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On completeness of invariant measures defined by differential equations

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On completeness of invariant measures defined by differential equations;

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I. — Introduction.

Let the real, first order, non-singular, ordinary differential system

$$(1)\qquad \frac{dx}{dt}=f(x, y), \qquad \frac{dy}{dt}=g(x, y) \qquad (f^2+g^2>0),$$

with

$$f, g \in C^{(\alpha)} \qquad (\alpha=1, 2, 3, \ldots, \infty, A) \quad (^1)$$

be defined in an open plane set R. Then through each point P_0 : $(x_0, y_0) \in R$ there exists an unique solution curve

$$x=x(t; x_0, y_0), \qquad y=y(t; x_0, y_0),$$

initiating at P_0 for $t=0$, and defined for some maximal « time » interval

$$\tau_-(P_0)<t<\tau_+(P_0).$$

(1) $f(x_1, x_2, \ldots, x_n) \in C^{(0)}$ in R, an open set of Euclidean n-space, in case $f(x)$ is continuous, in n variables in R : $f(x_1, x_2, \ldots, x_n)=f(x) \in C^{(K)}$ $(K=1, 2, \ldots)$ in R in case all the partial derivatives of $f(x)$ up to and including those of order K exist (are finite) and are continous in R. $f(x) \in C^{(\infty)}$ in case all partial derivatives of $f(x)$ exist and are continuous in R. $f(x) \in C^{(A)}$ in case $f(x)$ is analytic in R, that is, near each point in R, $f(x)$ has an absolutely convergent, real, n-variable power series representation. We define $0<1<2<\ldots<\infty<A$ and also $\infty \pm K=\infty$ and $A \pm K=A$.

The subset of Euclidean 3-space in which $x(t; x_0, y_0)$ and $y(t; x_0, y_0)$ are defined is open and therein these functions, as well as $\frac{\partial x}{\partial t}$ and $\frac{\partial y}{\partial t}$, are in class $C^{(\alpha)}$. The transformations T_t

$$(x_0, y_0) \rightarrow x(t; x_0, y_0), y(t; x_0, y_0),$$

from an open subset $R_0 \subset R$ onto open subsets $R_t \subset R$, from a one-parameter local group, or stream, of $C^{(\alpha)}$-homeomorphisms ([2]).

Suppose the ordinary differential system (1) were exact, that is, $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \equiv 0$ and R were a simply-connected region (open, connected set ([3])). Then $\{T_t\}$ forms a measure true stream, that is, T_t is measurability preserving (T_t and T^{-1} preserve Lebesgue measurability of sets) and also T_t preserves the magnitude of the measure. In this case, there exists a stream function $\psi(x, y) \in C^{(\alpha+1)}$ in R such that $f = \frac{\partial \psi}{\partial y}$, $g = -\frac{\partial \psi}{\partial x}$ and $\psi(x, y)$ is a principal integral ([4]) of the first order, partial differential equation

$$f(x, y)\frac{\partial \psi}{\partial x} + g(x, y)\frac{\partial \psi}{\partial y} = 0. \tag{2}$$

That is, $\psi(x, y)$ is a solution of (2) which is constant on no open set and therefore the class of all solutions of (2) are precisely those functions in $C^{(1)}$ in R which are functionally dependent on $\psi(x, y)$.

Even if the ordinary differential system (1) is not exact, there may exist a principal integral $\psi(x, y) \in C^{(2)}$ of the corresponding partial differential equation (2). Then

$$\mu(x, y) = \left[\frac{\psi_x^2 + \psi_y^2}{f^2 + g^2}\right]^{\frac{1}{2}} \in C^{(1)}$$

([2]) A homeomorphism T of an open set R_1 of Euclidean n-space onto a second open set R_2 of the same space is called a $C^{(\beta)}$-homeomorphism, $\beta = 0, 1, 2, \ldots, \infty$, A in case both T and T^{-1} are expressed by functions of class $C^{(\beta)}$.

([3]) For notation, *see* E. Hopf, *Ergodentheorie* (Berlin, 1937).

