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PAR JOSEPH LIOUVILLE

J. L. WALSH

Polynomial expansions of functions defined by Cauchy's integral

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Polynomial expansions of functions defined by Cauchy's integral;

By J. L. WALSH.

In the plane of the complex variable z, let the analytic Jordan curve C contain the origin in its interior, and let the function $f(z)$ satisfy a Lipschitz condition on C. Then, as Plemelj has shown [1908; compare also Privaloff 1918], there exist functions $f_1(z)$ and $f_2(z)$ analytic respectively in the interior and exterior of C, continuous in the corresponding closed regions, $\overline{C}_1$ and $\overline{C}_2$, such that on C we have $f(z) \equiv f_1(z) - f_2(z)$ These functions are represented by the Cauchy integral of $f(z)$ over C. The function $f_1(z)$ can be expressed in $\overline{C}_1$ as a uniformly convergent series of polynomials in z, and the function $f_2(z)$ can be expressed in $\overline{C}_2$ as a uniformly convergent series of polynomials in $\frac{1}{z}$; consequently $f(z)$ can be expressed on C as the difference of these two series. It is the object of the present paper to study under various hypotheses on $f(z)$ the definition of these two series *directly in terms of* $f(z)$ rather than in terms of $f_1(z)$ and $f_2(z)$, and where the series are the classical ones of Faber and Szegö. Thus we study expansions of the *components* of a given function $f(z)$. A corresponding study has previously been made by the present writer [1924] for special series of polynomials artificially defined for the purpose.

To be more explicit, in Section **1** we consider the boundary values of the Cauchy integral over C of an arbitrary function $f(z)$ either of

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class L^2 (that is, $f(z)$ and $[f(z)]^2$ are Lebesgue integrable with respect to arc-length on C) or of class $L(k, \alpha)$, namely a function whose k^{th} derivative with respect to arc-length on C satisfies a Lipschitz condition of order α, $0 < \alpha < 1$. We include the study of a useful class of functions recently introduced by Zygmund. In Section **2** we expand these boundary values by the Faber polynomials and associated functions, and in Section **3** by the Szegö polynomials and associated functions; this treatment seems to be the first proof of the convergence on C of a development in Szegö's polynomials without assuming analyticity on C. In Section **4** we show that except when C is a circle there exists no weight function with respect to which the two classes of functions $f_1(z)$ and $f_2(z)$ are mutually orthogonal on C. In Section **5** we discuss briefly degree of convergence.

It is especially appropriate that this article should appear in a volume dedicated to M. Paul Montel, for his work [1910] on series of polynomials has been the immediate inspiration for the greater part of the subsequent work on the subject, his study of degree of convergence [1919] is classical, and his theory of normal families of functions is one of the most important tools in the study of expansion problems.

1. Boundary Values. — For the case of the unit circle $C : |z| = 1$, explicit formulas can be given for the functions $f_1(z)$ and $f_2(z)$; we use polar coordinates (r, θ).

Theorem **1**.**1**. — *Let the function* $U(\theta)$ *be of class* L^2 *on* C, *let* $V(\theta)$ *be the function conjugate to* $U(\theta)$ *on* C *with* $\int_C V(\theta)\,d\theta = 0$, *and let* $u(z)$ *and* $v(z)$ *respectively be the functions harmonic interior to* C *defined by Poisson's Integral, taking on the boundary values* $U(\theta)$ *and* $V(\theta)$ *almost everywhere on* C. *Then we have*

$$(1.1)\quad f_1(z) \equiv \frac{1}{2\pi i}\int_C \frac{U(\theta)\,dt}{t-z} \equiv \frac{1}{2}[u(z) + i v(z) + u(0)], \quad z \text{ interior to } C,$$

$$(1.2)\quad f_2(z) \equiv \frac{1}{2\pi i}\int_C \frac{U(\theta)\,dt}{t-z}$$

$$\equiv -\frac{1}{2}\left[u\left(\frac{1}{z}\right) - i v\left(\frac{1}{z}\right) - u(0)\right], \quad z \text{ exterior to } C.$$

In these formulas θ is a function of t, and in both (1.1) and (1,2) as in all subsequent formulas, all integrals over C are to be taken counterclockwise. It is natural here, although by no means necessary, to interpret $U(\theta)$ as real. The conjugate function $V(\theta)$ can be defined on C either by means of Fourier series or the Poisson integral, and is of class L^2 on C since $U(\theta)$ is of class L^2 on C.

We denote by H_2 the class of functions $\sum_0^\infty a_n z^n$ analytic interior to C with $\sum_0^\infty |a_n|^2$ convergent, and by G_2 the class of functions $\sum_1^\infty a_{-n} z^{-n}$ analytic exterior to C and vanishing at infinity with $\sum_1^\infty |a_{-n}|^2$ convergent. For an analytic function $F(z)$ to belong to H_2 or G_2 it is respectively necessary and sufficient that the integral $\int_0^{2\pi} |F(re^{i\theta})|^2 \, d\theta$ be uniformly bounded as r approaches unity, $r < 1$ or $r > 1$. It then follows [Fatou; see for instance Walsh 1935, Sec. 6.10] that boundary values for approach " in angle " exist almost everywhere on C, are of class L^2 there, and that Cauchy's integral formula is valid in terms of these boundary values.

The Taylor expansion of a function of class H_2 converges in the mean (of order two) on C to the boundary values. If a sequence of functions of class H_2 converges in the mean on C, the limit function is necessarily of class H_2, and the sequence converges to the limit function throughout the interior of C, uniformly on any closed set interior to C; compare Section 3, lemma 1 below.

The function $u(z) + iv(z)$ of theorem 1.1 is of class H_2, so we have

$$(1.3) \qquad u(z) + iv(z) \equiv \frac{1}{2\pi i} \int_C \frac{U(\theta) + iV(\theta)}{t - z}, \quad z \text{ interior to C},$$

$$(1.4) \qquad 0 \equiv \frac{1}{2\pi i} \int_C \frac{U(\theta) + iV(\theta)}{t - z}, \quad z \text{ exterior to C};$$

each of these formulas can be established by expressing $U(\theta) + iV(\theta)$ as the limit in the mean on C of its formal expansion in positive powers of z.

