was taken in the previous memoir, where the theory of the simple Stokesian fluid was developed as the most general fluid obeying Stokes's principle and possessed of a natural viscosity and a reference temperature. While this theory reveals dependance of the viscous stresses upon the pressure in a fashion characteristic for phenomena at low pressures, it does not describe other effects, such as thermal transpiration, which are equally characteristic of rarefied gases. Thus to generalize the Stokesian fluid we should expect to propose that the stress $t^i{}_j$ depend not only upon the rate of deformation $d^i{}_j$ and the thermodynamic state, but also upon the thermal gradient $\theta_{,i}$, and, for that matter, upon the gradients of the other variables of state : $p_{,i}$, $\rho_{,i}$, $\varepsilon_{,i}$, $\eta_{,i}$. Now in general, however, it is assumed that even for fluids in motion there exist equations of state giving any one of the

---

any given term included in the latter alternatively may be forced into the former. Since $\dim \frac{\alpha}{\mu} = \mathsf{T}$, Verschaffelt's theory contains a natural time (*cf.* § **11**), and hence is not included in the theory of the present Memoir.

([43]) [1948, **6**, equat. (26 *a*)].

([44]) [1948, **7**]. Some special situations are considered in [1949, **6**].

([45]) [1949, **1**, § 4-5], [1950, **1**, § **31-32**].

state variables in terms of any other two ([46]); thus we shall confine ourselves to the two gradients $\theta_{,i}$ and $p_{,i}$, though taking care to note that in more general situations where no equations of state exist our theory will require generalization.

Once having gone so far, however, we may equally expect that if gradients of θ and p can give rise to stresses, in a still higher approximation gradients of these gradients can similarly engender stresses of their own. At the same time, as suggested by Levy's principle (18.4), since the first approximation to the viscous stress reflects the presence of velocity gradients, the gradients of these gradients may well affect the higher approximations. Thus we may propose that the stress t^i_j be a function of these gradients up to any specified order k.

Suppose now we embody these remarks in a formal définition of a fluid generalizing that of paragraph 5, and proceed with an analysis of the resulting terms in the stress analogous to that of paragraphs 6-7. After some labor, we reach the at first surprising conclusion that all the coefficients of the new terms are zero, and that thus our intended generalization simply reduces to the Stokesian fluid !

What, then, have we neglected ? An answer is suggested by our casual remark above that in general the existence of equations of state enabling us to give a complete description of the thermodynamic state in terms of two variables, such as p and θ, is assumed. Suppose we do *not* make such an assumption. Then we shall need to propose that the stress depends at least upon ρ, $\rho_{,i}$, ..., in addition to the other variables mentioned above. When applied to this rather elaborate structure, our methods will now yield a positive response, and enforce certain groupings of the terms. Consider, however, the second alternative : assuming that we *do* have an equation of state such as $f(p, \rho, \theta) = 0$ so that two state variables furnish a sufficient description, let us inquire what added information about the fluid this equation supplies. Now the simplest case is that of an ideal gas : $p = R\rho\theta$, where the constant R varies from one gas to another

([46]) And, for heterogeneous fluids, the instantaneous concentrations of the components of the mixture.

and thus is a material constant, whose dimensions are those of specific entropy : $\dim R = \mathsf{L}^2\mathsf{T}^{-2}\Theta^{-1}$. Of course we do not wish our theory to be limited to any special type of gas or liquid. The case of the ideal gas simply suggests the missing element in our summary of the phenomenological properties of fluids, namely, the existence of a material constant R_n, which might well be called the *fluid constant* or *natural entropy*, whose dimensions are

$$\dim R_n = \frac{\mathsf{L}^2}{\mathsf{T}^2\Theta}. \tag{19.1}$$

Indeed, once we have (19.1), we shall not need to suppose our fluid to have an equation of state.

It will not be necessary to postulate the existence of the fluid constant R_n. Notice that if $\varkappa_n$ be the *natural conductivity* (§ 13) and μ_n the *natural viscosity* (§ 4-5) of the fluid, we have

$$\dim \frac{\varkappa_n}{\mu_n} = \frac{\dfrac{\mathsf{ML}}{\mathsf{T}^3\Theta}}{\dfrac{\mathsf{M}}{\mathsf{LT}}} = \frac{\mathsf{L}^2}{\mathsf{T}^2\Theta} = \dim R_n. \tag{19.2}$$

Thus

$$R_n = c\,\frac{\varkappa_n}{\mu_n}, \tag{19.3}$$

where c is a dimensionless material constant. Introducing the fluid constant R_n into the list of variables upon which the stress may depend is then equivalent to introducing similarly the ordinary thermal conductivity.

We have seen, then, that in general the stress may be expected to depend not only upon the quantities μ_n and $d^i{}_j$ which give rise to the classical first approximation, but also upon the variables wich appear in the classical first approximation for the heat flux, viz., $\varkappa_n$ and $\theta_{,i}$. Conversely, we should expect that the heat flux in the general case should depend not only upon $\varkappa_n$ and $\theta_{,i}$ but also upon μ_n and $d^i{}_j$. Indeed, from the phenomenological point of view the stress and the heat flux represent but different aspects of the same principle. The stress tensor is a tensor whose divergence is added to the extraneous force in order to represent a resultant force equivalent to the

unspecified mutual forces and so to balance the momentum equation ([47]). The heat flux is a vector whose divergence is added to the stress power in order to represent a resultant power equivalent to that supplied by unspecified exchanges of mutual energies without performance of mechanical work and so to balance the energy equation ([48]). For those who prefer a molecular point of view, both stress and heat flux are but the averages of purely mechanical actions, the former representing the average momentum transfer across an imaginary surface, the latter the average energy transfer. We owe this observation to M. Brillouin ([49]) : " ... pour un corps formé de molécules animées de mouvements individuels, dont aucun élément de volume ne conserve une composition moléculaire invariable, cette séparation des phénomènes en mécanique et thermique est tout à fait insuffisante pour fournir le phénomène complet ". A gross model of a continuous medium is defined by specification of these gross equivalents in term of other gross variables. The conceptual analogy between stress and heat flux suggests that for *any* general model the gross variables selected for this purpose might well be expected to be the same for each. Among these variables there will be some representing the *thermodynamic state*, some specifying the type of *kinematical* condition to which the medium responds, and some supplying units of the *dimensions* necessary for consistency. For a *fluid*, the kinematical variables are *velocity gradients only*, and the principal dimensional constant is the *natural viscosity*. A general model of a solid would employ instead displacement gradients and a natural elasticity.

Thus, steadily clarifying the classical phenomenological point of view by stripping it of superfluous subsidiary assumptions, we have finally attained a very general and very simple concept of fluidity, which we may call *Brillouin's principle : the heat flux q_i and the stress $t^i{}_j$ are functions of the same list of variables*, viz. the velocity gradients

$$\dot{x}^i{}_{,j}, \quad \dot{x}^i{}_{,jk}, \quad \ldots,$$

([47]) *Cf.* (A.5).
([48]) *Cf.* (A.7).
([49]) [1900, **1**, § **37**].

and the thermodynamical gradients

$$p_{,i}, \quad p_{,ij}, \quad \ldots, \quad \theta_{,i}, \quad \theta_{,ij}, \quad \ldots,$$

to which we may add also the gradients of the extraneous force field ([50]):

$$f_{i,j}, \quad f_{i,jk}, \quad \ldots.$$

But the example of the Stokesian fluid shows that *no less important is the specification of the dimensional scalars*, for which we shall select

$$\mu_n, \quad \varkappa_n, \quad p, \quad p_m, \quad \theta, \quad \theta_0;$$

that is, the list of scalars is the same as that for the Stokesian fluid, with the important (and indeed essential) addition of $\varkappa_n$ for the stress and μ_n for the heat flux. Any further quantities entering the stress and heat flux equations beyond those given in the four lists above are perforce *dimensionless*.

It might be expected that so general a concept of fluidity as that we have now attained might be too vague to yield definite results, but on the contrary we shall shortly demonstrate that the principles of invariance quickly and easily lead to perfectly specific expressions for both the stress and the heat flux.

Our list of scalars

$$\mu_n, \quad \varkappa_n, \quad p, \quad p_m, \quad \theta, \quad \theta_0,$$

or, alternatively,

$$\mu_n, \quad \mathrm{R}_n, \quad p, \quad p_m, \quad \theta, \quad \theta_0,$$

is now sufficient to enable the construction of scalar quantities of all four fundamental dimensions :

$$(19.4) \qquad \left\{ \begin{aligned} &\dim \frac{\mu_n}{p} = \mathbf{T}, \\ &\dim \frac{\mu_n^2 (\mathrm{R}_n \theta)^{\frac{1}{2}}}{p^2} = \mathbf{M}, \\ &\dim \frac{\mu_n (\mathrm{R}_n \theta)^{\frac{1}{2}}}{p} = \mathbf{L}, \\ &\dim \theta_0 = \Theta. \end{aligned} \right.$$

([50]) Since $\rho f_i = \rho \ddot{x}_i - t_i^j{}_{,j}$, the addition of $f_{i,j}$, etc., to the list of variables upon which $t^k{}_l$ may depend is equivalent to adding the acceleration gradients $\ddot{x}_{i,j}$, etc. *Cf.* the remarks on Brillouin's theory in paragraph 18.

20. The method of this Memoir. — Our method consists now in embodying the foregoing heuristic remarks in a formal definition of a fluid, from which we shall deduce formulae for the stress and the heat flux, just as in the theory of the Stokesian fluid, *by applying the principles of tensorial and dimensional invariance.*

Since the stresses so calculated turn out to contain the terms (18.6) first derived by Maxwell, we shall call this fluid the *Maxwellian fluid*, although our approach to the subject is closer akin to Duhem's than to Maxwell's. The concept of fluidity embodied in the Maxwellian fluid consists partly in Brillouin's principle, which is a broad generalization of Stokes's principle (§ 3), Boussinesq's principle (18.2), and Levy's principle (18.4); but, equally important, our concept also specifies and restricts the number of dimensional material constants which a fluid may possess.