([4]) E. Kamke, *Differentialgleichungen Reeller Funktionen* (Chelsea, 1947), p. 323.

is a non-negative integrating factor for (1), that is,

$$\mu f = \psi_y, \qquad \mu g = -\psi_x.$$

In case (1) is exact then $\mu(x, y)$ is a constant.

Because of the relation [5].

$$(3) \qquad \frac{d}{dt}\left[\iint_{R_0} \mu(x, y)\frac{\partial(x, y)}{\partial(x_0, y_0)}\,dx_0\,dy_0\right] = \iint_{R_0} [(\mu f)_x + (\mu g)_y]\,dx_0\,dy_0 = 0$$

we can define a new « invariant » measure m_μ on the σ-ring [6] L of Lebesgue measurable sets of R by

$$m_\mu(\Lambda) = \iint_\Lambda \mu(x, y)\,dx,\,dy$$

for $\Lambda \in L$. Then the local group of $C^{(\alpha)}$-homeomorphisms $\{T_t\}$ is a m_μ-measure true stream. Here m_μ is a non-negative, completely additive set function on L. For any compact set $K \subset R$, $m_\mu(K) < \infty$. m_μ is an absolutely continuous measure in terms of the Lebesgue measure m_L, that is, if $m_L(\Lambda) = 0$, then $m_\mu(\Lambda) = 0$, The purpose of this note is to point out that if $\mu(x, y) \geqq 0$ and vanishes on no open set, even if $\mu(x, y) \in C^{(\infty)}$, there can exist a set $Z \in L$ such that $m_\mu(Z) = 0$ but $m_L(Z) > 0$. In this case the invariant measure m_μ is not complete since there exist non-measurable subsets of Z. However in case $\mu(x, y)$ is analytic in R, $m_\mu(\Lambda) = 0$ if and only if $m_L(\Lambda) = 0$ and m_μ is a complete measure.

II. — Homeomorphisms which preserve measurability.

Any simply-connected plane region R can be mapped by a $C^{(A)}$-homeomorphism onto the entire plane. Thus we can replace R by the entire plane in any measure-theoretic considerations which are invariant under such a map.

Let θ be a Borel set of Euclidean n-space. Let S be the σ-ring of

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[5] H. Poincaré, *Méthodes nouvelles de la Mécanique céleste*, t. III, chap. 26.

[6] Sometimes called σ-field. For notation, *see* P. Halmos, *Measure Theory* (New-York, 1950).

all subsets of θ. S actually forms an algebraic commutative Boolean (idempotent) ring with unity element under the operations of symetric difference $\triangle$ for « addition » and intersection $\cap$ for « multiplication ». Because of the simple, well-known formulae for union, difference, and complementation

$$(4) \qquad \begin{cases} \varphi \cup \psi = (\varphi \triangle \psi) \triangle (\varphi \cap \psi), \\ \varphi - \psi = \varphi \triangle (\varphi \cap \psi), \\ \quad \varphi' = \varphi \triangle \theta \end{cases}$$

all of the set-theoretic properties of S are determined by its structure as an algebraic ring under $\triangle$ and $\cap$. Let B be the σ-ring of Borel sets of θ (smallest σ-ring containing the open sets of θ), with the Borel measure m_B. B is an algebraic subring of S. The subset $b \subset B$ of Borel sets with measure zero forms an ideal in the ring B. The σ-ring L of Lebesgue measurable subsets of θ (all sets of the form $\eta \triangle \nu$ where $\eta \in B$ and ν is a subset of some set in b) with Lebesgue measure $m_{(L)}$ is an algebraic subring of S and a superring of B. The ideal $l \subset L$ of sets with Lebesgue measure zero is a superring of b, and consists of all sets of the form $\beta \triangle \nu$ where $\beta \in b$ and ν is a subset of some set of b.

Let θ_1 and θ_2 be two Borel sets of Euclidean spaces and let S_1, B_1, b_1, L_1, l_1 and S_2, B_2, b_2, L_2, l_2 be the corresponding σ-rings described above. Let T be a homeomorphism of θ_1 onto θ_2. Since T is a one-to-one point transformation, T induces a set mapping τ from S_1 onto S_2. Clearly τ is an algebraic isomorphism of S_1 onto S_2.