We can reflect the function $u(z)+iv(z)$ in the circle C; the function $u\left(\frac{1}{\bar{z}}\right)-iv\left(\frac{1}{\bar{z}}\right)$ is analytic exterior to C and takes on C the boundary values $U(\theta)-iV(\theta)$; the function $u\left(\frac{1}{\bar{z}}\right)-iv\left(\frac{1}{\bar{z}}\right)-u(0)$ is of class G_2, whence as in the proof of (1.3) and (1.4)

$$(1.5)\quad u\left(\frac{1}{\bar{z}}\right)-iv\left(\frac{1}{\bar{z}}\right)-u(0)\equiv\frac{-1}{2\pi i}\int_C\frac{U(\theta)-iV(\theta)}{t-z}\,dt,\quad z \text{ exterior to C},$$

$$(1.6)\quad u(0)\equiv\frac{1}{2\pi i}\int_C\frac{U(\theta)-iV(\theta)}{t-z}\,dt,\quad z \text{ interior to C}.$$

Equation (1.1) now follows from (1.3) and (1.6), and (1.2) follows from (1.4) and (1.5). The boundary values almost everywhere on C of $f_1(z)$ and $f_2(z)$ are given by

$$(1.7)\quad \begin{cases} f_1(z)\equiv\frac{1}{2}[U(\theta)+iV(\theta)+u(0)], \\ f_2(z)\equiv-\frac{1}{2}[U(\theta)-iV(\theta)-u(0)], \\ u(0)\equiv\frac{1}{2\pi i}\int_C U(\theta)\,d\theta. \end{cases}$$

We have also on C

$$(1.8)\quad \begin{cases} f_1(z)-f_2(z)\equiv U(\theta), \\ f_1(z)+f_2(z)\equiv iV(\theta)+u(0). \end{cases}$$

Following Zygmund [1945], we define $U(\theta)$ as of class Λ^* or $\Lambda^*(0)$ on C if, on C, $U(\theta)$ is continuous and we have

$$(1.9)\quad |U(\theta+h)+U(\theta-h)-2U(\theta)|\leq M|h|$$

where M is independent of θ and h. If $U^{(k)}(\theta)$ satisfies a condition on C similar to (1.9), we say that $U(\theta)$ is of class $\Lambda^*(k)$ on C. We have

THEOREM 1.2. — *If the function* $U(\theta)$ *is of class* $L(k, \alpha)$, $0<\alpha<1$, *on* C : $|z|=1$, *so also are* $f_1(z)$ *and* $f_2(z)$, *the latter functions being continuous in* $|z|\leq 1$ *and* $|z|\geq 1$ *respectively. If* $U(\theta)$ *is of class* $\Lambda^*(k)$ *on* C, *so also are* $f_1(z)$ *and* $f_2(z)$, *the latter functions being continuous in* $|z|\leq 1$ *and* $|z|\geq 1$ *respectively.*

It is a well known theorem due to Privaloff [1916] that if $U(\theta)$ is $L(0, \alpha)$, so also is $V(\theta)$; the first part of theorem **1**.2 then follows readily [compare Walsh, Sewell and Elliott, 1949; th. **3**.1]; indeed it follows [Sewell, 1942, p. 29] that $f_1^{(k)}(z)$ and $f_2^{(k)}(z)$ satisfy Lipschitz conditions of order α in the respective closed regions $|z| \leqq 1$ and $|z| \geqq 1$. Zygmund [1945] shows that If $U(\theta)$ is of class Λ^* on C, so also is $V(\theta)$, so the second part of theorem **1**.2 follows [compare Walsh and Elliott, 1950; theorem **2**.2]; here too the functions $f_1^{(k)}(z)$ and $f_2^{(k)}(z)$ satisfy conditions in z similar to (**1**.9) in the respective closed regions $|z| \leqq 1$ and $|z| \geqq 1$.

We proceed to extend theorem **1**.1 to an arbitrary analytic Jordan curve C. For such a curve, the class of functions H_2 is defined as the transform of the class H_2 under a conformal map of the interior of $\gamma : |w| = 1$ on to the interior of C. Let $F(z)$ be a function of class H_2 with respect to C. In the w-plane there exists a sequence of functions analytic in the closed interior of γ converging interior to γ to the function $F[\Phi(w)]$ and in the mean on γ to the boundary values of $F[\Phi(w)]$, where $w = \varphi(z)$, $z = \Phi(w)$ maps the interiors of C and γ on to each other. Thus in the z-plane there also exists a sequence of functions analytic in the closed interior of C converging in the mean on C to the boundary values of $F(z)$ and converging interior to C to the function $F(z)$. Consequently Cauchy's integral formula is valid

$$F(z) = \frac{1}{2\pi i}\int_C \frac{F(t)\,dt}{t-z}, \quad z \text{ interior to C.}$$

A similar proof and conclusion hold for the class G_2 for C, defined as the transform of the class G_2 under a conformal map of the exterior of γ on to the exterior of C with the points at infinity corresponding to each other. Let us prove

THEOREM **1**.3. — *Let* C *be an analytic Jordan curve, and let the function* $f(z)$ *be of class* L^2 *on* C. *Then we have*

$$(\mathbf{1}.10)\qquad \begin{cases} f_1(z) \equiv \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(t)\,dt}{t-z}, & z \text{ interior to C,} \\ f_2(z) \equiv \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(t)\,dt}{t-z}, & z \text{ exterior to C,} \end{cases}$$

where $f_1(z)$ *and* $f_2(z)$ *are respectively of classes* H_2 *and* G_2 *for* C; *the boundary values almost everywhere on* C *satisfy the equation*

$$f_1(z) - f_2(z) \equiv f(z). \tag{1.11}$$

In particular if $f(z)$ *is of class* $L(k, \alpha)$, $0 < \alpha < 1$ *or* $\Lambda^*(k)$ *on* C, *so also are* $f_1(z)$ *and* $f_2(z)$.

With the map $w = \varphi(z)$ already introduced, the function $f[\Phi(w)]$ is of class L^2 on γ, and we have by (1.8) almost everywhere on γ

$$f[\Phi(w)] \equiv \varphi_1(w) - \varphi_2(w),$$

where $\varphi_1(w)$ and $\varphi_2(w)$ are of classes H_2 and G_2 for γ. Under the map $w = \varphi(z)$, a certain closed annulus A in the z-plane, bounded by C and by an analytic Jordan curve C_1 containing C in its interior, is transformed one-to-one and conformally. If z is an interior point of A, we have

$$\varphi_2[\varphi(z)] \equiv -\varphi_3(z) + \varphi_4(z), \tag{1.12}$$

$$\begin{cases} \varphi_3(z) \equiv -\displaystyle\int_{C'_1} \frac{\varphi_2[\varphi(t)]\,dt}{t-z}, & z \text{ interior to } C'_1, \\ \varphi_4(z) \equiv -\displaystyle\int_{C'} \frac{\varphi_2[\varphi(t)]\,dt}{t-z}, & z \text{ exterior to } C', \end{cases} \tag{1.13}$$