To Brillouin we owe the notion of defining a fluid in terms of an infinite list of tensors, but in this Memoir we direct his method to the continuum itself, dispensing with the kinetic theory as intermediary. Unlike Boussinesq, Levy and Brillouin, we do not restrict our attention to the isotropic case, and our theory yields perfectly definite formulae for anisotropic fluids. Moreover, our application of the principles of dimensional invariance determines the specific form of each coefficient up to a dimensionless constant; in particular, from the assumption that the stress and heat flux are power series in the various vectors and tensors upon which they depend we shall prove that *they are necessarily also power series in the viscosity* μ_n, as was suggested by the result of Maxwell (18.6). We shall easily demonstrate that our method yields *all* the terms of a given degree in μ_n, and our resulting formulae include those of Maxwell, Enskog, Lennard-Jones, Burnett and Chapman-Cowling as special cases. Thus our method is in one sense more general and more exhaustive, yet in another sense more definite, than those used previously.

21. Definition of the Maxwellian fluid. — *A Maxwellian fluid is a continuous medium such that :*

I. *There exist material constants* μ_n, χ_n *and* θ_0, *called respectively the*

natural viscosity, *the* natural thermal conductivity, *and the* reference temperature, *whose dimensions are*

$$(21.1)\qquad \begin{cases} \dim \mu_n = \dfrac{\mathsf{M}}{\mathsf{LT}}, \\ \dim \varkappa_n = \dfrac{\mathsf{ML}}{\mathsf{T}^3\Theta}, \\ \dim \theta_0 = \Theta; \end{cases}$$

II. *The stress $t^i{}_j$ and the heat flux q_i may be considered functions of the following variables only*

$$(21.2)\qquad \begin{cases} \mu_n, \quad \varkappa_n, \quad p, \quad p_m, \quad \theta, \quad \theta_0, \quad \text{and dimensionless material constants;} \\ d^i{}_j, \quad \omega^i{}_j, \quad p_{,i}, \quad \theta_{,i}, \quad f_{i,j}; \\ \dot{x}^{i_1}{}_{,i_2i_3}, \quad p_{,i_1i_2}, \quad \theta_{,i_1i_2}, \quad f_{i_1,i_2i_3}; \\ \dots\dots\dots\dots\dots\dots, \\ \dot{x}^{i_1}{}_{,i_2\dots i_{k+1}}, \quad p_{,i_1\dots i_k}, \quad \theta_{,i_1\dots i_k}, \quad f_{i_1,i_2\dots i_k}; \\ \dots\dots\dots\dots\dots\dots\dots\dots, \end{cases}$$

and are analytic functions of all the vectors and tensors listed.

III. *If all the vectors and tensors listed vanish, then $t^i{}_j$ reduces to $-p\,\delta^i{}_j$, and q_i reduces to zero.* (In paragraph 30 we shall find cause to narrow our definition slightly by strengthening this restriction).

Remark 1. — The Maxwellian fluid has no natural time. Thus it is a generalization of the Stokesian fluid, rather than of the Reiner-Rivlin fluid (§ 11).

Remark 2. — In the list (21.2), the first velocity gradient $\dot{x}^i{}_{,j}$ has been decomposed into the rate of deformation $d^i{}_j$ and the vorticity $\omega^i{}_j$ so that later (§ 30) the effect of rotation may easily be separated.

Remark 3. — Our notations, and the definitions and distinctions sketched in the appendix to the first paper, render our definition valid and meaningful for compressible and incompressible fluids alike.

Remark 4. — For the subsequent analysis we shall prefer to introduce a *fluid constant* R_n, defined by (19.3), where c is any convenient number.

Remark 5. — Note that *no mention* is made of isotropy at this point. A theory of mathematical physics must be founded upon some mechanical or physical *concept*, sufficiently specific to lead to definite equations without further geometrical or kinematical assumptions. Isotropy is an idealization whose only value lies in the purely mathematical simplifications it effects. In paragraphs 22-25 we shall show that the present theory yields definite expressions for the stress and the heat flux in anisotropic fluids, reserving the reductions valid in the isotropic case for paragraphs 26-27, where it will appear that the necessity for abandoning the compact tensorial notation for the coefficients and instead writing down every one explicitly will render the formulae actually more lengthy.

22. Anisotropic fluids. — I. *Power series for the stress.* — Our method is to develop the stress $t^i{}_j$ in a power series in the vector and scalar variables in the list (21.2). It is convenient to divide the procedure into two steps, beginning first with the velocity gradients:

$$
\begin{aligned}
(22.1)\quad t^i{}_j = \; & {}^{(0)}_{1}\mathrm{A}^i{}_j + {}^{(1)}_{1}\mathrm{A}^i{}_j{}^k{}_l\, d^l{}_k + {}^{(1)}_{2}\mathrm{A}^i{}_j{}^k{}_l\, \omega^l{}_k + {}^{(2)}_{1}\mathrm{A}^i{}_j{}^k{}_l{}^m{}_n\, d^n{}_m\, d^l{}_k + {}^{(2)}_{2}\mathrm{A}^i{}_j{}^k{}_l{}^m{}_n\, d^n{}_m\, \omega^l{}_k \\
& + {}^{(2)}_{3}\mathrm{A}^i{}_j{}^k{}_l{}^m{}_n\, \omega^n{}_m\, \omega^l{}_k + {}^{(2)}_{4}\mathrm{A}^i{}_j{}^k{}_l{}^m{}_m\, \dot{x}^m{}_{,lk} + {}^{(3)}_{1}\mathrm{A}^i{}_j{}^k{}_l{}^m{}_n{}^p{}_q\, d^q{}_p\, d^n{}_m\, d^l{}_k \\
& + {}^{(3)}_{2}\mathrm{A}^i{}_j{}^k{}_l{}^m{}_n{}^p{}_q\, d^q{}_p\, d^n{}_m\, \omega^l{}_k + {}^{(3)}_{3}\mathrm{A}^i{}_j{}^k{}_l{}^m{}_n{}^p{}_q\, d^q{}_p\, \omega^n{}_m\, \omega^l{}_k \\
& + {}^{(3)}_{4}\mathrm{A}^i{}_j{}^k{}_l{}^m{}_n{}^p{}_q\, \omega^q{}_p\, \omega^n{}_m\, \omega^l{}_k + {}^{(3)}_{5}\mathrm{A}^i{}_j{}^k{}_l{}^m{}_n{}^p\, d^p{}_n\, \dot{x}^m{}_{,lk} + {}^{(3)}_{6}\mathrm{A}^i{}_j{}^k{}_l{}^m{}_n{}^p\, \omega^p{}_n\, \dot{x}^m{}_{,lk} \\
& + {}^{(3)}_{7}\mathrm{A}^i{}_j{}^k{}_l{}^m{}_n\, \dot{x}^n{}_{,mlk} + \cdots
\end{aligned}
$$

The ordering of the terms and the notation are determined by the following scheme. Let the dimensions of the term whose coefficient is $\mathrm{A}^{\cdots}_{\cdots}$ be $\mathsf{L}^{-l''}\mathsf{T}^{-t''}$, then the number $l''+t''=b$ is written in parentheses at the upper left of the coefficient, ${}^{(b)}_{\mathrm{A}}{}^{\cdots}_{\cdots}$, and ${}^{(b)}\mathrm{A}^{\cdots}_{\cdots}$ is written down before ${}^{(c)}\mathrm{A}^{\cdots}_{\cdots}$ if $b<c$. The lower left index is simply enumerative. [It will be proved later (§ 23) that this method results in the ordering of the terms according to powers of the natural viscosity μ_n.] In the expansion (22.1), all terms for which $b=0, 1, 2, 3$ are actually written down.

The coefficients ${}^{(b)}_{h}\mathrm{A}^{\cdots}_{\cdots}$ are functions of the variables which remain in the list (21.2) after $d^i{}_j$, $\omega^i{}_j$, $\dot{x}^i{}_{,jk}, \ldots, \dot{x}^{i_1}{}_{,i_2\ldots i_{k+1}}, \ldots$, have been struck off.

We now expand each of these coefficients in a series in the remaining vectors and tensors

