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par Jun-ichi TAMURA

1. Introduction.

Davison proved in [8] that

(1)
$$\psi := \sum_{n=1}^{\infty} 2^{\lfloor \alpha n \rfloor} = [0; 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, 2^8, 2^{13}, \ldots] \quad (\alpha = (1+\sqrt{5})/2),$$

and that the number ψ is transcendental. Here, $\lfloor x \rfloor$ denotes the integer part of a real number x and the right-hand side denotes a simple continued fraction, where the power of 2 appearing in the partial quotients are Fibonacci numbers.

The binary expansion of ψ can be described by the fixed point of a substitution.* For this purpose, we introduce some definitions.

K^* denotes the set of all finite words over an alphabet $K = \{a, b, c, \cdots, d\}$, i.e. K^* is the free monoid generated by K with the operation of concatenation and the empty word λ as its unit. K^∞ denotes the set of all infinite words $w_1 w_2 w_3 \cdots (w_n \in K)$.

A substitution σ (over K) is a monoid endomorphism σ on K^* extended to K^∞, defined by $\sigma(w) = \sigma(w_1)\sigma(w_2)\sigma(w_3)\cdots$ for $w = w_1 w_2 w_3 \cdots \in K^\infty$. A fixed point of σ is a word $w \in K^\infty$ such that $\sigma(w) = w$. Any substitution σ over K of the form

$$\sigma(a) = au \quad (u \neq \lambda), \quad \sigma(x) \neq \lambda \quad (\forall x \in K),$$

has the unique fixed point w prefixed by a, namely, $w = au\, \sigma(u)\, \sigma^2(u) \cdots$. Here, the product $\tau\, \sigma$ denotes the composition of τ and σ, and σ^n indicates the n-fold iteration $\sigma^{n-1}\sigma$ $(n \geqq 1)$ with $\sigma^0(u) = u$ $(u \in K^* \cup K^\infty)$.

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*Cf. Böhmer [4], Mahler [13], Danilov [7], Adams-Davison [1], Bundschuh [5], Nishioka-Shiokawa-Tamura [15], see also Allouche [2], Stolarsky [21].

The base-2 expansion of the number ψ is given by

$$\psi = 0.1011010110110101101011011010110110\cdots = 0.w_1w_2w_3\cdots,$$

where the word $w = w_1w_2w_3\cdots \in \{0,1\}^\infty$ is the unique fixed point of the substitution σ over $\{0,1\}$ defined by $\sigma(1) = 10,\ \ \sigma(0) = 1$.

The main purpose of this paper is to give a higher dimensional version of (1), where the Fibonacci sequence is replaced by a linear recurrence sequence of order 3, the so-called "Tribonacci" sequence**; the continued fraction expansion is replaced by its dimension 2 analogue, the simplest of the Jacobi-Perron algorithms; and the fixed point of a different substitution

$$\sigma(a) = ab,\ \sigma(b) = ac,\ \sigma(c) = a. \tag{2}$$

In what follows, $\omega = abacabaabacababa\cdots$ denotes the fixed point of the substitution σ over $K := \{a,b,c\}$ defined by (2)*** We shall prove that

$$\begin{pmatrix} 0.10101011010101010101011010\cdots \\ 0.11101111110111110111110\cdots \end{pmatrix} =$$

$$\begin{bmatrix} 0; 2^0, 2^0, 2^1, 2^1, 2^2, 2^4, 2^7, 2^{13}, \cdots, 2^{f_{n-3}}, & \cdots \\ 0; 2^0, 2^1, 2^2, 2^3, 2^6, 2^{11}, 2^{20}, 2^{37}, \cdots, 2^{f_{n-2}+f_{n-3}}, & \cdots \end{bmatrix},$$

where the left-hand side denotes the vector of two real numbers $0.\tau(\omega)$ and $0.v(\omega)$ having the sequences $\tau(\omega)$ and $v(\omega)$ as their digits in the binary expansion respectively with the coding

$$\tau(a) := 1,\ \tau(b) = \tau(c) := 0,\ v(a) = v(b) := 1,\ v(c) := 0;$$

and the right-hand side denotes the Jacobi-Perron algorithm, and f_n $(n \geqq -2)$ denotes the sequence with the recurrence relation

$$f_n = f_{n-1} + f_{n-2} + f_{n-3}\ (n \geqq 1),\ f_{-2} = f_{-1} = 0,\ f_0 = 1. \tag{3}$$