where C'_1 and C' are analytic Jordan curves interior to A near C_1 and C respectively. These integrals may be taken over the curves C_1 and C themselves; for instance choose C'_1 as the image of a variable circle $|w_1| = r < 1$ under the conformal map of the interior of C_1 on to the region $|w_1| < 1$; then the first integral in (1.13) is a Stieltjes-Lebesgue integral with respect to the parameter $\arg w_1 = \theta$, an integral which is constant for fixed z as r varies and approaches unity, and whose integrand converges in the mean to the integrand for $r = 1$ (compare the w_1-plane); the integral approaches the corresponding integral taken over C_1. The integrals (1.13), now taken over the curves C_1 and C, define functions $\varphi_3(z)$ and $\varphi_4(z)$ analytic respectively throughout the interior of C_1 and throughout the exterior of C; on the curve C the function $\varphi_3(z)$ is analytic and for the boundary values $\varphi_2(w)$, $\varphi_3(z)$, $\varphi_4(z)$, equation (1.12) is valid. The function $\varphi_2(w)$ has boundary values almost everywhere on γ of class L^2, as has the function $\varphi_4[\Phi(w)]$, and $\int |\varphi_2(re^{i\theta})|^2 d\theta$ and $\int |\varphi_4[\Phi(re^{i\theta})]|^2 d\theta$ are uniformly bounded ($r > 1$), so $\varphi_4(z)$ is of class G_2 for C; the equation $\varphi_4(\infty) = 0$ follows from (1.13).

We now introduce the definitions

$$f_1(z) \equiv \varphi_1[\varphi(z)] + \varphi_3(z), \quad z \text{ interior to C},$$
$$f_2(z) \equiv \varphi_4(z), \qquad z \text{ exterior to C},$$

from which (1.11) follows. The functions $f_1(z)$ and $f_2(z)$ are respectively of classes H_2 and G_2 for C, so we have (1.10).

The remainder of theorem **1**.3 is an immediate consequence of theorem **1**.2; compare Walsh and Elliott [1950], theorem **3**.4. Compare also Davydov [1949].

2. Faber polynomials. — If C is an analytic Jordan curve of the z-plane, Faber [1903] maps the exterior of C onto the exterior of γ: $|w| = 1$ by the transformation $z = \psi(w)$, $w = \Psi(z)$, with $\psi(\infty) = \infty$, and studies the kernel of Cauchy's integral

$$(2.1) \qquad \frac{dt}{t - z} \equiv \frac{\psi'(w)\, dw}{\psi(w) - z} \equiv \sum_0^\infty F_n(z)\, w^{-n-1}\, dw,$$

where $F_n(z)$ is a polynomial in z of degree n, and the development is valid for z interior to C and w on γ. Indeed, if we set $z = \psi(w_0)$, we have

$$(2.2) \qquad \psi(w) = \psi(w_0) + \psi'(w_0)(w - w_0) + \frac{\psi''(w_0)(w - w_0)^2}{2!} + \ldots,$$

$$(2.3) \qquad \Theta(w) \equiv \frac{\psi'(w)}{\psi(w) - \psi(w_0)} - \frac{1}{w - w_0} \equiv \frac{-\dfrac{\psi''(w_0)}{2!} - \dfrac{\psi'''(w_0)(w - w_0)}{3!} - \ldots}{\psi'(w_0) + \dfrac{\psi''(w_0)(w - w_0)}{2!} + \ldots}.$$

The last member of (2.3) is uniformly bounded for all w_0 on γ and for all w on γ, and even for all $|w_0| \geq r < 1$ and for all $|w| \geq r$, provided $\psi(w)$ is analytic with $\psi'(w) \neq 0$ for $|w| \geq r$. The coefficients of the expansion

$$\Theta(w) \equiv \frac{A_1}{w} + \frac{A_2}{w^2} + \ldots$$

are uniformly bounded for $|w_0| \geqq r$, $A_n = O(r^n)$. We have also

$$\frac{1}{w - w_0} = \frac{1}{w}\left(1 + \frac{w_0}{w} + \frac{w_0^2}{w^2} + \ldots\right), \qquad |w_0| < |w|,$$

whose coefficients are $O(r^n)$ if $|w_0| \leqq r$. Thus it follows from (2.1) that the coefficients $F_n(z)$ are uniformly $O(r^n)$ if z lies on C_r: $|\psi(w)| = r < 1$.

We are now in a position to prove

THEOREM **2.1**. — *Let* C *be an analytic Jordan curve of the z-plane, and let* $f(z)$ *be* L^2 *on* C. *Then for the functions of theorem* **1**.3 *we have*

$$(2.4) \qquad f_1(z) \equiv \sum_0^\infty a_n F_n(z), \qquad a_n = \frac{1}{2\pi i}\int_\gamma \frac{f[\psi(w)]\,dw}{w^{n+1}},$$

where the expansion is valid uniformly on any closed set interior to C, *and*

$$(2.5) \quad f_2(z) \equiv \sum_1^\infty a_{-n}\Psi'(z)\Psi^{-n}(z), \qquad a_{-n} = \frac{-1}{2\pi i}\int_C f(z)F_{n-1}(z)\,dz,$$

where the expansion is valid uniformly on any closed set exterior to C.

Equation (2.1) is valid uniformly for z on C_r and w on γ, so (2.4) follows from the first of equations (1.10) by integration. Equation (2.1) is likewise valid uniformly for z on C and $|w| \geqq R > 1$, and can be written in the form

$$\frac{dz}{t - z} \equiv \sum_0^\infty \Psi'(t)[\Psi(t)]^{-n-1} F_n(z)\,dz,$$

so (2.5) follows from the second of equations (1.10).

Equations (2.4) and (2.5) are expansions of $f_1(z)$ and $f_2(z)$ directly in terms of $f(z)$ without the explicit intervention of Cauchy's integral formula.

Equation (2.5) has the disadvantage of being an expansion in terms of the powers of the mapping function $\Psi(z)$ instead of polynomials, a disadvantage that we now overcome by mapping the interior of C onto the interior of γ, with the mapping function

$w=\varphi(z)$, $z=\Phi(w)$; we suppose $z=0$ to lie interior to C, and assume $\varphi(0)=0$. A procedure entirely analogous to the one just used defines an expansion

$$\frac{dt}{t-z}=\frac{\Phi'(w)\,dw}{\Phi(w)-z}\sum_{1}^{\infty}F_{-n}\left(\frac{1}{z}\right)w^{n-1}\,dw, \tag{2.6}$$

where $F_{-n}\left(\frac{1}{z}\right)$ is a polynomial in $\frac{1}{z}$ of degree n without constant term, and where the convergence properties of (2.6) are similar to those of (2.1). From the second of equations (1.10) we have at once the second part of

THEOREM 2.2. — *Let* C *be an analytic Jordan curve of the z-plane containing the origin in its interior, and let $f(z)$ be of class* L^2 *on* C. *Then in the notation already introduced we have*

$$f_1(z)\equiv\sum_{0}^{\infty}a_nF_n(z),\qquad a_n=\frac{1}{2\pi i}\int_C\frac{f(z)\Psi'(z)\,dz}{\Psi^{n+1}(z)}, \tag{2.7}$$

where the expansion is valid uniformly on any closed set interior to C, *and*

$$f_2(z)\equiv\sum_{1}^{\infty}c_nF_{-n}\left(\frac{1}{z}\right),\qquad c_n=\frac{1}{2\pi i}\int_C f(z)\varphi'(z)\varphi^{n-1}(z)\,dz, \tag{2.8}$$

where the expansion is valid uniformly on any closed set exterior to C.