THEOREM I. — *Let* T *be a homeomorphism of* θ_1 *onto* θ_2, *Borel sets of Euclidean spaces. Then the induced set-map* τ *is an isomorphism of* B_1 *onto* B_2. *Furthermore the following four conditions are equivalent :*

1. T *is a* m_L*-measurability preserving transformation;*
2. τ *induces an isomorphism of* L_1 *onto* L_2*;*
3. τ *induces an isomorvhism of* b_1 *onto* b_2*;*
4. τ *induces an isomorphism of* l_1 *onto* l_2.

Proof. — Since T is a homoemorphism, open sets of θ_1 correspond to open sets of θ_2 under τ and thus there is a one-to-one correspondence between the sets of B_1 and B_2. Since τ is an isomorphism of S_1 onto S_2, τ is therefore an isomorphism of B_1 onto B_2.

Since τ is an isomorphism of S_1 onto S_2, conditions 1 and 2 are equivalent by the definition of measurability preserving transformations.

Suppose condition 3 holds; we shall deduce condition 2. If $\Lambda_1 \in L_1$, then $\Lambda_1 = \eta_1 \triangle \nu_1$ where $\eta_1 \in B_1$ and $\nu_1 \subset \beta_1 \in b_1$. Then

$$\tau(\Lambda_1) = \Lambda_2 = \tau(\eta_1) \triangle \tau(\nu_1) = \eta_2 \triangle \nu_2,$$

with

$$\tau(\eta_1) = \eta_2 \in B_2 \quad \text{and} \quad \tau(\nu_1) = \nu_2 \subset \tau(\beta_1) = \beta_2 \in b_2.$$

Thus $\tau(\Lambda_1) = \Lambda_2 \in L_2$ and τ maps L_1 into L_2. Using the inverse homeomorphism T^{-1} we see that τ^{-1} maps L_2 into L_1 and since τ is one-to-one on S_1, τ is an isomorphism of L_1 onto L_2.

Conversely suppose τ is an isomorphism of L_1 onto L_2. Suppose $\tau(\beta_1) = \beta_2 \in B_2 - b_2$ for some $\beta_1 \in b_1$. Then $m_B(\beta_2) > 0$ and there exists a set (7) $\chi_2 \subset \beta_2$ such that $\chi_2 \in S_2 - L_2$. But then $\tau^{-1}(\chi_2) = \chi_1 \subset \beta_1$ and thus $\chi_1 \in L_1$. But $\tau(\chi_1) \notin L_2$ and this contradicts the hypothesis that τ maps L_1 onto L_2. Thus τ maps b_1 into b_2 and, as above, τ is an isomorphism of b_1 onto b_2.

Since $b_i = B_i \cap l_i$ $(i = 1, 2)$, condition 4 clearly implies 3. Because l_i consists of those sets of L_i which are contained in sets of b_i $(i = 1, 2)$, conditions 2 and 3 imply 4. Q. E. D.

Corollary. — *If* T *is a* $C^{(1)}$*-homeomorphism of* θ_1 *onto* θ_2, *open sets of an Euclidean n-space, then* T *is* m_L*-measurability preserving.*

Proof. — We need show only that $\tau(b_1) \subset b_2$. If $\beta_1 \in b_1$ then β_1 has a covering by a countable number of compact closed n-balls $\bar{R}_i$ with

$$\sum_{i=1}^{\infty} m_B(\bar{R}_i) = 1.$$

In $\bar{R}_i$ the continuous Jacobian of T is bounded, $|J(T)| < K_i$ Then

$$m_B(\beta_1 \cap \bar{R}_i) = 0,$$

(7) *See* P. Halmos, *op. cit.*, p. 70. The existence of a non-measurable linear set combined with an application of Fubini's theorem gives this result.

and we cover $\beta_1 \cap \overline{R}_i$ by an open set o_i with $m_B(o_i) = \frac{\delta}{2 K_i}$. Then

$$m_B(\tau(o_i)) \leqq \frac{\delta}{2^i}$$

and thus

$$\sum_{i=1}^{\infty} m_B(\tau(o_i)) \leqq \delta.$$

Thus $m_B(\tau(\beta_1)) \leqq \delta$ and therefore, since δ is an arbitrary positive number, $\tau(\beta_1) = \beta_2 \in l_2$. But certainly $\beta_2 \in B_2$. Thus $\beta_2' \in b_2$. Therefore $\tau(b_1) \subset b_2$ and using T^{-1}, which also has a continuous Jacobian, $\tau^{-1}(b_2) \subset b_1$ and therefore τ maps b_1 onto b_2.