$$
\begin{aligned}
(22.2)\qquad {}^{(0)}_{1}\mathrm{A}^{i}{}_{j} = {} & {}^{(0)(0)}_{1\,|\,1}\mathrm{A}^{i}{}_{j} + {}^{(0)(1)}_{1\,|\,1}\mathrm{A}^{i}{}_{j}{}^{k}p_{,k} + {}^{(0)(1)}_{1\,|\,2}\mathrm{A}^{i}{}_{j}{}^{k}\theta_{,k} + {}^{(0)(2)}_{1\,|\,1}\mathrm{A}^{i}{}_{j}{}^{kl}p_{,k}p_{,l} \\
& + {}^{(0)(2)}_{1\,|\,2}\mathrm{A}^{i}{}_{j}{}^{kl}p_{,k}\theta_{,l} + {}^{(0)(2)}_{1\,|\,3}\mathrm{A}^{i}{}_{j}{}^{kl}\theta_{,k}\theta_{,l} + {}^{(0)(2)}_{1\,|\,4}\mathrm{A}^{i}{}_{j}{}^{kl}p_{,kl} \\
& + {}^{(0)(2)}_{1\,|\,5}\mathrm{A}^{i}{}_{j}{}^{kl}\theta_{,kl} + {}^{(0)(2)}_{1\,|\,6}\mathrm{A}^{i}{}_{j}{}^{kl}f_{k,l} + {}^{(0)(3)}_{1\,|\,1}\mathrm{A}^{i}{}_{j}{}^{klm}p_{,k}p_{,l}p_{,m} \\
& + {}^{(0)(3)}_{1\,|\,2}\mathrm{A}^{i}{}_{j}{}^{klm}p_{,k}p_{,l}\theta_{,m} + {}^{(0)(3)}_{1\,|\,3}\mathrm{A}^{i}{}_{j}{}^{klm}p_{,k}\theta_{,l}\theta_{,m} \\
& + {}^{(0)(3)}_{1\,|\,4}\mathrm{A}^{i}{}_{j}{}^{klm}\theta_{,k}\theta_{,l}\theta_{,m} + {}^{(0)(3)}_{1\,|\,5}\mathrm{A}^{i}{}_{j}{}^{klm}p_{,k}f_{l,m} \\
& + {}^{(0)(3)}_{1\,|\,6}\mathrm{A}^{i}{}_{j}{}^{klm}\theta_{,k}f_{l,m} + {}^{(0)(3)}_{1\,|\,7}\mathrm{A}^{i}{}_{j}{}^{klm}p_{,k}p_{,lm} \\
& + {}^{(0)(3)}_{1\,|\,8}\mathrm{A}^{i}{}_{j}{}^{klm}\theta_{,k}p_{,lm} + {}^{(0)(3)}_{1\,|\,9}\mathrm{A}^{i}{}_{j}{}^{klm}p_{,k}\theta_{,lm} \\
& + {}^{(0)(3)}_{1\,|\,10}\mathrm{A}^{i}{}_{j}{}^{klm}\theta_{,k}\theta_{,lm} + {}^{(0)(3)}_{1\,|\,11}\mathrm{A}^{i}{}_{j}{}^{klm}p_{,klm} \\
& + {}^{(0)(3)}_{1\,|\,12}\mathrm{A}^{i}{}_{j}{}^{klm}\theta_{,klm} + {}^{(0)(3)}_{1\,|\,13}\mathrm{A}^{i}{}_{j}{}^{klm}f_{k,lm} + \ldots, \\
{}^{(1)}_{h}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l} = {} & {}^{(1)(0)}_{h\,|\,1}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l} + {}^{(1)(1)}_{h\,|\,1}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{m}p_{,m} + {}^{(1)(1)}_{h\,|\,2}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{m}\theta_{,m} \\
& + {}^{(1)(2)}_{h\,|\,1}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{mn}p_{,m}p_{,n} + {}^{(1)(2)}_{h\,|\,2}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{mn}p_{,m}\theta_{,n} \\
& + {}^{(1)(2)}_{h\,|\,3}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{mn}\theta_{,m}\theta_{,n} + {}^{(1)(2)}_{h\,|\,4}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{mn}p_{,mn} \\
& + {}^{(1)(2)}_{h\,|\,5}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{mn}\theta_{,mn} + {}^{(1)(2)}_{h\,|\,6}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{mn}f_{m,n} + \ldots, \\
{}^{(2)}_{h}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{m}{}_{n} = {} & {}^{(2)(0)}_{h\,|\,1}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{m}{}_{n} + {}^{(2)(1)}_{h\,|\,1}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{m}{}_{n}{}^{q}p_{,q} + {}^{(2)(1)}_{h\,|\,2}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{m}{}_{n}{}^{q}\theta_{,q} + \ldots, \\
{}^{(2)}_{4}\mathrm{A}^{i}{}_{j}{}^{kl}{}_{m} = {} & {}^{(2)(0)}_{4\,|\,1}\mathrm{A}^{i}{}_{j}{}^{kl}{}_{m} + {}^{(2)(1)}_{4\,|\,1}\mathrm{A}^{i}{}_{j}{}^{kl}{}_{m}{}^{n}p_{,n} + {}^{(2)(1)}_{4\,|\,2}\mathrm{A}^{i}{}_{j}{}^{kl}{}_{m}{}^{n}\theta_{,n} + \ldots, \\
{}^{(3)}_{h}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{m}{}_{n}{}^{p}{}_{q} = {} & {}^{(3)(0)}_{h\,|\,1}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{m}{}_{n}{}^{p}{}_{q} + \ldots, \qquad {}^{(3)}_{h}\mathrm{A}^{i}{}_{j}{}^{kl}{}_{m}{}^{n}{}_{p} = {}^{(3)(0)}_{h\,|\,1}\mathrm{A}^{i}{}_{j}{}^{kl}{}_{m}{}^{n}{}_{p} + \ldots, \\
{}^{(3)}_{7}\mathrm{A}^{i}{}_{j}{}^{klm}{}_{n} = {} & {}^{(3)(0)}_{7\,|\,1}\mathrm{A}^{i}{}_{j}{}^{klm}{}_{n} + \ldots, \qquad \ldots.
\end{aligned}
$$

In these series the coefficients ${}^{(b)(a)}_{h\,|\,d}\mathrm{A}^{\cdots}_{\cdots}$ are functions only of the scalars μ_n, R_n, p, p_m, θ, θ_0, and dimensionless constants. The first upper index (b) in parentheses is the same as that of the quantity being expanded. The second, (a), is determined by the following scheme. Let the dimensions of the term in the new series whose coefficient is ${}^{(b)(a)}_{h\,|\,d}\mathrm{A}^{\cdots}_{\cdots}$ be $\mathsf{L}^{-l'}\mathsf{T}^{-t'}\mathsf{M}^{-m}\Theta^{q}$; then put $a = l' + t' + 3m$, and order according to increasing a. The second lower index, separated from the first by a vertical bar, is merely enumerative. We have actually written out in the expansions (22.1) and (22.2) combined all the terms for which $a + b \leqq 3$. [It will be proved later (§ 23) that consequently we have written out all terms of order 3 or less in a power series expansion in the viscosity coefficient μ_n].

The symmetry of the stress tensor $t^{i}{}_{j}$ imposes the general requirement

$$
(22.3)\qquad {}^{(b)(a)}_{h\,|\,d}\mathrm{A}^{i}{}_{j}{}^{\cdots}_{\cdots} = {}^{(b)(a)}_{h\,|\,d}\mathrm{A}^{j}{}_{i}{}^{\cdots}_{\cdots},
$$

upon all the coefficients. Other simplifications result from such symmetries as $d_{ij}=d_{ji}$, $p_{,ij}=p_{,ji}$, $\omega_{ij}=-\omega_{ji}$, but we shall not work them out.

23. Anisotropic fluids. — II. *The form of the stress coefficients.* — Now as we have said, all the coefficients ${}^{(b)(a)}_{h\,|\,d}\mathrm{A}^{\cdots}_{\cdots}$ are of the functional form

$$(23.1)\qquad {}^{(b)(a)}_{h\,|\,d}\mathrm{A}^{\cdots}_{\cdots}=f(\mu_n, \mathrm{R}_n, p, p_m, \theta, \theta_0).$$

For the following dimensional argument, let the space co-ordinates be rectangular Cartesian, so that all tensorial components assume their natural physical dimensions. Our resulting conclusions will be expressed in invariant form, and hence valid in any co-ordinate system. From (22.1) and (22.2) the dimensions of each of the ${}^{(b)(a)}_{h\,|\,d}\mathrm{A}^{\cdots}_{\cdots}$ may be determined easily. For example

$$(23.2)\qquad \left\{\begin{aligned} &\dim {}^{(0)(0)}_{1\,|\,1}\mathrm{A}^{i}{}_{j}=\dim {}^{(0)}_{1}\mathrm{A}^{i}{}_{j}=\dim t^{i}{}_{j}=\frac{\mathsf{M}}{\mathsf{L}\mathsf{T}^2},\\ &\dim {}^{(1)(2)}_{h\,|\,2}=\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}{}^{mn}=\frac{\dim {}^{(1)}_{h}\mathrm{A}^{i}{}_{j}{}^{k}{}_{l}}{\dim p_{,m}\dim \theta_{,m}}=\frac{\dfrac{\dim t^{i}{}_{j}}{\dim d^{i}{}_{j}}}{\dim p_{,m}\dim \theta_{,m}}\\ &\qquad\qquad=\frac{\dfrac{\dfrac{\mathsf{M}}{\mathsf{L}\mathsf{T}^2}}{\dfrac{1}{\mathsf{T}}}}{\dfrac{\mathsf{M}}{\mathsf{L}^2\mathsf{T}^2}\,\dfrac{\Theta}{\mathsf{L}}}=\frac{\mathsf{L}^2\mathsf{T}}{\Theta}. \end{aligned}\right.$$

In general,

$$(23.3)\qquad \dim {}^{(b)(a)}_{h\,|\,d}\mathrm{A}^{\cdots}_{\cdots}=\mathsf{M}^m\mathsf{L}^l\mathsf{T}^t\Theta^q,$$

where m, l, t, q are known. By (19.4) it follows then that the ratio

$$\frac{{}^{(b)(a)}_{h\,|\,d}\mathrm{A}^{\cdots}_{\cdots}}{\left(\dfrac{\mu_n^2(\mathrm{R}_n\theta)^{\frac{1}{2}}}{p^2}\right)^m\left(\dfrac{\mu_n(\mathrm{R}_n\theta)^{\frac{1}{2}}}{p}\right)^l\left(\dfrac{\mu_n}{p}\right)^t\theta^q},$$

is dimensionless. If then we divide (23.1) by the denominator in the above expression, the resulting relation

$$(23.4)\qquad \frac{{}^{(b)(a)}_{h\,|\,d}\mathrm{A}^{\cdots}_{\cdots}}{\mu_n^{t+l+3m}\,\mathrm{R}_n^{\frac{l+m}{2}}\,\theta^{\frac{l+m}{2}+q}\,p^{-t-l-2m}} = g(\mu_n, \mathrm{R}_n, p, p_m, \theta, \theta_0),$$

connects six dimensional quantities composed of four fundamental dimensions, and hence must reduce to a relation connecting $6-4=2$ dimensionless ratios of these quantities. For the desired dimensionless ratios may be selected simply $\frac{p}{p_m}$ and $\frac{\theta}{\theta_0}$; that is, $g(\mu_n, \mathrm{R}_n, p, p_m, \theta, \theta_0) = g\left(\frac{p}{p_m}, \frac{\theta}{\theta_0}\right)$.

We have demonstrated then that

$$(23.5)\qquad {}^{(b)(a)}_{h\,|\,d}\mathrm{A}^{\cdots}_{\cdots} = \frac{\mu_n^{t+l+3m}\,\mathrm{R}_n^{\frac{l+m}{2}}\,\theta^{\frac{l+m}{2}+q}}{p^{t+l+2m}}\,{}^{(b)(a)}_{h\,|\,d}\mathrm{K}^{\cdots}_{\cdots},$$

where the tensor ${}^{(b)(a)}_{h\,|\,d}\mathrm{K}^{\cdots}_{\cdots}$ *is one having the same tensorial variance as the original coefficient* ${}^{(a)(b)}_{h\,|\,d}\mathrm{A}^{\cdots}_{\cdots}$, *but whose physical components are dimensionless functions only of dimensionless constants and of the dimensionless ratios* $\frac{p}{p_m}$ *and* $\frac{\theta}{\theta_0}$. Thus we have reduced the stresses in a Maxwellian fluid to a perfectly definite form. Furthermore, the formula (23.5) demonstrates that the *stress tensor is a power series in the viscosity coefficient*, since $t+2+3m$ is always a non-negative integer.