We shall also show the linear independence, and transcendence of the numbers $0.\tau(\omega)$, $0.v(\omega)$ in Theorem 1. We give a more general transcendence result in Theorem 3, for instance, the number

$$0.\xi(\omega) = 0.3232033233232033232320332332320332323203\cdots,$$

**Cf. Sloane [20], p. 60, ♯ 406; Carlitz, Hoggatt and Scoville [6].

***Cf. Rauzy [18]. He set up a link between the distribution of the sequence $\{(\theta n, \eta n)\}_{n=1,2,3,\cdots}$ modulo $\mathbb{Z}^2$ $(\theta, n$ belong to a cubic field) and the sequence ω.

which is considered as an expression in base g ($\geqq 4$), is transcentental, where ξ is the morphism defined by $\xi(a) = 3$, $\xi(b) = 2$, $\xi(c) = 203$. For certain functions connected with the values $0.\tau(\omega)$ and $0.\upsilon(\omega)$, we can show results similar to those in Theorem 1 using an algorithm defined by Parusnikov, which is the counterpart of Jacobi-Perron algorithm for functions. The Jacobi-Perron algorithms used in Theorem 1 and Theorem 2 will be introduced in § 2. Theorem 1-3 will be stated in § 3. In this version, we use the theory of representation of numbers by Tribonacci sequence**** for the proof of the transcendence results in § 4.

Instead of the fixed point ω, we can state theorems similar to Theorem 1-3 for the fixed point of the substitution τ over $\{a_1, a_2, \cdots, a_{s+1}\}$ ($s \geqq 1$) given by

$$\tau\ (a_j) := a_1^{k_s - j + 1} a_{j+1}\ (1 \leqq j \leqq s),$$
$$\tau\ (a_{s+1}) := a_1$$

with $k_s \geqq k_{s-1} \geqq \cdots \geqq k_1 \geqq 1$ ($k_j \in \mathbb{N}$), where the Jacobi-Perron algorithms turn out to be of dimension s. We will give such results in a forthcoming paper.

2. Jacobi-Perron Algorithm.

In this section, we define two kinds of continued fraction expansions of higher dimension due to Jacobi, and Parusnikov.***** We use the following notation:

$\mathbb{L} = \mathbb{C}\ ((z^{-1}))$, the field of formal Laurent series with complex coefficients.

$\mathbb{L}$ is a metric space with the distance function $\|\ \theta - \eta\ \|$ $(\theta\ ,\ \eta \in \mathbb{L})$, where $\|\ \|$ is the usual non-archimedian norm defined by $\|\ \theta\ \| = e^{-k}$ for $\theta = \sum_{m=k}^{\infty} c_m z^{-m} \in \mathbb{L}$ with $c_k \neq 0$, $k \in \mathbb{Z}$, and $\|\ 0\ \| = 0$.

$[\theta] :=$ the polynomial part of $\theta \in \mathbb{L}$, *i.e.* $[\theta] = \sum_{m=k}^{0} c_m z^{-m}$ for $\theta = \sum_{m=k}^{\infty} c_m z^{-m}$.

****This is a generalization of the Fibonacci representation due to Zeckendorf [22]. A large and interesting class of representations is introduced by Shallit [19], see also Fraenkel [10], Knuth [12].