The function $f_2(z)$ vanishes at infinity, as does each of the polynomials $F_{-n}\left(\frac{1}{z}\right)$. Both (2.7) and (2.8) are expansions expressed directly in terms of $f(z)$ without explicit use of Cauchy's integral. We emphasize the fact that the expansion (2.8) is derived from the Cauchy integral itself, and is not merely a transformation of the expansion (2.7); such a transformation would have the disadvantage for our present purpose of yielding polynomials in $\frac{1}{z}$ of degrees 0, 1, 2, ... *with constant terms not necessarily zero*, instead of polynomials in $\frac{1}{z}$ of degrees 1, 2, ... with constant terms zero; the formal expansion of a non-zero constant (a function of class H_2) in terms of the transformed polynomials would not vanish identically.

From known properties of the Faber polynomials [Faber, 1903; Sewell, 1942] and from theorem **1**.3 we have the first part of the

COROLLARY. — *If the function* $f_1(z)$ *is analytic throughout the interior of the Jordan curve* $C_R : |\Psi(z)| = R > 1$, *then the expansions* (2.4) *and* (2.7) *are also valid throughout the interior of* C_R, *uniformly on any closed set interior to* C_R. *If the function* $f_2(z)$ *is analytic throughout the exterior of the Jordan curve* $C_r : |\varphi(z)| = r < 1$, *then the expansions* (2.5) *and* (2.8) *are also valid throughout the exterior of* C_r, *uniformly on any closed set exterior to* C_r.

If the function $f(z)$ *is of class* $L(0, \alpha)$, *on* C, $0 < \alpha < 1$, *the expansions* (2.4), (2.5), (2.7) *and* (2.8) *are also valid uniformly on* C.

To prove the second part of this corollary we use theorem **1**.3. Only equation (2.5) requires supplementary discussion; interpreted in the plane of $w = \Psi(z)$, this expansion is a Laurent development of a function analytic $|w| > 1$, known to be valid for $|w| > 1$; the function developed is of class $L(0, \alpha)$ on γ, so the development is also valid uniformly on γ.

If C is itself the unit circle, theorems **2**.1 and **2**.2 are identical with each other and with classical theorems.

3. SZEGÖ'S POLYNOMIALS. — In our study of orthogonal polynomials, it will be convenient to prove several preliminary propositions,

LEMMA **1**. — *Let* C *be an analytic Jordan curve, and let* $\varphi_0(z)$ *be of class* H_2 *for* C. *If the sequence of functions* $\varphi_n(z)$ *each of class* H_2 *converges in the mean to* $\varphi_0(z)$ *on* C, *then the sequence* $\varphi_n(z)$ *converges to* $\varphi_0(z)$ *throughout the interior of* C, *uniformly on any closed point set interior to* C.

Cauchy's integral formula is valid for C and each of the functions $\varphi_0(z)$ and $\varphi_n(z)$; the conclusion follows from Schwarz's inequality.

If we assume here that $\varphi_0(z)$ and $\varphi_n(z)$ are of class L^2 on C but not necessarily of class H_2, it remains true that the function $\varphi_{01}(z)$ defined as the limit interior to C of the functions $\varphi_{n1}(z)$ is also of class H_2

$$\varphi_{01}(z) \equiv \frac{1}{2\pi i}\int_C \frac{\varphi_0(t)\,dt}{t-z}, \qquad \varphi_{n1}(z) \equiv \frac{1}{2\pi i}\int_C \frac{\varphi_n(t)\,dt}{t-z}, \qquad z \text{ interior to C.}$$

For the case that C is the unit circle, this conclusion follows from the equation $\varphi_0(z) \equiv \varphi_{01}(z) - \varphi_{02}(z)$ on C, where $\varphi_{01}(z)$ and $\varphi_{02}(z)$ may be found from the formal Laurent expansion of $\varphi_0(z)$ on C and are of classes H_2 and G_2 respectively; for the case that C is a more general analytic Jordan curve, this conclusion follows from theorem 1.3.

LEMMA 2. — *Let* C *be an analytic Jordan curve, and let* $F(z)$ *be of class* H_2 *for* C. *Let* $\sum_0^\infty a_n p_n(z)$ *be the formal expansion of* $F(z)$ *on* C *in terms of the polynomials* $p_n(z)$ *normal and orthogonal on* C *with respect to the weight function* $n(z)$ *positive and continuous on* C. *Then this formal development converges to* $F(z)$ *in the mean on* C, *and converges to* $F(z)$ *throughout the interior of* C, *uniformly on any closed set interior to* C.

If the function $w = \varphi(z)$, $z = \Phi(w)$ maps the interior of C onto the interior of $\gamma : |w| = 1$, the function $F[\Phi(w)]$ can be approximated in the mean on γ as closely as desired by a polynomial in $w = \varphi(z)$. Any polynomial in $w = \varphi(z)$ can be uniformly approximated on C as closely as desired by a polynomial in z [Walsh, 1935, p. 36], so on C the function $F(z)$ can be approximated in the mean as closely as desired by a polynomial in z, whether approximation is measured with or without the weight function. Hence the measure of approximation to $F(z)$ on C by the polynomial of degree n *of best approximation* in the sense of least squares, namely by the sum of the first $n+1$ terms of the formal expansion of $F(z)$, approaches zero as n becomes infinite; the remainder of lemma 2 follows from lemma 1.

LEMMA 3. — *Let* C *be an analytic Jordan curve, and let the norm function* $n(z)$ *be positive and analytic on* C. *Let the function* $F(z)$ *be continuous in the closed interior of* C *and of class* $L(0, \alpha)$ *on* C, $0 < \alpha < 1$. *Then the formal expansion of* $F(z)$ *in the polynomials* $p_n(z)$ *normal and orthogonal on* C *with respect to* $n(z)$, *converges uniformly to* $F(z)$ *in the closed interior of* C.