Applying our general formula (23.5) to the examples (23.2) we immediately obtain

$$(23.6)\qquad \begin{cases} {}^{(0)(0)}_{1\,|\,1}\mathrm{A}^i{}_j = p\,{}^{(0)(0)}_{1\,|\,1}\mathrm{K}^i{}_j, \\ {}^{(1)(2)}_{h\,|\,2}\mathrm{A}^i{}_j{}^k{}_e{}^{mn} = \dfrac{\mu_n^3\,\mathrm{R}_n}{p^3}\,{}^{(1)(2)}_{h\,|\,2}\mathrm{K}^i{}_j{}^k{}_l{}^{mn}. \end{cases}$$

The first of these formulae can be still further simplified, for by III of paragraph 21 we have $\mathrm{K}^i{}_j = -\delta^i{}_j$, so that ${}^{(0)(0)}_{1\,|\,1}\mathrm{A}^i{}_j = -p\,\delta^i{}_j$. Our notation is such that no confusion can arise if we abbreviate the second of (23.6) by the symbolic formula

$$(23.7)\qquad {}^{(1)(2)}_{h\,|\,2} = \frac{\mu_n^3\,\mathrm{R}_n}{p^3}.$$

Similar application of (23.5) to the remaining coefficients actually written down in (22.2) yields the following similarly abbreviated table of forms

$$
(23.8)\quad
\begin{aligned}
&{}^{(0)(1)}_{1\,|\,1} = \frac{\mu_n(\mathrm{R}_n\theta)^{\frac{1}{2}}}{p}, &&{}^{(0)(1)}_{1\,|\,2} = \frac{\mu_n \mathrm{R}_n^{\frac{1}{2}}}{\theta^{\frac{1}{2}}}, &&{}^{(0)(2)}_{1\,|\,1} = \frac{\mu_n^2 \mathrm{R}_n\theta}{p^3},\\
&{}^{(0)(2)}_{1\,|\,2} = \frac{\mu_n^2 \mathrm{R}_n}{p^2}, &&{}^{(0)(2)}_{1\,|\,3} = \frac{\mu_n^2 \mathrm{R}_n}{p\theta}, &&{}^{(0)(2)}_{1\,|\,4} = \frac{\mu_n^2 \mathrm{R}_n\theta}{p^2},\\
&{}^{(0)(2)}_{1\,|\,5} = \frac{\mu_n^2 \mathrm{R}_n}{p}, &&{}^{(0)(2)}_{1\,|\,6} = \frac{\mu_n^2}{p}, &&{}^{(0)(3)}_{1\,|\,1} = \frac{\mu_n^3(\mathrm{R}_n\theta)^{\frac{3}{2}}}{p^5},\\
&{}^{(0)(3)}_{1\,|\,2} = \frac{\mu_n^3 \mathrm{R}_n^{\frac{3}{2}}\theta^{\frac{1}{2}}}{p^4}, &&{}^{(0)(3)}_{1\,|\,3} = \frac{\mu_n^3 \mathrm{R}_n^{\frac{3}{2}}}{p^3\theta^{\frac{1}{2}}}, &&{}^{(0)(3)}_{1\,|\,4} = \frac{\mu_n^3 \mathrm{R}_n^{\frac{3}{2}}}{p^2\theta^{\frac{3}{2}}},\\
&{}^{(0)(3)}_{1\,|\,5} = \frac{\mu_n^3(\mathrm{R}_n\theta)^{\frac{1}{2}}}{p^3}, &&{}^{(0)(3)}_{1\,|\,6} = \frac{\mu_n^3 \mathrm{R}_n^{\frac{1}{2}}}{p^2}, &&{}^{(0)(3)}_{1\,|\,7} = \frac{\mu_n^3(\mathrm{R}_n\theta)^{\frac{3}{2}}}{p^4},\\
&{}^{(0)(3)}_{1\,|\,8} = \frac{\mu_n^3 \mathrm{R}_n^{\frac{3}{2}}\theta^{\frac{1}{2}}}{p^3}, &&{}^{(0)(3)}_{1\,|\,9} = \frac{\mu_n^3 \mathrm{R}_n^{\frac{3}{2}}\theta^{\frac{1}{2}}}{p^3}, &&{}^{(0)(3)}_{1\,|\,10} = \frac{\mu_n^3 \mathrm{R}_n^{\frac{3}{2}}}{p^2\theta^{\frac{1}{2}}},\\
&{}^{(0)(3)}_{1\,|\,11} = \frac{\mu_n^3(\mathrm{R}_n\theta)^{\frac{3}{2}}}{p^3}, &&{}^{(0)(3)}_{1\,|\,12} = \frac{\mu_n^3 \mathrm{R}_n^{\frac{3}{2}}\theta^{\frac{1}{2}}}{p^2}, &&{}^{(0)(3)}_{1\,|\,13} = \frac{\mu_n^3(\mathrm{R}_n\theta)^{\frac{1}{2}}}{p^2},\\
&{}^{(1)(0)}_{h\,|\,1} = \mu_n, &&{}^{(1)(1)}_{h\,|\,1} = \frac{\mu_n^2(\mathrm{R}_n\theta)^{\frac{1}{2}}}{p^2}, &&{}^{(1)(1)}_{h\,|\,2} = \frac{\mu_n^2 \mathrm{R}_n^{\frac{1}{2}}}{p\theta^{\frac{1}{2}}},\\
&{}^{(1)(2)}_{h\,|\,1} = \frac{\mu_n^3 \mathrm{R}_n\theta}{p^4}, &&{}^{(1)(2)}_{h\,|\,2} = \frac{\mu_n^3 \mathrm{R}_n}{p^3}, &&{}^{(1)(2)}_{h\,|\,3} = \frac{\mu_n^3 \mathrm{R}_n}{p^2\theta},\\
&{}^{(1)(2)}_{h\,|\,4} = \frac{\mu_n^3 \mathrm{R}_n\theta}{p^3}, &&{}^{(1)(2)}_{h\,|\,5} = \frac{\mu_n^3 \mathrm{R}_n}{p^2}, &&{}^{(1)(2)}_{h\,|\,6} = \frac{\mu_n^3}{p^2},\\
&{}^{(2)(0)}_{h\,|\,1} = \frac{\mu_n^2}{p}, &&{}^{(2)(1)}_{h\,|\,1} = \frac{\mu_n^3(\mathrm{R}_n\theta)^{\frac{1}{2}}}{p^3}, &&{}^{(2)(1)}_{h\,|\,2} = \frac{\mu_n^3 \mathrm{R}_n^{\frac{1}{2}}}{p^2\theta^{\frac{1}{2}}} \quad (h \neq 4),\\
&{}^{(2)(0)}_{4\,|\,1} = \frac{\mu_n^2(\mathrm{R}_n\theta)^{\frac{1}{2}}}{p}, &&{}^{(2)(1)}_{4\,|\,1} = \frac{\mu_n^3 \mathrm{R}_n\theta}{p^3}, &&{}^{(2)(1)}_{4\,|\,2} = \frac{\mu_n^3 \mathrm{R}_n}{p^2},\\
&{}^{(3)(0)}_{h\,|\,1} = \frac{\mu_n^3}{p^2} \text{ for } h = 1, 2, 3, 4, &&{}^{(3)(0)}_{h\,|\,1} = \frac{\mu_n^3(\mathrm{R}_n\theta)^{\frac{1}{2}}}{p^2} \text{ for } h = 5, 6, &&\\
&{}^{(3)(0)}_{7\,|\,1} = \frac{\mu_n^3 \mathrm{R}_n\theta}{p^2}. &&&&
\end{aligned}
$$

The reader may easily verify that our method of ordering in paragraph 22 is such that

$$
(23.9)\qquad b + a = t + l + 3m.
$$

Thus by (23.5) *the sum of the indices* (*a*) *and* (*b*) *in parentheses at the upper left of the coefficient* ${}^{(b)(a)}_{h|d}\mathrm{A}^{\cdots}_{\cdots}$ *is the power of the viscosity coefficient* μ_n *appearing in its final form.* Therefore the expansions (22.2), and consequently also the table of coefficients (23.8), contain all terms in μ_n^0, μ_n^1, μ_n^2, and μ_n^3 in the general expression for the stresses in a Maxwellian fluid. The same method easily and quickly yields all the terms of any desired order ([51]).

24. Anisotropic fluids. — III. *Power series for the heat flux.* — Now let us write

$$q_j = {}^{(0)}_{1}\mathrm{B}_j + {}^{(1)}_{1}\mathrm{B}_j{}^k{}_l d^l{}_k + {}^{(1)}_{2}\mathrm{B}_j{}^k{}_l \omega^l{}_k + \ldots, \tag{24.1}$$

where the expansion is the direct analogue of (22.1), except that the heat flux coefficient ${}^{(b)}_{h}\mathrm{B}_j^{\cdots}$ is a tensor of order one less than its stress counterpart ${}^{(b)}_{h}\mathrm{A}^i{}_j^{\cdots}$. Then each of the coefficient ${}^{(b)}_{h}\mathrm{B}_j^{\cdots}$ is to be expanded according to its analogue in (22.2) :

$${}^{(0)}_{1}\mathrm{B}_j = {}^{(0)(0)}_{1|1}\mathrm{B}_j + {}^{(0)(1)}_{1|1}\mathrm{B}_j{}^k p_{,k} + \ldots, \qquad \ldots. \tag{24.2}$$

The ordering of the terms is according to precisely the same scheme as for the stress expansion, so we need not write out the result.

25. Anisotropic fluids. — IV. *The form of the heat flux coefficients.* — The reasoning of paragraph 23 may be applied to the heat flux coefficients ${}^{(a)(b)}_{h|d}\mathrm{B}^{\cdots}_{\cdots}$, which thus may be expressed in terms of dimensionless cofficients ${}^{(b)(a)}_{h|d}\mathrm{L}^{\cdots}_{\cdots}$ by the same formula (23.5). Since our expansions of paragraph 24 are of precisely the same form as those of paragraph 22, and since $\dim t^i{}_j = \mathbf{ML^{-1}T^{-2}}$, while $\dim q_i = \mathbf{MT^{-3}}$, we have always

$$\dim\left[{}^{(b)(a)}_{h|d}\mathrm{B}^{\cdots}_{\cdots}\right] = \frac{\mathbf{L}}{\mathbf{T}}\dim\left[{}^{(b)(a)}_{h|d}\mathrm{A}^{\cdots}_{\cdots}\right]. \tag{25.1}$$

Instead of applying (23.5) to each of the heat flux coefficients ${}^{(b)(a)}_{h|d}\mathrm{B}^{\cdots}_{\cdots}$

([51]) In [1948, 1], which is a preliminary version of the present paper, all terms in μ_n^4 resulting from a somewhat less general definition are explicitly written down.

separately we may notice simply that (25.1) has the effect of increasing l by one and decreasing t by one. Hence by (23.5) we have

$$(25.2)\qquad {}^{(b)(a)}_{h\,|\,d}\mathrm{B}^{\cdots}_{\cdots} = (\mathrm{R}_n\theta)^{\frac{1}{2}}\left[{}^{(b)(a)}_{h\,|\,d}\right]{}^{(b)(a)}_{h\,|\,d}\mathrm{L}^{\cdots}_{\cdots},$$

where the quantity in square brackets may be read off from the table (23.8), and the tensor ${}^{(b)(a)}_{h\,|\,d}\mathrm{L}^{\cdots}_{\cdots}$ is a function only of dimensionless constants and of the dimensionless ratios $\frac{p}{p_m}$ and $\frac{\theta}{\theta_0}$, and in a rectangular Cartesian system is itself dimensionless. Thus all the heat flux coefficients are determined at one blow.