Q. E. D.

An example of a $C^{(0)}$-homeomorphism of the linear interval $[0, 1]$ onto itself which is not measurability preserving is given by T

$$x \to f(x) = \frac{1}{2}[x + \psi(x)]$$

where $\psi(x)$ is the continuous, but not absolutely continuous, non-decreasing, Cantor function [8]. If K is the compact Cantor set, then $m_B(K) = 0$ but $m_B(f(K)) = \frac{1}{2}$.

III. — Completeness of invariant measures.

We shall construct a function $\mu(x, y)$ such that :

1. $\mu(x, y) \in C^{(\infty)}$ in the entire plane;
2. $\mu(x, y) \geqq 0$ and the plane set Z defined by $\mu(x, y) = 0$ contains no interior points;
3. $m_B(Z) = \infty$.

Let z be a compact, nowhere dense point set of the real line such that the linear Borel measure $m'_B(z) > 0$. A closed point set is nowhere dense if and only if it contains no interior point. Such a

(8) P. HALMOS, *op. cit.*, p. 83.

set z is given in Hobson ([9]) and can be described briefly as follows.

Let $[0, 1]$ be divided into $m > 2$ equal parts and the last exempted from further division. Then let the remaining $m - 1$ parts each be divided into m^2 equal parts, the last of each being exempted from further division. Let the remaining parts be then divided into m^3 equal parts, the last of these in each case being exempted from further division. If this process is carried out a countable number of times, the endpoints of the divisions, together with their accumulation points, form a nowhere dense, compact set z.

This set z has mesure

$$0 < \prod_{i=1}^{\infty}\left(1 - \frac{1}{m^i}\right) < 1$$

for after i operations, the measure of the union of the exempted segments is

$$\frac{1}{m} + \frac{m-1}{m^3} + \frac{(m-1)(m^2-1)}{m^6} + \ldots + \frac{(m-1)(m^2-1)\ldots(m^{i-1}-1)}{m^{\frac{i(i+1)}{2}}}$$

or

$$1 - \left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{m^2}\right)\cdots\left(1 - \frac{1}{m^i}\right).$$

Thus $m'_B(z) > 0$ and indeed, may be chosen arbitrarily close to one if the dividing ratio m be large enough. The complement of z in $[0, 1]$ is a countable union of disjoint open intervals I_n and z consists of the closure of the endpoints of the I_n.

The distance from a real number x to the compact set z is $d(x) \geqq 0$ and $d(x) = 0$ if and only if $x \in z$.

Then for any point (x, y) of the plane we define

$$(5)\quad \begin{cases} \mu(x, y) = 0 & \text{in case } x \in z. \\ \mu(x, y) = e^{-\frac{1}{x^2}} & \text{in case } x < 0. \\ \mu(x, y) = e^{-\frac{1}{(x-1)^2}} & \text{in case } x > 1. \\ \mu(x, y) = e^{-\frac{1}{\frac{\Delta_n^2}{4} - \left(\frac{\Delta_n}{2} - d(x)\right)^2}} & \text{in case } x \in I_n \end{cases}$$

([9]) A. E. Hobson, *The theory of functions a real variable* (Cambridge, 1921), p. 119 and also p. 164.

where $A_n = b_n - a_n$ is the length of $I_n = (a_n, b_n)$. Then $\mu(x, y) \geq 0$ and, defining the plane set $(x, y) \in Z$ in case $x \in z$, $\mu(x, y) = 0$ if and only if $(x, y) \in Z$. The set Z is closed and contains no interior points of the plane. Moreover $m_B^n(Z) = \infty$. We shall show that $\mu(x, y) \in C^{(\infty)}$ and then clearly

$$m_\mu(Z) = \iint_Z \mu(x, y)\, dx\, dy = 0.$$

If $x < 0$ or $x > 1$, then $\mu(x, y) \in C^{(\infty)}$. If $x \in I_n$, that is, $a_n < x < b_n$, then

$$\frac{\Delta_n^2}{4} - \left(\frac{\Delta_n}{2} - d(x)\right)^2 = (x - a_n)(b_n - x)$$

and thus $\mu(x, y) \in C^{(\infty)}$ for these regions.