Szegö has shown [1939, p. 365] that under the conditions of lemma 3 we have the formula

$$(3.1)\qquad p_n(z) \equiv \frac{1}{(2\pi)^{\frac{1}{2}}} [\Psi'(z)]^{\frac{1}{2}} [\Psi(z)]^n [\Delta(z)]^{-1} + O(h^n),$$

uniformly in a neighborhood of C, where $0 < h < 1$ and h is independent of z, where $\Delta(z)$ is a function analytic and different from zero in the closed exterior of C, with $\Delta(\infty) > 0$, where $|\Delta(z)|^2$ is equal to $n(z)$ on C, and where $w = \Psi(z)$ is the mapping function of Section **2**. We write (3.1) in the form

$$(3.2)\qquad p_n(z) \equiv N(z)\, w^n + O(h^n),$$

where we have $w = \Psi(z)$. The formal expansion of $F(z)$ on C in the polynomials $p_n(z)$ can then be written

$$(3.3)\quad F(z) \sim \sum_0^\infty p_n(z) \int_C F(z_1)\, \overline{p}_n(z_1)\, |dz_1|$$

$$\equiv \sum_0^\infty [N(z)\, w^n + O_z(h^n)] \int_C F(z_1) [\overline{N}(z_1)\, \overline{w}_1^n + O_{z_1}(h^n)]\, |dz_1|.$$

The subscripts z and z_1 serve merely to indicate functional dependence. We express the last member of (3.3) as the sum of four infinite series, which we study in order. With $w_1 = \Psi(z_1)$, the coefficient

$$\int_C F(z_1)\, \overline{N}(z_1)\, \overline{w}_1^n\, |dz_1| = \int_\gamma \frac{F(z_1)\, \overline{N}(z_1)}{|\Psi'(z_1)|} \frac{|dw_1|}{w_1^n} = \int_\gamma \frac{F(z_1)\, \overline{N}(z_1)}{|\Psi'(z_1)|} \frac{dw_1}{i w_1^{n+1}},$$

where γ is $|w_1| = 1$, is except for the factor 2π the coefficient in the formal expansion of the function $\frac{F(z_1)\overline{N}(z_1)}{|\Psi'(z_1)|}$ in powers of w. It is not necessary for the present purpose to relate this latter function to the boundary values of an analytic function; it is sufficient merely that this function is of class $L(0, \alpha)$ on γ. It then follows from the corollary to theorem **2**.2 applied where C (notation of theorem **2**.2) is identical with γ, that the series

$$\sum_0^\infty w^n \int_C F(z_1)\, \overline{N}(z_1)\, \overline{w}_1^n\, |dz_1|$$

and hence also the series

$$\sum_{0}^{\infty} \mathrm{N}(z)\, w^n \int_{\mathrm{C}} \mathrm{F}(z_1)\, \overline{\mathrm{N}}(z_1)\, \overline{w}_1^n \,|\, dz_1 \,|$$

converges uniformly on C. It is obvious that the three series

$$\sum_{0}^{\infty} \mathrm{O}_z(h^n) \int_{\mathrm{C}} \mathrm{F}(z_1)\, \overline{\mathrm{N}}(z_1)\, \overline{w}_1^n \,|\, dz_1 \,|,$$

$$\sum_{0}^{\infty} \mathrm{N}(z)\, w^n \int_{\mathrm{C}} \mathrm{F}(z_1)\; \mathrm{O}_{z_1}(h^n) \,|\, dz_1 \,|,$$

$$\sum_{0}^{\infty} \mathrm{O}_z(h^n) \int_{\mathrm{C}} \mathrm{F}(z_1)\, \mathrm{O}_{z_1}(h^n) \,|\, dz_1 \,|,$$

also converge uniformly on C, so the series (3.3) converges uniformly on C and therefore converges uniformly in the closed interior of C. Interior to C the sum of (3.3) is $\mathrm{F}(z)$, by lemma 2. Lemma 3 follows.

We are now in a position to establish the analogue of theorem **2.1** :

THEOREM **3.1**. — *Let the norm function $n(z)$ be positive and of class* $\mathrm{L}(0, \alpha)$, $0 < \alpha < 1$, *on the analytic Jordan curve* C, *and let $f(z)$ be of class* L^2 *on* C. *We introduce the notation*

$$(3.4) \qquad q_{k1}(z) \equiv \frac{1}{2\pi i} \int_{\mathrm{C}} \frac{n(t)\, \overline{p}_k(t)}{t - z} \frac{|\, dt \,|}{dt}\, dt, \quad z \text{ interior to C},$$

$$(3.5) \qquad q_{k2}(z) \equiv \frac{1}{2\pi i} \int_{\mathrm{C}} \frac{n(t)\, \overline{p}_k(t)}{t - z} \frac{|\, dt \,|}{dt}\, dt, \quad z \text{ exterior to C},$$

and also use the notation $q_{k1}(z)$ and $q_{k2}(z)$ for the boundary values on C *of these functions; by theorem* **1**.3 *the functions $q_{k1}(z)$ and $q_{k2}(z)$ are of class* $\mathrm{L}(0, \alpha)$ *on* C.

If $f(z)$ is of class L^2 *on* C, *then we have (notation of theorem* **1**.3)

$$(3.6) \qquad f_1(z) \equiv -\sum_{0}^{\infty} p_k(z) \int_{\mathrm{C}} f(t)\, q_{k2}(t)\, dt,$$

uniformly on any closed set interior to C,

$$(3.7)\qquad f_2(z) \equiv \sum_0^\infty q_{k2}(z) \int_C f(t) p_k(t)\, dt,$$

uniformly on any closed set exterior to C.

With the notation (3.5), we have [Walsh, 1935, p. 137]

$$(3.8)\qquad \frac{-1}{t-z} \equiv 2\pi i \sum_0^\infty p_k(z) q_{k2}(t),$$

uniformly for z on C and for t on any closed set exterior to C. Equation (3.7) follows at once.

By theorem 1.3 we have on C

$$(3.9)\qquad q_{k1}(z) - q_{k2}(z) \equiv \frac{n(z)\bar{p}_k(z)\,|dz|}{dz},$$

and by lemma 2 we have

$$(3.10)\qquad f_1(z) \equiv \sum_0^\infty a_k p_k(z), \qquad a_k = \int_C n(z) f_1(z) \bar{p}_k(z)\,|dz|$$

uniformly on any closed set interior to C. But we have [compare Walsh, 1935, p. 145]

$$\int_C f_1(z) q_{k1}(z)\, dz = 0, \qquad \int_C f_2(z) q_{k2}(z)\, dz = 0,$$

whence from (3.10) and (3.9)

$$a_k = -\int_C f_1(z) q_{k2}(z)\, dz = -\int_C f(z) q_{k2}(z)\, dz,$$

and (3.6) is a consequence of (3.10). There follows incidentally

Corollary 1. — *If in theorem* 3.1 *the only requirement on* $n(z)$ *is that of being positive and measurable, with* $n(z)$ *and* $\frac{1}{n(z)}$ *bounded, then* (3.6) *and* (3.7) *persist.*

To prove corollary 1, we note (*loc. cit.*) that the second member of (3.8) is dominated by a geometric series of constant terms, uniformly for z on C and for t on any closed set exterior to C, so (3.7) follows. The identity of (3.6) and (3.10) follows as before from the fact [compare Walsh, 1935, p. 145] that the integral over C is zero of the product of any two functions of class H_2 or of class G_2.