In the general case, *the forms of the heat flux and the stress expansions differ but by the one tensorial index i and the factor* $(\mathrm{R}_n\theta)^{\frac{1}{2}}$. The former distinction is a very important one, however, as shall be illustrated presently in the isotropic case.

26. Isotropic fluids. — I. *The stress.* — A fluid is *isotropic* if the coefficients ${}^{(b)(a)}_{h\,|\,d}\mathrm{A}^{\cdots}_{\cdots}$ and ${}^{(b)(a)}_{h\,|\,d}\mathrm{B}^{\cdots}_{\cdots}$ in the expansions (22.2) and (24.2) reduce to numerical tensors. We cannot now use the simple and elegant formula (6.5) which Reiner derived as an expression for Stokes's principle in the isotropic case; an analogous but more elaborate treatment is required.

We shall not give this analysis in detail, but shall be content to indicate an example. Suppose that the symmetric tensor $a^i{}_j$ is to be an isotropic function of the two symmetric tensors $b^i{}_j$ and $c^i{}_j$. That is, there is a matrix relation of the form

$$(26.1)\qquad \mathbf{a} = \mathrm{A}_{\alpha\beta}[\mathbf{b}^\alpha\mathbf{c}^\beta + \mathbf{c}^\beta\mathbf{b}^\alpha],$$

where the $\mathrm{A}_{\alpha\beta}$ are functions of those scalars which can be formed from the powers $\mathbf{b}^\alpha\mathbf{c}^\beta$. Among these scalars are the principal invariants $\mathrm{I}_\mathbf{b}$, $\mathrm{II}_\mathbf{b}$, $\mathrm{III}_\mathbf{b}$, $\mathrm{I}_\mathbf{c}$, $\mathrm{II}_\mathbf{c}$, $\mathrm{III}_\mathbf{c}$ of the two tensors, as well as higher invariants such as $b^i{}_j b^j{}_k b^k{}_l b^l{}_m$, and joint invariants $\Phi_{\mathbf{bc}}, \ldots$, defined by

$$(26.2)\qquad \Phi_{\mathbf{bc}} \equiv b^i{}_j c^j{}_i, \qquad \Phi_{\mathbf{b}^2\mathbf{c}} \equiv b^i{}_k b^k{}_j c^j{}_i \equiv \Phi_{\mathbf{cb}^2}, \qquad \ldots.$$

Each of the matrices $\mathbf{b}$ and $\mathbf{c}$ satisfies the Cayley-Hamilton equation, which in the three dimensional case assumes the forms

$$(26.3)\qquad \begin{cases} \mathbf{b}^3 = \mathrm{I}_\mathbf{b}\mathbf{b}^2 - \mathrm{II}_\mathbf{b}\mathbf{b} + \mathrm{III}_\mathbf{b}\mathbf{I}, \\ \mathbf{c}^3 = \mathrm{I}_\mathbf{c}\mathbf{c}^2 - \mathrm{II}_\mathbf{c}\mathbf{c} + \mathrm{III}_\mathbf{c}\mathbf{I}, \end{cases}$$

Thus all third and higher powers of **b** and **c** may be eliminated from (26.1), as well as from the various invariants, and we obtain

$$(26.4)\quad \mathbf{a} = \mathcal{F}_{00}\mathbf{I} + \mathcal{F}_{10}\mathbf{b} + \mathcal{F}_{01}\mathbf{c} + \mathcal{F}_{20}\mathbf{b}^2 + \mathcal{F}_{11}(\mathbf{bc} + \mathbf{cb}) + \mathcal{F}_{02}\mathbf{c}^2 + \mathcal{F}_{21}(\mathbf{b}^2\mathbf{c} + \mathbf{cb}^2) + \mathcal{F}_{12}(\mathbf{bc}^2 + \mathbf{c}^2\mathbf{b}) + \mathcal{F}_{22}(\mathbf{b}^2\mathbf{c}^2 + \mathbf{c}^2\mathbf{b}^2),$$

where the $\mathcal{F}_{\Upsilon\Phi}$ are scalar functions of the nine invariants $\mathrm{I}_{\mathbf{b}}$, $\mathrm{II}_{\mathbf{b}}$, $\mathrm{III}_{\mathbf{b}}$, $\mathrm{I}_{\mathbf{c}}$, $\mathrm{II}_{\mathbf{c}}$, $\mathrm{III}_{\mathbf{c}}$, $\Phi_{\mathbf{bc}}$, $\Phi_{\mathbf{b}^2\mathbf{c}}$, $\Phi_{\mathbf{bc}^2}$. The component form of (26.4) is

$$(26.5)\quad a^i{}_j = \mathcal{F}_{00}\delta^i{}_j + \mathcal{F}_{10}b^i{}_j + \mathcal{F}_{01}c^i{}_j + \mathcal{F}_{20}b^i{}_k b^k{}_j + \mathcal{F}_{11}(b^i{}_k c^k{}_j + c^i{}_k b^k{}_j) + \mathcal{F}_{02}c^i{}_k c^k{}_j + \mathcal{F}_{21}(b^i{}_k b^k{}_l c^l{}_j + c^i{}_k b^k{}_l b^l{}_j) + \mathcal{F}_{12}(c^i{}_k c^k{}_l b^l{}_j + b^i{}_k c^k{}_l c^l{}_j) + \mathcal{F}_{22}(b^i{}_k b^k{}_l c^l{}_m c^m{}_j + c^i{}_k c^k{}_l b^l{}_m b^m{}_j).$$

When **b** and **c** are not symmetric, or when **a** depends upon tensors of order higher than two, the analysis becomes more elaborate. In the case of the Maxwellian fluid, not only must such complications be considered, but also the stress $t^i{}_j$ depends upon an infinite list of vectors and tensors. It is thus not feasible to write out a complete formula for the stress generalizing Reiner's formula (6.5) employed in the theory of the Stokesian fluid. Instead, we shall simply content ourselves with writing out all terms of order 0, 1, 2 and 3 in the viscosity coefficient μ_n. Using (23.8) and the methods sketched above, for the coefficients ${}^{(s)}t^i{}_j$ in the power series

$$(26.6)\qquad t^i{}_j = {}^{(0)}t^i{}_j + \mu_n\,{}^{(1)}t^i{}_j + \mu_n^2\,{}^{(2)}t^i{}_j + \mu_n^3\,{}^{(3)}t^i{}_j + \ldots$$

we thus obtain

$$(26.7)\quad {}^{(0)}t^i{}_j = -p\,\delta^i{}_j, \qquad {}^{(1)}t^i{}_j = \mathrm{A}\,d^k{}_k\delta^i{}_j + 2\,\mathrm{B}\,d^i{}_j,$$

$$\begin{aligned}
{}^{(2)}t^i{}_j = {} & \frac{\mathrm{R}_n\theta}{p^3}[\mathrm{D}_1 p^{\cdot k}p_{,k}\delta^i{}_j + 2\,\mathrm{C}_1 p^{\cdot i}p_{,j}] + \frac{\mathrm{R}_n}{p^2}[\mathrm{D}_2 p^{\cdot k}\theta_{,k}\delta^i{}_j + \mathrm{C}_2(p^{\cdot i}\theta_{,j} + \theta^{\cdot i}p_{,j})] \\
& + \frac{\mathrm{R}_n}{p\theta}[\mathrm{D}_3\theta^{\cdot k}\theta_{,k}\delta^i{}_j + 2\,\mathrm{C}_3\theta^{\cdot i}\theta_{,j}] + \frac{\mathrm{R}_n\theta}{p^2}[\mathrm{D}_4 p^{\cdot k}{}_{,k}\delta^i{}_j + 2\,\mathrm{C}_4 p^{\cdot i}{}_{,j}] \\
& + \frac{1}{p}[\mathrm{D}_5 f^k{}_{,k}\delta^i{}_j + \mathrm{C}_5(f^i{}_{,j} + f_j{}^{\cdot i})] + \frac{\mathrm{R}_n}{p}[\mathrm{D}_6\theta^{\cdot k}{}_{,k}\delta^i{}_j + 2\,\mathrm{C}_6\theta^{\cdot i}{}_{,j}] \\
& + \frac{1}{p}[\mathrm{F}_1(d^k{}_k)^2\delta^i{}_j + \mathrm{F}_2 d^l{}_k d^k{}_l\delta^i{}_j \\
& \qquad + \mathrm{F}_3\omega^l{}_k\omega^k{}_l\delta^i{}_j + 2\,\mathrm{E}_1 d^k{}_k d^i{}_j \\
& \qquad + 2\,\mathrm{E}_2 d^i{}_k d^k{}_j + 2\,\mathrm{E}_3\omega^i{}_k\omega^k{}_j + \mathrm{E}_4(d^i{}_k\omega^k{}_j + \omega_k{}^i d^k{}_j)],
\end{aligned}$$