For $x \in I_n$,

$$\frac{\Delta_n^2}{4} - \left(\frac{\Delta_n}{2} - d(x)\right)^2 = 2\,\Delta_n d - d^2 < d - d^2 < d.$$

Thus

$$|\mu(x, y)| \leq e^{-\frac{1}{d(x)}} < \frac{1}{d(x)^2} e^{-\frac{1}{d(x)}}.$$

By induction it is easy to show that

$$\frac{\partial^k \mu}{\partial x^k} = \frac{P_k((x - a_n)), (b_n - x))}{[(x - a_n)(b_n - x)]^{2k}} e^{-\frac{1}{(x) - a_n)(b_n - x)}}$$

where P_k is a polynomial and since

$$d(x) \leq |x - a_n| < 1, \qquad d(x) \leq |b_n - x| < 1,$$

we have

$$|P_k((x - a_n)(b_n - x))| < \overline{P}_k$$

an upper bound independent of I_n and thus

$$(6) \qquad \left|\frac{\partial^k \mu}{\partial x^k}\right| < \frac{\overline{P}_k}{d(x)^{2k+1}} e^{-\frac{1}{d(x)}} \qquad \text{for} \quad k = 0, 1, 2, \ldots.$$

For sufficiently small positive d, the function $\frac{1}{d^{2k+1}} e^{-\frac{1}{d}}$ is strictly increasing.

Finally consider a point $(x_0, y_0) \in Z$. Then $\mu(x_0, y_0) = 0$ and for $|x-x_0|<\delta$, a sufficiently small positive number, $d(x)<\delta$ and

$$0 \leqq |\mu(x, y)-\mu(x_0, y_0)| \leqq \frac{\overline{P}_0}{d(x)^2} e^{-\frac{1}{d(x)}} < \frac{\overline{P}_0}{\delta^2} e^{-\frac{1}{\delta}}.$$

Thus $\mu(x, y)$ is continuous at each point of Z and $\mu(x, y) \in C^{(0)}$ in the plane. We prove $\mu(x, y) \in C^{(\infty)}$ by induction. Suppose

$$\mu(x, y) \in C^{(k)} \quad \text{and} \quad \frac{\partial^k \mu}{\partial x^k}(x_0, y_0) = 0.$$

Then for $0<|x-x_0|<\delta$, a sufficiently small positive number, $d(x)<\delta$ and

$$\left| \frac{\dfrac{\partial^k \mu(x, y)}{\partial x^k} - \dfrac{\partial^k \mu(x_0, y_0)}{\partial x^k}}{x-x_0} \right| < \frac{\overline{P}_k e^{-\frac{1}{\delta}}}{\delta \delta^{2k+1}}.$$

Thus $\dfrac{\partial^{k+1} \mu(y_0, x_0)}{\partial x^{k+1}} = 0$. From the inequality

$$\left| \frac{\partial^{k+1} \mu}{\partial x^{k+1}} \right| < \frac{\overline{P}_{k+1}}{d(x)^{2k+2}} e^{-\frac{1}{d(x)}}$$

it is clear that

$$\lim_{\substack{x=x_0 \\ y=y_0}} \frac{\partial^{k+1} u(x, y)}{\partial x^{k+1}} = 0$$

and thus $\dfrac{\partial^{k+1} \mu(x, y)}{\partial x^{k+1}}$ exists and is continuous at each point of Z and therefore at every point of the plane. Since $\dfrac{\partial \mu}{\partial y} \equiv 0$ we have

$$\mu(x, y) \in C^{(k+1)} \quad \text{and} \quad \frac{\partial^{k+1} \mu(x_0, y_0)}{\partial x^{k+1}} = 0.$$

Therefore the induction is complete and $\mu(x, y) \in C^{(\infty)}$ in the plane.