COROLLARY 2. — *If in theorem* 3.1 *the function* $n(z)$ *is analytic on* C, *and if either* $f(z)$ *or* $f_1(z)$ *is assumed of class* $L(0, \alpha)$ *on* C, *then* (3.6) *is valid uniformly in the closed interior of* C.

Corollary 2 follows from lemma 2.

We have not shown (3.7) to be valid uniformly in the closed exterior of C. To obtain an expansion of $f_2(z)$ valid in that closed exterior, we introduce the polynomials $P_n\left(\frac{1}{z}\right)$ in $\frac{1}{z}$ of respective degrees n orthogonal on C with respect to the norm function $n(z)$. We prove

THEOREM 3.2. — *Let the norm function* $n(z)$ *be positive and of class* $L(0, \alpha)$, $0 < \alpha < 1$, *on the analytic Jordan curve* C, *let the origin be interior to* C, *and let* $f(z)$ *be of class* L^2 *on* C. *We introduce the notation*

$$(3.11) \qquad P_{k1}(z) \equiv \frac{1}{2\pi i}\int_C \frac{n(t)\bar{P}_k\left(\frac{1}{t}\right)}{t-z}\frac{|dt|}{dt}\,dt, \quad z \text{ interior to C},$$

$$(3.12) \qquad P_{k2}(z) \equiv \frac{1}{2\pi i}\int_C \frac{n(t)\bar{P}_k\left(\frac{1}{t}\right)}{t-z}\frac{|dt|}{dt}\,dt, \quad z \text{ exterior to C},$$

and also use the notation $P_{k1}(z)$ *and* $P_{k2}(z)$ *for the boundary values on* C *of these functions, boundary values of class* $L(0, \alpha)$ *on* C.

If $f(z)$ *is of class* L^2 *on* C, *then we have (notation of theorem* 1.3)

$$(3.13) \qquad f_1(z) \equiv \sum_0^\infty P_{k1}(z)\int_C f(t)\,P_k\left(\frac{1}{t}\right)dt,$$

uniformly on any closed set interior to C,

$$(3.14) \qquad f_2(z) \equiv -\sum_0^\infty P_k\left(\frac{1}{z}\right)\int_C f(t)\,P_{k1}(t)\,dt,$$

uniformly on any closed set exterior to C.

The expansion properties of the polynomials $P_n\left(\frac{1}{z}\right)$ are readily deduced from the classical results by the substitution $w = \frac{1}{z}$. Thus we have [Walsh, 1935, p. 137]

$$(3.15) \qquad \frac{1}{t-z} = -2\pi i \sum_{0}^{\infty} P_k\left(\frac{1}{z}\right) P_{k1}(t),$$

uniformly for z on C and for t on any closed set interior to C. Equation (3.13) follows.

By theorem 1.3 we have on C

$$P_{k1}(z) - P_{k2}(z) \equiv \frac{n(z)\,\bar{P}_k\left(\frac{1}{z}\right) |dz|}{dz},$$

and by lemma 2 we have

$$(3.16) \qquad f_2(z) \equiv \sum_{0}^{\infty} b_k P_k\left(\frac{1}{z}\right), \qquad b_k = \int_C n(z) f_2(z)\,\bar{P}_k\left(\frac{1}{z}\right) |dz|,$$

uniformly on any closed set exterior to C. We have also

$$\int_C f_1(z)\,P_{k1}(z)\,dz = 0, \qquad \int_C f_2(z)\,P_{k2}(z)\,dz = 0,$$

whence we have

$$b_k = -\int_C f(z)\,P_{k1}(z)\,dz,$$

so (3.16) is essentialy (3.14).

Although $f_2(z)$ vanishes at infinity, the individual terms of the second member of (3.14) do not necessarily vanish at infinity. Here three remarks are appropriate. First, let $Q_k\left(\frac{1}{z}\right)$ denote the polynomials in $\frac{1}{z}$ of respective degrees k obtained by orthogonalizing and normalizing on C the functions $z^{-1}, z^{-2}, \ldots$; it is still true [compare Walsh, 1935, p. 310] that the formal expansion of $f_2(z)$ on C in terms of the $Q_k\left(\frac{1}{z}\right)$ converges in the mean to $f_2(z)$ on C. If we set

$$Q_{k1}(z) \equiv \frac{1}{2\pi i} \int_C \frac{n(t)\,\bar{Q}_k\left(\frac{1}{t}\right)}{t-z} \frac{|dt|}{dt}\,dt, \quad z \text{ interior to C},$$

and use the notation $Q_{k1}(z)$ also for the boundary values on C of this function, the method of proof of (3.14) establishes

$$(3.17) \qquad f_2(z) \equiv -\sum_1^\infty Q_k\left(\frac{1}{z}\right) \int_C f(t)\, Q_{k1}(t)\, dt,$$

uniformly on any closed set exterior to C; each term of the second member of (3.17) vanishes at infinity. Second, the function $z f_2(z)$ is analytic exterior to C, and we have the expansion

$$(3.18) \qquad z f_2(z) \equiv \sum_0^\infty P_k\left(\frac{1}{z}\right) \int_C n(z)\, z f_2(z)\, \bar{P}_k\left(\frac{1}{z}\right) |dz|,$$

uniformly on any closed set exterior to C. The coefficient in (3.18) cannot necessarily be written as $\int_C z f(z) P_{k1}(z)\, dz$, however, for the equation $\int_C z f_2(z) P_{k2}(z)\, dz = 0$ may not be valid.

Third, suppose $n(z)$ continuous on C and so chosen that the function unity is orthogonal with respect to $n(z)$ on C to each of the functions $\frac{1}{z}, \frac{1}{z^2}, \cdots$:

$$(3.19) \qquad \int_C \frac{n(z)\,|dz|}{z^n} = 0 \qquad (n = 1, 2, \ldots).$$

For this it is necessary and sufficient [Walsh, 1935, p. 41] that there exist a fuction analytic exterior to C, continuous in the closed exterior of C, zero at infinity, with the boundary values $\frac{n(z)\,|dz|}{dz}$ on C. A consequence of (3.19) is that the polynomials $P_n\left(\frac{1}{z}\right)$ obtained by orthogonalizing and normalizing with respect to $n(z)$ on C the functions $1, \frac{1}{z}, \frac{1}{z^2}, \cdots$, satisfy the equation $P_n(0) = 0$ for $n > 0$. In the notation previously introduced we have $P_n\left(\frac{1}{z}\right) \equiv Q_n\left(\frac{1}{z}\right)$ for $n > 0$, and (3.17) holds. Moreover in the expansion (3.16) known to be valid for $z = \infty$, we set $z = \infty$ and deduce $b_0 = 0$, the proof of (3.16) is valid, and establishes (3.14), now in the form

$$(3.20) \qquad f_2(z) \equiv -\sum_1^\infty P_k\left(\frac{1}{z}\right) \int_C f(t)\, P_{k1}(t)\, dt,$$

uniformly on any closed set exterior to C. It will be noticed that like $f_2(z)$, each term of the second member of (3.20) vanishes at infinity.