$$
\begin{aligned}
{}^{(3)}t^i{}_j = \frac{R_n\theta}{p^4}\big[& G_1 d^k{}_k p^{,l} p_{,l}\delta^i{}_j + G_2 d^k{}_l p^{,l} p_{,k}\delta^i{}_j \\
& + G_3(d^i{}_k p^{,k} p_{,j} + d_j{}^k p_{,k} p^{,i}) \\
& + G_4 d_l{}^l p^{,k} p_{,k} + G_5(\omega^i{}_k p^{,k} p_{,j} + \omega_j{}^k p_{,k} p^{,i})\big] \\
+ \frac{R_n}{p^3}\big[& G_6 d^k{}_k p^{,l}\theta_{,l}\delta^i{}_j + G_7 d^k{}_l p^{,l} p_{,k}\delta^i{}_j \\
& + G_8(d^i{}_k p^{,k}\theta_{,j} + d_j{}^k p_{,k}\theta^{,i}) + G_9(d^i{}_k\theta^{,k} p_{,j} + d_j{}^k\theta_{,k} p^{,i}) \\
& + G_{10} d^i{}_j p^{,k}\theta_{,k} + G_{11}\omega^{kl} p_{,k}\theta_{,l}\delta^i{}_j \\
& + G_{12}(\omega^i{}_k p^{,k}\theta_{,j} + \omega_j{}^k p_{,k}\theta^{,i}) + G_{13}(\omega^i{}_k\theta^{,k} p_{,j} + \omega_j{}^k\theta_{,k} p^{,i})\big] \\
+ \frac{R_n}{p^2\theta}\big[& G_{14} d^k{}_k\theta^{,k}\theta_{,k}\delta^i{}_j + G_{15} d^k{}_l\theta^{,l}\theta_{,k}\delta^i{}_j \\
& + G_{16}(d^i{}_k\theta^{,k}\theta_{,j} + d_j{}^k\theta_{,k}\theta^{,i}) \\
& + G_{17} d^i{}_j\theta^{,k}\theta_{,k} + G_{18}(\omega^i{}_k\theta^{,k}\theta_{,j} + \omega_j{}^k\theta_{,k}\theta^{,i})\big] \\
+ \frac{R_n\theta}{p^3}\big[& G_{19} d^k{}_k p^{,l}{}_{,l}\delta^i{}_j + G_{20} d^k{}_l p^{,l}{}_{,k}\delta^i{}_j + G_{21}(d^i{}_k p^{,k}{}_{,j} + d_j{}^k p_{,k}{}^{,i}) \\
& + G_{22} d^k{}_k p^{,i}{}_{,j} + G_{23} d^i{}_j p^{,k}{}_{,k} + G_{24}(\omega^i{}_k p^{,k}{}_{,j} + \omega_j{}^k p_{,k}{}^{,i})\big] \\
+ \frac{1}{p^2}\big[& G_{25} d^k{}_k f^l{}_{,l}\delta^i{}_j + G_{26} d^k{}_l f^l{}_{,k}\delta^i{}_j \\
& + G_{27}(d^i{}_k f^k{}_{,j} + d_j{}^k f_k{}^{,i}) + G_{28}(d^i{}_k f_j{}^{,k} + d_j{}^k f^i{}_{,k}) \\
& + G_{29} d^k{}_k(f^i{}_{,j} + f_j{}^{,i}) + G_{30} d^i{}_j f^k{}_{,k} + G_{31}\omega_k{}^l f_l{}^{,k}\delta^i{}_j \\
& + G_{32}(\omega^i{}_k f^k{}_{,j} + \omega_j{}^k f_k{}^{,i}) + G_{33}(\omega^i{}_k f_j{}^{,k} + \omega_j{}^k f^i{}_{,k})\big] \\
+ \frac{R_n}{p^2}\big[& G_{34} d^k{}_k\theta^{,l}{}_{,l}\delta_{ij} + G_{35} d^k{}_l\theta^{,l}{}_{,k}\delta^i{}_j + G_{36}(d^i{}_k\theta^{,k}{}_{,j} + d_j{}^k\theta_{,k}{}^{,i}) \\
& + G_{37} d^i{}_k\theta^{,i}{}_{,j} + G_{38} d^i{}_j\theta^{,k}{}_{,k} + G_{39}(\omega^i{}_k\theta^{,k}{}_{,j} + \omega_j{}^k\theta_{,k}{}^{,i})\big] \\
+ \frac{1}{p^2}\big[& G_{40}(d^k{}_k)^3\delta^i{}_j + G_{41} d^k{}_l d^l{}_k d^m{}_m\delta^i{}_j \\
& + G_{42} d^k{}_l d^l{}_m d^m{}_k\delta^i{}_j + G_{43} d^k{}_k d^i{}_l d^l{}_j + G_{44}(d^k{}_k)^2 d^i{}_j \\
& + G_{45} d^k{}_l d^l{}_k d^i{}_j + G_{46}\omega^k{}_l d^l{}_m d^m{}_k\delta^i{}_j \\
& + G_{47}(\omega^i{}_k d^k{}_l d^l{}_j + \omega_j{}^k d_k{}^l d_l{}^i) + G_{48}\omega^k{}_l\omega^l{}_k d^m{}_m\delta^i{}_j \\
& + G_{49}\omega^k{}_l\omega^l{}_m d^m{}_k\delta^i{}_j + G_{50}\omega^i{}_l\omega^l{}_j d^k{}_k \\
& + G_{51}(\omega^i{}_k\omega^k{}_l d^l{}_j + \omega_j{}^k\omega_k{}^l d_l{}^i) + G_{52}\omega^k{}_l\omega^l{}_m\omega^m{}_k\delta^i{}_j\big] \\
+ \frac{R_n\theta}{p_3}\big[& G_{53}(\dot{x}^i{}_{,jk} p^{,k} + \dot{x}_j{}^{,ik} p_{,k}) + G_{54}\dot{x}^l{}_{,lk} p^{,k}\delta^i{}_j \\
& + G_{55}\dot{x}^{l,i}{}_{,j} p_{,l} + G_{56}(\dot{x}^l{}_{,lj} p^{,i} + \dot{x}_l{}^{,li} p_{,j}) \\
& + G_{57}(\dot{x}^i{}_{,k}{}^{,k} p_{,j} + \dot{x}_j{}^{,k}{}_{,k} p^{,i}) + G_{58}\dot{x}^l{}_{,k}{}^{,k} p_{,l}\delta^i{}_j\big] \\
+ \frac{R_n}{p_2}\big[& G_{59}(\dot{x}^i{}_{,jk}\theta^{,k} + \dot{x}_j{}^{,ik}\theta_{,k}) + G_{60}\dot{x}^l{}_{,lk}\theta^{,k}\delta^i{}_j \\
& + G_{61}\dot{x}^{l,i}{}_{,j}\theta_{,l} + G_{62}(\dot{x}^l{}_{,lj}\theta^{,i} + \dot{x}_l{}^{,li}\theta_{,j}) \\
& + G_{63}(\dot{x}^i{}_{,k}{}^{,k}\theta_{,j} + \dot{x}_j{}^{,k}{}_{,k}\theta^{,i}) + G_{64}\dot{x}^l{}_{,k}{}^{,k}\theta_{,l}\delta^i{}_j\big] \\
+ \frac{R_n\theta}{p^2}\big[& G_{65}\dot{x}^k{}_{,k}{}^{,l}{}_{,l}\delta^i{}_j + G_{66}\dot{x}^k{}_{,k}{}^{,i}{}_{,j} + G_{67}(\dot{x}^i{}_{,k}{}^{,k}{}_{,j} + \dot{x}_j{}^{,k}{}_{,k}{}^{,i})\big].
\end{aligned}
$$

All the coefficients A, B, C_1, ... are dimensionless functions of $\frac{p}{p_m}$, $\frac{\theta}{\theta_0}$ and dimensionless material constants only.

To conform with classical notations let us write $\lambda \equiv A\mu_n$, $\mu \equiv B\mu_n$. Then the terms ${}^{(0)}t^i{}_j$ and ${}^{(1)}t^i{}_j$ yield the classical Newton-Cauchy-Poisson law (18.1). Thus for the Maxwellian fluid as for the Stokesian fluid, in the case of isotropy the classical theory of viscous compressible fluids is the first approximation.

Notice that not only is the number of terms greatly reduced in the isotropic case, but also whole classes of terms, and consequently certain types of phenomena, which appear for anisotropic fluids, disappear altogether ([52]). For example, for an anisotropic Maxwellian fluid, by (22.2) and (23.8) the terms of zero and first order in the viscosity μ_n are

$$(26.8)\quad t^i{}_j = -p\,\delta^i{}_j + \mu_n\left[\frac{(R_n\theta)^{\frac{1}{2}}}{p}\,{}^{(0)(1)}_{1\,|\,1}K^{i}{}_{j}{}^{k}p_{,k} + \left(\frac{R_n}{\theta}\right)^{\frac{1}{2}}{}^{(0)(1)}_{1\,|\,2}K^{i}{}_{j}{}^{k}\theta_{,k} + {}^{(1)(0)}_{1\,|\,1}K^{i}{}_{j}{}^{k}{}_{l}\,d^{l}{}_{k} + {}^{(1)(0)}_{2\,|\,1}K^{i}{}_{j}{}^{k}{}_{l}\,\omega^{l}{}_{k}\right],$$

while in the isotropic case the three tensors ${}^{(0)(1)}_{1\,|\,1}$, ${}^{(0)(1)}_{1\,|\,2}$ and ${}^{(1)(0)}_{2\,|\,1}$ necessarily reduce to zero. Thus thermal transpiration, for example, is an effect of first order in the viscosity for anisotropic fluids, but only of second order for isotropic fluids.

Notice also that in ${}^{(2)}t^i{}_j$ only second order velocity gradients occur, but in ${}^{(3)}t^i{}_j$ not only second but also both third and fourth order velocity gradients enter ([53]).

27. Isotropic fluids. — II. *The heat flux.* — In the isotropic case the fact that the heat flux q_i is a vector while the stress $t^i{}_j$ is a tensor of second order induces a striking difference between these two quantities, for it is precisely those terms in (22.2) which disappear

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([52]) This observation is illustrated by Korteweg's equations (18.10) and (18.11).

([53]) My earlier results [1948, 1], [1948, 2] concern a theory less general in that only second order velocity gradients are considered. Thus my present and former expressions for ${}^{(2)}t^i{}_j$ are the same, but my present ${}^{(3)}t^i{}_j$ is more general than the former one.

in the isotropic case whose counterparts for the heat flux survive, and conversely. We obtain an expression of the form

$$q_i = {}^{(0)}q_i + \mu_n {}^{(1)}q_i + \mu_n^2 {}^{(2)}q_i + \mu_n^3 {}^{(3)}q_i + \ldots, \tag{27.1}$$

where

$$\begin{aligned} {}^{(0)}q_i &= 0, \qquad {}^{(1)}q_i = \mathrm{R}_n \mathrm{P}_1 \theta_{,i} + \frac{\mathrm{R}_n \theta}{p} \mathrm{P}_2 p_{,i}, \\ {}^{(2)}q_i &= \frac{\mathrm{R}_n \theta}{p^2} [\mathrm{S}_1 p_{,j} d^j{}_i + \mathrm{S}_2 p_{,j} \omega^j{}_i + \mathrm{S}_3 p_{,i} d^k{}_k] \\ &\quad + \frac{\mathrm{R}_n}{p} [\mathrm{S}_4 \theta_{,j} d^j{}_i + \mathrm{S}_5 \theta_{,j} \omega^j{}_i + \mathrm{S}_6 \theta_{,i} d^k{}_k] \\ &\quad + \frac{\mathrm{R}_n \theta}{p} [\mathrm{U}_1 \dot{x}_i{}^j{}_{,j} + \mathrm{U}_2 d^k{}_{k,i}]. \end{aligned} \tag{27.2}$$

The expression for ${}^{(3)}q_i$ is long, so we shall not write it down, leaving its construction as an exercise for any reader who may wish to learn by experience the simplicity of our method. For conformity with classical notations, let us write $-\varkappa \equiv \mu_n \mathrm{R}_n \mathrm{P}_1$.