THEOREM II. — *Let* R *be simply-connected plane region. Then there exists a real, first oder non-singular ordinary differential equation.*

$$\text{(1)} \qquad \frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y) \qquad (f^2+g^2>0)$$

with $f, g \in C^{(\infty)}$ *in* R, *with an integrating factor* $\mu(x, y) \in C^{(\infty)}$, *defining an invariant measure*

$$m_\mu(\Lambda) = \iint_\Lambda \mu(x, y)\, dx\, dy$$

on the plane Lebesgue m_L-measurable sets, such that

1. $\mu(x, y) \in C^{(\infty)}$ in R;
2. $\mu(x, y) \geq 0$;
3. $\mu(x, y)$ *vanishes on no open set;*
4. $\frac{\partial}{\partial x}(\mu f) + \frac{\partial}{\partial y}(\mu, g) \equiv 0$ in R

and the m_μ-measure is not complete.

If f, g are only in $C^{(1)}$ with $f^2 + g^2 > 0$ in R, and if $\mu(x, y) \in C^{(A)}$ in R is an integrating factor, that is, satisfies conditions 2, 3, and 4, then the m_μ-measure is complete, that is, $m_\mu(\Lambda) = 0$ if and only if $m_L(\Lambda) = 0$.

Proof. — We first prove the theorem in the case R is the entire plane and then use theorem I for the general simply-connected region.

Consider the differential system

$$\frac{dx}{dt} = \mu(x, y), \qquad \frac{dy}{dt} = -2\frac{\partial \mu}{\partial x}y + 1 \tag{7}$$

where $\mu(x, y)$ is the function described above. This system is $C^{(\infty)}$ and also non-singular since $\mu(x_0, y_0) = 0$ if and only if $(x_0, y_0) \in Z$ in which case $\frac{\delta\mu}{\delta x} = 0$ and $(-2\mu_x y + 1) = 1$. Moreover, it is not exact but has an integrating factor of $\mu(x, y)$ since

$$\frac{\partial}{\partial x}(\mu^2) + \frac{\partial}{\partial y}(-2\mu_x\mu_y + \mu) \equiv 0.$$

To show the m_μ-measure is not complete let $\chi \in Z$ be a non-m_L-measurable set. This is possible since $m_L(Z) = \infty$. But

$$m_\mu(Z) = \iint_Z \mu(x, y)\, dx\, dy = 0$$

and thus χ is a non-m_μ-measurable subset of a set Z of m_μ-measure zero. Therefore the m_μ-measure is not complete.

Next let $\mu(x, y) \in C^{(A)}$ be any integrating factor of 1 in the plane. Clearly if

$$m_L(\Lambda) = 0$$

then

$$m_\mu(\Lambda) = \iint_\Lambda \mu \, dx \, dy = 0.$$

Conversely let $m_\mu(\Lambda) = 0$. Then there exist two sets $\Lambda_1, \Lambda_2 \in L$ such that $m_L(\Lambda_1) = 0$, and

$$\Lambda_1 \cap \Lambda_2 = 0, \qquad \Lambda_1 \cup \Lambda_2 = \Lambda, \quad \text{and} \quad \Lambda_2 \subset Z,$$

the set of zeros of $\mu(x, y)$. We shall show that $m_L(Z) = 0$.

Since Z is closed, $Z \in L$. Then for almost all horizontal lines $h_c : y = y_c$, the linear measure of $Z \cap h_c$ exists. If $Z \cap h_c$ contains only a countable number of points, it has linear measure zero. If $Z \cap h_c$ contains a non-countable number of points, then since $\mu(x, y_c)$ is an analytic function of one argument, $Z \cap h_c = h_c$ and thus has infinite linear measure. However there are only a countable number of lines h_c on which $m'_L(Z \cap h_c) = \infty$. For if otherwise, there would be a finite accumulation point $\bar{y}$ of the corresponding ordinates and then, for each fixed x_0, $\mu(x_0, y)$ is an analytic function of y and must vanish. Thus $\mu(x, y) \equiv 0$ which contradicts the hypothesis of the theorem that $\mu(x, y)$ vanishes on no open set. Therefore on almost all horizontal lines $m'_L(Z \cap h_c) = 0$. Therefore by Fubini's theorem $m_L(Z) = 0$. Since $\Lambda \subset \Lambda_1 \cup Z$ and $m_L(\Lambda_1) = 0$, we have $m_L(\Lambda) = 0$.