Equation (3.19) is both necessary and sufficient that we have $P_n(0)=0$ for $n>0$, and (3.19) itself is not difficult to study. As before we set $z=\psi(w)$, $dz=\psi'(w)\,dw$, whence on $\gamma:|w|=1$ we have $dw=iw\,|dw|$. Thus (3.19) becomes

$$\int_\gamma \frac{n(z)\,|\psi'(w)|\,.\,|dw|}{[\psi(w)]^n}=0 \qquad (n=1,2,\ldots).$$

On γ any function of the set $\frac{1}{w}, \frac{1}{w^2}, \cdots$ can be uniformly approximated by a linear combination of functions of the set $[\psi(w)]^{-1}, [\psi(w)]^{-2}, \ldots,$ and conversely, so (3.19) is equivalent to

$$\int_\gamma \frac{n(z)\,|\psi'(w)|\,.\,|dw|}{w^n}=0 \qquad (n=1,2,\ldots), \tag{3.21}$$

and by taking conjugates we have also

$$\int_\gamma n(z)\,|\psi'(w)|\,w^n\,|dw|=0 \qquad (n=1,2,\ldots). \tag{3.22}$$

It follows from (3.21) and (3.22) that all the coefficients on γ of the function $n(z)\,|\psi'(w)|$ vanish, for the expansion in terms of the complete set of normal orthogonal functions $e^{in\theta}$ $(n=\ldots,-2,-1,0,1,2,\ldots)$, except the coefficient corresponding to $n=0$. Consequently the formal expansion on γ of that function reduces to a constant, so we have $n(z)\,|\psi'(w)|\equiv\text{const.}$ Except for a multiplicative constant we have $n(z)\equiv\frac{1}{|\psi'(w)|}$, an equation which thus is essentially necessary and sufficient for (3.19) and (3.20); the function $n(z)$ is uniquely determined except for a multiplicative constant ([1]).

To theorem 3.2 we add the

([1]) This method establishes also the following result :

Let C be an analytic Jordan curve containing the origin in its interior, and let the polynomials $p_n(z)$ $(n=0,1,2,\ldots)$, of respective degrees n, be normal and orthogonal on C with respect to the positive continuous norm

COROLLARY. — *Under the conditions of theorems* **3.1** *and* **3.2**, *if* $n(z)$ *is positive and analytic on* C, *and if* $f(z)$ *is of class* $L(0, \alpha)$ *on* C, *then* (3.6) *and* (3.14) *are valid respectively in the closed interior and the closed exterior of* C.

This corollary follows from theorem **1**.3 and lemma 3. Under the hypothesis (3.19), équation (3.14) can be replaced here by (3.20). Although we have used the same notation, the norm function $n(z)$ need not be the same in theorems **3.1** and **3.2** nor in equations (3.6) and (3.14).

4. ORTHOGONALITY OF INNER AND OUTER FUNCTIONS. — Each of the theorems **2.1**, **2.2**, **3.1**, **3.2** furnishes a sieve for the separation of a given function $f(z)$ of class L^2 on C into its two components $f_1(z)$ and $-f_2(z)$ of respective classes H_2 and G_2; the sieve involves integration over C with respect to dz. The question naturally arises as to whether there exists a similar sieve involving integration over C with respect to $|dz|$, corresponding to orthogonalization with a norm function on C. The answer here is negative unless C is essentially the unit circle :

THEOREM **4.1**. — *Let* C *be an analytic Jordan curve containing the origin in its interior, and let there exist a positive norm function* $n(z)$ *continuous on* C *such that we have*

$$(4.1)\qquad \int_C \frac{n(z)\, z^n\, |dz|}{\bar{z}^k} = 0 \qquad (n = 0, 1, 2, \ldots;\ k = 1, 2, \ldots).$$

Then C *is a circle whose center is* $z = 0$ *and* $n(z)$ *is identically constant on* C.

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function $n(z)$. *Then a necessary and sufficient condition for the equation* $p_n(0) = 0\ (n = 1, 2, \ldots)$, *is that* $n(z)$ *be a constant multiple of the function* $|\varphi'(z)|$, *where* $w = \varphi(z)$ *with* $\varphi(0) = 0$ *maps the interior of* C *onto the interior of* $\gamma : |w| = 1$.

Of course the mapping function $w = \varphi(z)$, $z = \Phi(w)$ is not uniquely determined by the condition $\varphi(0) = 0$, but any two functions $\varphi(z)$ differ only by a multiplicative constant of modulus unity, and $|\varphi'(z)|$ is uniquely determined.

We treat first the case that C is the unit circle $|z| = 1$. On C we have $\bar{z} = \frac{1}{z}$, so (4.1) can be written ($z = e^{i\theta}$)

$$(4.2) \qquad \int_C n(z) z^k |dz| = \int_C n(z) (\cos k\theta + i \sin k\theta)\, d\theta = 0 \qquad (k = 1, 2, \ldots).$$

Here we set

$$c = \frac{1}{2\pi} \int_C n(z)\, d\theta, \qquad n_0(z) \equiv n(z) - c,$$

whence $\int_C n_0(z)\, d\theta = 0$. It follows from (4.2) that $n_0(z)$, a real continuous function on C, is orthogonal on C to each of the functions 1, $\cos k\theta$, $\sin k\theta$ ($k = 1, 2, \ldots$). Thus $n_0(z)$ vanishes identically and we have $n(z) \equiv c$ on C. Incidentally, this conclusion is a consequence of (4.2) rather than (4.1).

Let now C be an arbitrary analytic Jordan curve in the z-plane whose exterior is mapped onto the exterior of γ : $|w| = 1$ by the transformation $z = \psi(w)$, $w = \Psi(z)$, with $\psi(\infty) = \infty$. We set $n(z)|dz| \equiv n_1(w)|dw|$, so (4.1) now becomes

$$(4.3) \qquad \int_\gamma n_1(w) [\psi(w)]^n [\bar{\psi}(w)]^{-k} |dw| = 0 \qquad (n = 0, 1, 2, \ldots; k = 1, 2, \ldots).$$

The function $\bar{w}^{-m}$ ($m \geqq 1$), can be uniformly approximated [Walsh, 1935, p. 37] on γ by a linear combination of the functions $[\bar{\psi}(w)]^{-k}$ ($k \geqq 1$); it follows from (4.3) with $n = 0$ that we have

$$\int_\gamma n_1(w)\, \bar{w}^{-m} |dw| = 0, \qquad (m = 1, 2, \ldots).$$

This equation is essentially (4.2), so we conclude that $n_1(w)$ is identically constant.