Some of the interesting new phenomena predicted by this expression will be discussed in paragraph 31.

28. Order of magnitude considerations. — In the foregoing analysis we have ordered the terms according to the power of the viscosity coefficient they contain. Since that coefficient is not dimensionless, no order of magnitude consideration in general can be founded upon such an expansion, although it serves to specify the asymptotic properties of $t^i{}_j$ as $\mu_n \to 0$. We have been guided rather by a formal extension of our results for the Stokesian fluid (§ 8), where in reality the terms are ordered in powers of the dimensionless number

$$\mathfrak{J} \equiv \frac{\mu_n d}{p}, \tag{28.1}$$

where d is a typical rate of deformation. For the Maxwellian fluid the analogous ordering is necessarily more complicated, being in fact a power series in many dimensionless variables, one of which is $\mathfrak{J}$, another of which is

$$\mathfrak{J}_0 \equiv \frac{\mu_n \varkappa_n \theta_{,i} \theta^{,i}}{p^2 \theta}, \tag{28.1}$$

where now $\theta_{,i}\theta^{,i}$ is the squared magnitude of a typical thermal gradient, etc. The number $\mathfrak{J}_0$ is one of the two numbers whose magnitude indicates the importance of thermal transpiration in isotropic fluids.

A full analysis of the order of magnitude of terms, and hence of the effects they represent, requires not only the introduction of such dimensionless characteristic numbers, but also a knowledge of the experimental values of the dimensionless higher order coefficients C_1, D_1, ..., knowledge not at present available.

Of all the many dimensionles similarity parameters arising from the higher order terms in the theory of the Maxwellian fluid, there is one and only one, the number $\mathfrak{J}$ defined by (8.1), which represents *the effect of viscosity alone*. All the others are of the type (28.1), being proportional to the product $\mu_u \varkappa_n$ and thus indicating the importance of the *interaction between the effects of viscosity and of thermal conduction*.

29. Remarks on the Clausius-Duhem inequality. — The Clausius-Duhem entropy inequality (A.18) requires that in any admissible motion the dissipated power $\Phi = \mathfrak{s}^i{}_j d^j{}_i = t^i{}_j d^j{}_i + p\, d^k{}_k$ shall be positive or zero. In the classical theory of viscous fluids, Φ is a quadratic form in $d^i{}_j$; conditions that this form shall be essentially positive yield the Duhem ([54])-Stokes ([55]) conditions on the first order viscosity coefficients

$$\mu \geqq 0, \qquad 3\lambda + 2\mu \geqq 0. \tag{29.1}$$

The counterpart of these relations for the Maxwellian fluid is not clear. In general, there are no ranges of values for the higher order viscosity coefficients such that $\Phi \geqq 0$ for *arbitrary* ([56]) values of its various arguments $d^i{}_j$, $p_{,i}$, etc., but such a condition would in any case be too strong, for the various arguments are not independent, being connected in a complicated way through the equations of

([54]) [1901, 1, part I, Chap. I, § 3].

([55]) (Note, p. 136-137, of the 1901 reprint of [1851, 1]).

([56]) The second order theory, for example, yields a cubic form for Φ, and a cubic form can never be positive definite.

motion. It is not reasonable to expect that the requirement $\Phi \geqq 0$ can be satisfied through the agency of inequalities upon the coefficients alone. In ordinary gas dynamics the differential equations admit solutions representing both condensation and rarefaction shocks, and the law of entropy increase is used to show that the latter are inadmissible. Likewise the general theory of fluids may well lead to several possible solutions for a given problem ([57]), and the condition $\Phi \geqq 0$ may then be used to determine which of them be admissible.

Similarly, in the classical theory the Clausius-Duhem inequality yields $\varkappa \geqq 0$, but its implications for the Maxwellian fluid are not plain.

30. Comparison with Burnett's stress equations. The effect of vorticity. — In order to compare our expression (26.7$_3$) for ${}^{(2)}t^i_j$ with the corresponding result (18.12) which Burnett deduced from the kinetic theory of gases, we first notice that the basic hypotheses of that theory as ordinarily presented ([58]) yield at once

$$p = \pi = p_m \quad \text{or} \quad t^i_i + 3p = 0, \tag{30.1}$$

$$p = R\rho\theta. \tag{30.2}$$

The kinetic theory is thus restricted to ideal gases whose mean pressure p_m always equals the thermodynamic pressure π.

The first consequence of (30.1) is that *all moduli* λ, μ, C_1, ... *become independent of pressure and reduce to functions of the temperature ratio* $\frac{\theta}{\theta_0}$ *only*. For the special case of the coefficient μ, this conclusion is one of the early consequences of the kinetic theory (*cf.* § 9). The second consequence of (30.1) is that the two first order and 19 second order coefficients of viscosity in our expressions (26.7) are reduced to eleven by the ten relations

$$\left\{\begin{array}{ll} 3\lambda + 2\mu = 0, & \\ 3D_i + 2C_i = 0 & (i = 1, 2, \ldots, 6), \\ 3F_i + 2E_i = 0 & (i = 1, 2, 3), \end{array}\right. \tag{30.3}$$

([57]) *Cf.* § 12.

([58]) E. G. [1939, 1, § 2.31, 2.32, 2.41, 2.42].

of which the first is the classical Stokes relation (*cf.* § 9). The foregoing remarks explain completely the reduction of first order coefficients from two to one, but in the Maxwellian fluid even after the reductions (30.3) are applied there remain ten second order coefficients, while in Burnett's equations there are but six. Thus still further special assumptions are concealed in the kinetic theory analysis. An immediate comparison of our results with Burnett's equations (13.12) is not possible, not only because of the awkward form in which these latter are expressed, but also since they contain the superfluous variable ρ, which must first be eliminated ([59]) by the equation of state (30.2). After this elimination is carried out, we find that we must write ([60])

$$(30.4)\qquad \left\{\begin{array}{ll} 2C_1 = -\varpi_2, & 2C_2 = \varpi_2 - \varpi_4, \\ 2C_3 = -\varpi_5, & 2C_4 = \varpi_2, \\ 2C_5 = -\varpi_2, & 2C_6 = -\varpi_3, \\ 2E_1 = -\varpi_1 - \frac{2}{3}\varpi_2 + \frac{2}{3}\varpi_6, & 2E_2 = 3\varpi_2 - \varpi_6, \\ 2E_3 = \varpi_2, & E_4 = -\varpi_2, \end{array}\right.$$

if we are to reduce our result to Burnett's. That is, in addition to the ten relations (30.3) we must impose also the following four

$$(30.5)\qquad C_1 = -C_4 = C_5 = -E_3 = \frac{1}{2}E_4,$$

whose physical meaning is not apparent, nor is there reason to believe them correct. Chapman and Cowling's claim ([61]) that Burnett's result contains " the only symmetrical and non-divergent tensors that can be formed from the elements involved " in the terms of one of their earlier expansions is simply false : in fact, the condition (30.5) implies that their result contains the five symmetrical non-divergent

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([59]) Conversely, to show that Korteweg's result (18.11) is included in the stresses for the Maxwellian fluid, put $\rho = f(\theta, p)$.

([60]) In this and the succeeding formula as presented in [1948, 1] and [1948, 2] there are errors in sign.

([61]) [1939, 1, § 15.3].

tensors ([62]) $\{p^{,i}{}_{,j}\}$, $\{p^{,i}p_{,j}\}$, $\{f^i{}_{,j}\}$, $\{\omega^i{}_k\omega^k{}_j\}$ and $\{d^i{}_k\omega^k{}_j\}$ only in the one combination

$$\frac{\{p^{,i}{}_{,j}\}}{\rho}+\frac{\{p^{,i}p_{,j}\}}{p}+\{f^i{}_{,j}\}-\{\omega^i{}_k\omega^k{}_j\}+2\{d^i{}_k\omega^k{}_j\}.$$

Consider now a state of pure rotation, in which the rate of deformation $d^i{}_j$ vanishes, the thermodynamic state is uniform, and the rotation $\omega^i{}_j$ represents a constant angular velocity ω about the z-axis. Then from (26.7_3) we have

$$(30.6)\qquad \begin{cases} t^x{}_x+p=t^y{}_y+p=-\dfrac{2\,\omega^2\mu_n^2}{p}(\mathrm{F}_3+\mathrm{E}_3), \\ t^z{}_z+p=-\dfrac{2\,\omega^2\mu_n^2}{p}\mathrm{F}_3, \end{cases}$$

or, when Burnett's relations (30.3) and (30.4) are employed,

$$(30.7)\qquad \begin{cases} t^x{}_x+p=t^y{}_y+p=-\dfrac{1}{3}\varpi_2\dfrac{\omega^2\mu_n{}^2}{p}, \\ t^z{}_z+p=\dfrac{2}{3}\varpi_2\dfrac{\omega^2\mu_n}{p}. \end{cases}$$

Thus, according to Burnett's result, a state of pure rotation induces stresses whose magnitude depends upon the viscosity of the substance. That is, if a cylindrical mass of fluid confined between two parallel plates be suddenly set into rigid rotation about an axis normal to the plates, the normal force upon the plates will depend upon the viscosity of the fluid. This result seems most improbable. While conceivably the rotation may affect the stress in the general case, it is hardly likely that rotation alone can produce an effect of viscosity.

In his review of a preliminary draft of this Memoir Professor Tsien ([62a]) wrote that " the author's doubt on Burnett's equations

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([62]) The notation is defined by (18.9).