*****Cf. Bernstein [3], Parusnikov [16], Nikišin-Sorokin [14].

$$< \theta > := \theta - [\theta].$$

$$[\underline{\theta}] := {}^t([\theta]\ ,\ [\eta])\ ,\ < \underline{\theta} > := {}^t(< \theta >\ ,\ < \eta >) \text{ for } \underline{\theta} = {}^t(\theta\ ,\ \eta) \in \mathbb{L}^2,$$

where ${}^t(\theta\ ,\ \eta)$ denotes the transpose of $(\theta\ ,\ \eta)$.

$T : \mathbb{L}^2 \rightarrow \mathbb{L}^2$ denotes the map defined by $T({}^t(\theta\ ,\eta)) := {}^t(1/\eta\ ,\ \theta/\eta)$. To be accurate, T is not a map on $\mathbb{L}^2$. For brevity, in what follows, we shall simply write $f : A \rightarrow B$ for a "map" f with some exceptional elements $x \in A$ for which f is not defined. We also write $T({}^t(\theta\ ,\ \eta))$ by

$$\frac{1}{{}^t(\theta\ ,\ \eta)}.$$

Now we define, following Parusnikov [16], the Jacobi-Perron algorithm for $\underline{\theta} = {}^t(\theta\ ,\ \eta) \in \mathbb{L}^2$. Let $\underline{b}_0 := [\underline{\theta}]$, $\underline{\theta}_0 = {}^t(\theta_0\ ,\ \eta_0) := < \underline{\theta} >$. If $\theta_0 \neq 0$, then, noting that $T^{-1}(\underline{\theta}_0) = (\eta_0/\theta_0,\ 1/\theta_0)$, we can write

$$\underline{\theta} = \underline{b}_0 + \underline{\theta}_0 = \underline{b}_0 + \frac{1}{T^{-1}(\underline{\theta}_0)}.$$

Let $\underline{b}_1 := [T^{-1}(\underline{\theta}_0)]$, $\underline{\theta}_1 = {}^t(\theta_1\ ,\ \eta_1) := < T^{-1}(\underline{\theta}_0) >$. If $\theta_1 \neq 0$, then

$$\underline{\theta} = \underline{b}_0 + \cfrac{1}{\underline{b}_1 + \cfrac{1}{T^{-1}(\underline{\theta}_1)}}.$$

Continuing the process, we get

$$\underline{\theta} = \underline{b}_0 + \cfrac{1}{\underline{b}_1 + \cfrac{1}{\underline{b}_2 + \ddots + \cfrac{1}{\underline{b}_m + \underline{\theta}_m}}}, \tag{4}$$

provided that $\theta_n \neq 0$ for all $0 \leqq n \leqq m-1$, where

$$\underline{\theta}_n := {}^t(\theta_n\ ,\ \eta_n) = < T^{-1}S^{n-1}(< \underline{\theta} >) >$$

and

$$\underline{b}_n := [T^{-1}(S^{n-1}(< \underline{\theta} >)]\ \ (n \geqq 1),\ \underline{b}_0 := [\underline{\theta}_0], \tag{5}$$

and S^n is the n-fold iteration of

$$S(\underline{\theta}) := < T^{-1}(\underline{\theta}) > .$$

If $\theta_m = 0$, then the algorithm terminates. If $\theta_n \neq 0$ for all $n \geqq 0$, then

$$\underline{\pi}_n = \underline{\pi}_n(\underline{\theta}) := \cfrac{1}{\underline{b}_1 + \cfrac{1}{\underline{b}_2 + \ddots + \cfrac{1}{\underline{b}_n}}},$$

which will be denoted by $[\underline{b}_0; \underline{b}_1, \underline{b}_2, \cdots, \underline{b}_n]$, converges componentwise to $\underline{\theta}$ as $n \longrightarrow \infty$ with respect to the metric induced by the norm $\|\ \|$, cf. Parusnikov [16].