From (4.3) with $n = 1$ we now write

$$\int_\gamma \psi(w)\, \bar{w}^{-k} |dw| = 0, \qquad (k = 1, 2, \ldots),$$

$$(4.4) \qquad \int_\gamma \psi(w)\, w^k \, dw = 0, \qquad (k = 0, 1, 2, \ldots).$$

It is a consequence of (4.4) [Walsh, 1935, p. 40] that there exists a function $\psi_1(w)$ analytic in $|w| < 1$ and continuous in $|w| \leqq 1$ which coincides with $\psi(w)$ on γ. The function $\psi(w)$ already defined for $|w| \geqq 1$ and now defined as $\psi_1(w)$ for $|w| < 1$ is uniquely defined and continuous on γ, hence analytic also on γ. Thus $\psi(w)$ is analytic at every finite point of the plane and has a simple pole at infinity, so we have $\psi(w) \equiv Aw$, where A is a non-vanishing constant. Then C: $|\Psi(z)| = 1$ is a circle whose center is $z = 0$ and $n(z) \equiv \frac{n_1(w)\,|dw|}{|dz|}$ is identically constant, so theorem **4.1** is established.

5. Degree of Convergence; summability. — If the function $f_1(z)$ is analytic $|z| < 1$, continuous $|z| \leqq 1$, and of class $L(k, \alpha)$ on C: $|z| = 1$, then it is known [Sewell, 1942, p. 90, 112] that there exist polynomials $\pi_n(z)$ of respective degrees n such that we have

$$(5.1) \qquad |f_1(z) - \pi_n(z)| \leqq \frac{M}{n^{k+\alpha}}, \quad z \text{ on C},$$

where M is independent of n and z. Moreover, the polynomials $\pi_n(z)$ can be found as a Jackson summation of order k of the Taylor development of $f_1(z)$. For the partial sums of the Taylor development, polynomials $p_n(z)$ of respective degrees n, we have the weaker inequality [de la Vallée Poussin, 1919, p. 27; Jackson, 1930, p. 21]

$$(5.2) \qquad |f_1(z) - p_n(z)| \leqq \frac{M_1 \log n}{n^{k+\alpha}}, \quad z \text{ on C}.$$

These known results extend directly to the polynomial expansions of Sections **2** and **3**.

From the results of Sewell [1942, *loc. cit.*] and Zygmund [1945] we have at once

Theorem 5.1. — *If* C *is an analytic Jordan curve and if* $f(z)$ *is of class* $L(k, \alpha)$ *on* C, $0 < \alpha < 1$, *then for the expansions* (2.4) *and* (2.7) *we have the inequalities* (5.1) *and* (5.2), *where* $p_n(z)$ *denotes the sum of the first* $n+1$ *terms of the expansion and* $\pi_n(z)$ *denotes the sum of the first* $n+1$ *terms of the Jackson summation of order* k; *we also have*

$$(5.3) \qquad |f_2(z) - \pi_n(z)| \leqq \frac{M}{n^{k+\alpha}}, \qquad z \text{ on C},$$

$$(5.4) \qquad |f_2(z) - p_n(z)| \leqq \frac{M_1 \log n}{n^{k+\alpha}}, \quad z \text{ on C},$$

in analogous notation for the expansions (2.5) *and* (2.8). *If* $f(z)$ *is of class* $\Lambda^*(k)$ *on* C, *the inequalities* (5.1)-(5.4) *are valid with* $\alpha = 1$.

Sewell [*loc. cit.* p. 100] proves essentially that if a series converges like a convergent geometric series, then its Jackson summation of arbitrary order k converges to the same sum, with an error $o(n^{-k-\alpha})$, $0 < a \leqq 1$. Consequently the results of Section 3 yield

THEOREM 5.2. — *Let* C *be an analytic Jordan curve containing the origin in its interior, let the norm function* $n(z)$ *be positive and analytic on* C, *and let* $f(z)$ *be of class* $L(k, \alpha)$ *on* C, $0 < \alpha < 1$. *Then for the expansion* (3.6) *we have the inequalities* (5.1) *and* (5.2), *where* $p_n(z)$ *denotes the sum of the first* $n+1$ *terms of the expansion and* $\pi_n(z)$ *denotes the sum of the first* $n+1$ *terms of its Jackson summation of order* k; *we also have* (5.3) *and* (5.4) *in analogous notation for the expansion* (3.14). *If* $f(z)$ *is of class* $\Lambda^*(k)$ *on* C, *the inequalities* (5.1)-(5.4) *are valid with* $\alpha = 1$.

This conclusion relative to (3.14) applies also to (3.20) under the hypothesis (3.19).

Inequalities (5.1) and (5.2), valid for z on C, are also valid for z in the closed interior of C, and inequalities (5.3) and (5.4) are valid in the closed exterior of C. Corresponding inequalities, obtained by addition of (5.1) and (5.3) and addition of (5.2) and (5.4) hold for $f(z)$ on C.

If the given function $f(z)$ is itself analytic on C, so also are the functions $f_1(z)$ and $f_2(z)$. To be more explicit, let $f(z)$ be analytic in the annulus bounded by the Jordan curves $C_R : |\Psi(z)| = R (> 1)$ and $C_r : |\varphi(z)| = r (< 1)$. The Cauchy integrals (1.10) defining $f_1(z)$ and $f_2(z)$ may be taken over any Jordan curve interior to C_R but containing z, and exterior to C_r and having z in its exterior, respectively. Thus $f_1(z)$ is analytic throughout the interior of C_R and $f_2(z)$ is analytic throughout the exterior of C_r. From the general theory of the Faber polynomials we then have for (2.4)

$$\limsup_{n \to \infty} \left[\max \left| f_1(z) - \sum_0^n a_k F_k(z) \right|, \ z \text{ on } C \right]^{\frac{1}{n}} \leqq \frac{1}{R};$$

convergence is uniform on any closed set interior to C_R; and for (2.8) we have

$$\limsup_{n\to\infty}\left[\max\left|f_2(z)-\sum_0^n c_n F_{-n}\left(\frac{1}{z}\right)\right|,\ z \text{ on } C\right]^{\frac{1}{n}} \leqq r;$$

convergence is uniform on any closed set exterior to C_r.

From the general theory of the Szegö polynomials we also have for (3.6)

$$\limsup_{n\to\infty}\left[\max\left|f_1(z)-\sum_0^n a_k p_k(z)\right|,\ z \text{ on } C\right]^{\frac{1}{n}} \leqq \frac{1}{R},$$

convergence being uniform on any closed set interior to C_R and for (3.14)

$$\limsup_{n\to\infty}\left[\max\left|f_2(z)-\sum_0^n b_k P_k\left(\frac{1}{z}\right)\right|,\ z \text{ on } C\right]^{\frac{1}{n}} \leqq r,$$

convergence being uniform on any closed set exterior to C_r.

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