([62a]) (*Math. Rev.*, t. 11, 1950, p. 623.) It may be worthwhile to take note of the second ill-taken criticism of Professor Tsien : " The equations of motion used by the author are Eckart's equations which are probably incorrect ". No equations of motion are *used* in this Memoir, which simply derives expressions

derived from the kinetic theory is unfounded and arises from the neglect of pressure gradient in a pure rotation ". Now I have never claimed that the stress (30.7) arise in a flow dynamically possible according to Burnett's equation. I mean simply to point out the nature of the *response* of the fluid-at bottom, the indisputable fact that if $p = \text{const.}$, $\rho = \text{const.}$, $d^i{}_j = 0$, but $\omega^i{}_j \neq 0$, Burnett's equation yield a stress proportional to $\mu_n{}^2\omega^2$, wich to me, at least, seems a very strange way for a fluid to react ([62b]).

In discussing results obtained by Enskog's method of integration of the Maxwell-Boltzmann equation, one must remember that the successive approximations are uniquely defined only in terms of a certain formal procedure which perhaps may not yield all terms involving μ_n^m at the mth stage. Thus the special relations (30.5) and some of the terms in ω_{ij} which appear in Burnett's equations may possibly be cancelled by terms from as yet undetermined higher approximations. These remarks are offered as mere suggestions; I have not repeated Burnett's calculations, which seem formidable. The anomaly noticed above is not inherent in the kinetic theory treatment, for Professor H. Grad assures me that results obtained by his method ([63]) of integrating the Maxwell-Boltzmann equation are free of it. While Boussinesq's equations (18.3) contain terms involving the vorticity, he expressly stated that the viscous stresses must vanish in a pure rotation. In M. Brillouin's equations (18.7) and (18.8) are vorticity terms, but, convinced from phenomenological considerations that the rate of deformation $d^i{}_j$ should be the only kinematic variable upon which $t^i{}_j$ dépends ([64]), Brillouin himself many times expressed his belief that for any conservative field of molecular forces it would be found eventually that the coefficients of

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for the stress and heat flux, expressions which can then be substituted into the equations of motion.

([62b]) So much so, indeed, that Professor Synge, misunderstanding my discussion of this fact in [1948, 2], called it a " rather startling and physically improbable consequence of [the author's] theory " (*Math. Rev.*, t. **10**, 1949, p. 214; see also the correction on p. 856).

([63]) *Loc. cit.* ([45]).

([64]) *Cf.* Stokes's principle, § 3.

all terms containing $\omega^i{}_j$ vanished ([65]). The whole subject of the derivation of hydrodynamical equations from the kinetic theory deserves to be re-evaluated, clarified, and presented from a more general point of view in the commonly accepted tensor notation.

In any case, in the present memoir our definition of the Maxwellian fluid was left sufficiently broad as to include the anomalous result (30.6) only for purposes of comparison with Burnett's equations. Following the example of Boussinesq (*cf.* § 18), we shall now complete our definition (§ 21) by strengthening the third part to read as follows :

If all the vectors and tensors in the list (21.2) *with the exception of* $\omega^i{}_j$ *vanish, then* $t^i{}_j$ *reduces to* $-p\,\delta^i{}_j$ *and* q_i *reduces to zero.*

For the anisotropic fluid it follows that

$$(30.8) \qquad {}^{(1)(0)}_{2\,|\,1} = {}^{(2)(0)}_{3\,|\,1} = {}^{(3)(0)}_{4\,|\,1} = \ldots = 0,$$

where the symbols refer to stress and heat flux coefficients alike. For the isotropic fluid we have

$$(30.9) \qquad E_3 = F_3 = N_1 = N_2 = \ldots = 0.$$

31. Comparison with Chapman and Cowling's heat flux equation. The Brillouin effect. — So much for the stress. Turning now to the heat flux, we find that our result (27.2) contains ten moduli, while the equations (18.14) derived from the kinetic theory by Chapman and Cowling contain but six. To force them into agreement we must put

$$(31.1) \qquad \begin{array}{l} \qquad\qquad P_2 = 0, \qquad S_2 = 0, \\ \left\{\begin{array}{lll} S_1 = \theta_3, & 3S_3 = -\theta_3, & S_4 = 2\theta_2 + \theta_5, \\ S_5 = 2\theta_2, & 3S_6 = 3\theta_1 + 2\theta_2 - \theta_5, & \\ U_1 = \theta_4, & 3U_2 = 2\theta_2 - \theta_4. & \end{array}\right. \end{array}$$

That is, if we are to reduce our results to Chapman and Cowling's,

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([65]) [1900, **1**, § 14 (footnote), **23** (end), **27** (end), **34** (end), **39** (end).]

we must impose the following four relations upon the ten coefficients :

$$S_1 + 3S_3 = 0, \qquad S_5 = 3U_2 + U_1, \tag{31.2}$$

$$P_2 = 0, \qquad S_2 = 0. \tag{31.3}$$

Now in the case of the stress, the special case derivable from the kinetic theory is sufficiently general that from it all the possible types of phenomena to be expected in the second approximation may be predicted. The specializations (30.3) and (30.5) simply express a certain linking of the magnitudes of the effects, a linking which is improbable but not impossible, and may even be correct for a monatomic gas in certain circumstances. The two conditions (31.2) imply two similar linkings for the heat flux. But the two conditions (31.3) are much more serious, for they indicate that *in the kinetic theory result, two phenomena possible in the Maxwellian fluid are not predicted at all.*

The first and more interesting of these phenomena is indicated by the term whose coefficient is P_2 : *at sufficiently low pressures and sufficiently high temperatures, heat flow results from a sufficiently large pressure gradient even in the absence of a thermal gradient.* This result is predicted by M. Brillouin's equation (18.8), since $\rho_{,i} = \frac{p_{,i}}{R\theta} - \frac{p\theta_{,i}}{R\theta^2}$, and I propose that it be called the *Brillouin effect.* The fact that in Brillouin's heat flux formula the term in question is but one of 25, not specially distinguished in any way, may explain its never having been discussed in the literature, but our present analysis shows that *the Brillouin effect is of first order in the viscosity coefficient,* to the best of my knowledge now remarked for the first time. Perhaps a neater form for our first order terms in the heat flux is given by the equivalent

$$q_i = -\varkappa\theta_{,i} + \frac{\mu}{\rho} P'_2 p_{,i} + \ldots \tag{31.4}$$

The magnitude of the coefficient P_2 should not be difficult to estimate experimentally. One might measure the rate at which energy has to be supplied in order to maintain the surface of a body at uniform high temperature when immersed in a high speed air flow at low pressure. Unless P_2 turns out to be very small, this effect may well

predominate over thermal conduction by Fourier's law in aerodynamical phenomena at high altitudes; in meteorological phenomena, however, it is probably nullified by the low temperatures of the upper atmosphere.

We should not omit to remark one example of the strange and arbitrary character of the kinetic theory result in that of all the six coefficients S_i only S_2 vanishes; that is, according to Chapman and Cowling's equations a pressure gradient or a thermal gradient in conjunction with deformation, expansion, or vorticity may give rise to second order thermal flux in quite analogous ways, except that of all the possible combinations it is only that of pressure gradient and vorticity which is insufficent.

32. Contrast of continuum and kinetic theory methods. — The foregoing comparison (§ 30-31) of the results of the two approaches to the problem of determining the stress and heat flux in a rarefied gas indicate that the kinetic theory becomes less and less adequate for the ordinary prediction of gross phenomena the higher the degree of approximation required. We may summarise the advantages of the two methods as follow :

Advantages of kinetic theory :

(*a*). In its broader features, the molecular model more closely resembles the accepted picture of physical matter. This advantage is seriously reduced, however, by the extreme idealizations which must be incorporated in the model if the calculations are actually to be carried out [*cf.* (A) and (B) below].

(*b*). The numerical values coefficients are predicted, while in a continuum theory these must be regarded as experimentally determined, even though their number may be so great as to put them beyond the reach of experimental methods. This argument is the strongest of all in favor of the kinetic theory. It can be answered only obliquely, in that the idealizations mentioned above are such as to render the kinetic theory results quantitatively inaccurate, though often still of very great qualitative usefulness.

(*c*). In principle, it should be possible to calculate boundary

conditions as well as equations of motion from the kinetic theory, and there have been several discussions and attempts in this direction ([66]), while to obtain boundary conditions in a continuum theory one must in effect postulate them. A satisfactory kinetic theory treatment has yet to be given, however.

(*d*). In a sufficiently general kinetic theory the model may be adjusted so as to yield different results for very rare gases ([67]), very dense gases ([68]), liquids, etc. [but *cf.* (A) below].

Advantages of continuum theory :

(A). It is not restricted to perfect gases, to compressible fluids, to homogeneous substances, or to isotropic fluids. The ordinary Newton-Cauchy-Poisson law (18.1), for example, is easily derived for all isotropic fluids by continuum methods. To obtain even a special case of this first approximation from the kinetic theory, however, it is customary to limit attention to a perfect moderately rarefied monatomic gas. The adjustments of the kinetic theory necessary to enable it to describe polyatomic gases, dense gases, or liquids are rather elaborate, and in some cases questionable. While indeed the values of the coefficients in the different cases differ from one another, the general form of the stress is the same, and it is not economical to employ so complicated a structure as the kinetic theory to obtain such gross results as the general form of stress and heat flux equations.

(B). Its results are free from the special relations among the coefficients which kinetic theory predicts but does not explain or justify. On the other hand, it gives no hint of the numerical values these coefficients assume [*cf.* (*b*) above].

(C). The terms of any given degree in μ_n may be written down easily with little labor, while even in the simplest kinetic theory for

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([66]) [1879, **1**, § **80**, **83**, **84**], [1924, **2**], [1929, **2**], [1932, **2**, Chap. IX-X], [1947, **2**, Chap. V]. The results given in the last cited paper were criticised by Professor Uhlenbeck at a lecture before the American Physical Society, New York, January, 1948, and have not received general acceptance.

([67]) [1930, **1**, § **4**], [1948, **7**].

([68]) [1922, **1**, p. 18-19], [1939, **1**, Chap. **16**].

monatomic perfect moderately rarefied gases the calculations incident upon use of Enskog's method of integrating the Maxwell-Boltzmann equation are elaborate, and increase enormously with the degree of approximation. With the continuum method, moreover, one may be sure of reaching at once all terms of a given degree in μ_n, while by Enskog's method of integration there appears to be some uncertainty in this regard.

(D). Continuum theory, being independent of any sort of deterministic mechanics of the ultimate particle, serves as a general guide with which any molecular model must be consistent.

The arguments on either side are strong, and neither approach should be neglected.

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