Next consider any subset $\Lambda' \subset \Lambda$. Then $\Lambda' \in l$ and $m_L(\Lambda') = 0$. Thus $m_\mu(\Lambda') = 0$ and the m_μ-measure is complete. The theorem is proved in case R is the entire plane.

Again return to the case of a general simply-connected plane region R. Let $T : (x, y) \to (u(x, y), v(x, y))$ be a $C^{(A)}$-homeomorphism (10) of the plane onto the (u, v)-region R. By T the

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(10) First map the plane onto the square $|x^1| < 1$, $|y^1| < 1$ by

$$x^1 = \frac{2}{\pi} \operatorname{tg}^{-1} x, \qquad y^1 = \frac{2}{\pi} \operatorname{tg}^{-1} y.$$

Then use the Riemann conformal mapping theorem.

differential system (7) is carried into a non-singular $C^{(\infty)}$-differential system defined in R

$$(8) \qquad \frac{du}{dt} = u_x f + u_y g = F(u, v), \qquad \frac{dv}{dt} = v_x f + v_y g = G(u, v)$$

where F, $G \in C^{(\infty)}$ in R and $F^2 + G^2 > 0$. The corresponding integrating factor is

$$M(u, v) = \mu \left| \frac{\partial(x, y)}{\partial(u, v)} \right|.$$

with the corresponding integral $\psi(x(u, v), y(u, v))$, as is seen from the following matrix equation

$$(9) \qquad \begin{pmatrix} \psi_u \\ \psi_v \end{pmatrix} = \begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix} \mu \begin{pmatrix} -g \\ f \end{pmatrix}$$

and

$$\begin{pmatrix} -g \\ f \end{pmatrix} = \begin{pmatrix} y_v & -y_u \\ x_v & x_u \end{pmatrix} \begin{pmatrix} -G \\ F \end{pmatrix}.$$

Thus

$$(10) \qquad \begin{pmatrix} \psi_u \\ \psi_v \end{pmatrix} = \mu(x(u, v), y(u, v)) \frac{\partial(x, y)}{\partial(u, v)} \begin{pmatrix} -G \\ F \end{pmatrix} = \pm M \begin{pmatrix} -G \\ F \end{pmatrix}.$$

Clearly $M(u, v) \geq 0$, $M \in C^{(\infty)}$ and vanishes only on the set T(Z which contains no interior points. Since T is a $C^{(1)}$-homeomorphism it preserves the set properties measurability and also of having zero measure. Thus $m_L(T(Z)) > 0$ and thus the m_M-measure is not complete.

A similar argument proves that, in case $\mu(x, y) \in C^{(A)}$ is an integrating factor of (1) in R, a simply-connected region, the resulting invariant measure is complete. Q. E. D.

Corollary. — *Let*

$$(1) \qquad \frac{dx}{dt} = f(x, y) \qquad \frac{dy}{dt} = g(x, y) \qquad (f^2 + g^2 > 0)$$

with $f, g \in C^{(A)}$ *in a simply-connected plane region* R *have a principal integral* $\psi(x, y)$ *such that*

1. $\psi(x, y) \in C^{(A)}$ *in* R;

2. $\psi(x, y)$ *is constant on no open set of* R;
3. $\psi(x, y)$ *is constant along each solution curve of* (1).

Then the differential system (1) *has a complete invariant measure.*

Proof. — Let

$$\mu(x, y) = \left[\frac{\psi_x^2 + \psi_y^2}{f^2 + g^2}\right]^{\frac{1}{2}}$$

be the integrating factor corresponding to the integral $\psi(x, y) \in C^{(A)}$. Then $\mu(x, y) \in C^{(A)}$ and, by the theorem, the invariant measure

$$m_\mu(\Lambda) = \iint_\Lambda \mu(x, y)\, dx\, dy$$

is complete. Q. E. D.

We shall discuss criteria that (1) should have an analytic integral in a later paper.