Hence we can write

$$\underline{\theta} = [\underline{b}_0; \underline{b}_1, \underline{b}_2, \cdots, \underline{b}_n, \cdots]. \tag{6}$$

In what follows, we also write

$$\underline{\pi}_n = \begin{bmatrix} b_0; b_1, b_2, \cdots, b_n \\ c_0; c_1, c_2, \cdots, c_n \end{bmatrix}, \underline{\theta} = \begin{bmatrix} b_0; b_1, b_2, \cdots, b_n \cdots \\ c_0; c_1, c_2, \cdots, c_n \cdots \end{bmatrix},$$

for $\underline{b}_n = {}^t(b_n, c_n)$. The algorithm given by (5) will be called the Jacobi-Perron-Parusnikov (abbr. JPP) algorithm (of dimension 2). Apart from the algorithm (5), we can consider the expression (6) for a given sequence $\underline{b}_n \in \mathbb{L}^2$ provided that its nth convergent $\underline{\pi}_n$ is well-defined (except for a finite number of them) and converges to some element in $\mathbb{L}^2$. The JPP expression (6) will be called admissible if it is derived from the algorithm (5). The admissible expressions satisfy

$$\begin{gathered} b_n,\ c_n \in \mathbb{C}[z] \text{ with} \\ 0 \leqq \deg\ b_n < \deg\ c_n, \text{ or } b_n = 0\ \ \&\ \ c_n \neq 0 \text{ for all } n \geqq 1. \end{gathered} \tag{7}$$

If we take the field $\mathbb{R}$, and the integral part $\lfloor \theta \rfloor$ of $\theta \in \mathbb{R}$ instead of $\mathbb{L}$, and the polynomial part $[\theta]$ of $\theta \in \mathbb{L}$, respectively, we have an algorithm which is the simplest one among the Jacobi-Perron algorithms, cf. [3], p. 49. This algorithm will be simply referred to as the Jacobi-Perron (abbr. JP) algorithm. When (6) is admissible in the JP algorithm, we have

$$b_n,\ c_n \in \mathbb{Z} \text{ with } 0 \leqq b_n \leqq c_n \neq 0 \text{ for all } n \geqq 1. \tag{8}$$

3. Main Results

THEOREM 1. *Let g be a positive integer, and let $\theta(g)$, $\eta(g)$ be the real numbers defined by the expression in the JP algorithm*

$$(9)\qquad \begin{pmatrix}\theta(g)\\ \eta(g)\end{pmatrix} := \begin{bmatrix} 0; & g^{f_{-2}}, & g^{f_{-1}}, & \cdots, & g^{f_{n-3}} & , \cdots \\ 0; & g^{f_{-1}+f_{-2}}, & g^{f_0+f_{-1}}, & \cdots, & g^{f_{n-2}+f_{n-3}}, & \cdots \end{bmatrix},$$

where f_n $(n \geqq -2)$ is the Tribonacci sequence defined by (3). Then the following assertions are valid:

(i) If $g \geqq 2$, the g-adic expansions of $\theta(g)$ and $\eta(g)$ are given by

$$\theta(g) = 0.\tau(\omega), \quad \eta(g) = 0.v(\omega),$$

where ω is the fixed point of σ defined by (2), τ and v are codings defined by $\tau(a) = v(a) = v(b) = g-1$, $\tau(b) = \tau(c) = v(c) = 0$.

(ii) If $g \geqq 2$, then $\theta(g)$ and $\eta(g)$ are transcendental numbers.

(iii) 1, $\theta(g)$, $\eta(g)$ are linearly independent over $\mathbb{Q}$ for all $g \geqq 1$.

Remark 1: (9) is admissible in the JP algorithm.

Remark 2: θ (1), η (1) are algebraic numbers of degree 3.

THEOREM 2. *Let $\psi_x(z)$ $(x \in \{a,b,c\})$ be analytic functions on $|z| > 1$ defined by $\psi_x(z) := \sum_{n\in\mathbb{N},\omega_n=x} z^{-n}$, where $\omega = \omega_1\,\omega_2\,\omega_3\cdots(\omega_n \in \{a,b,c\})$ is the fixed point in Theorem 1. Then we have the following statements:*

(i) The functions $\psi_a(z)$, $\psi_b(z)$, $\psi_c(z)$ are transcendental over $\mathbb{C}(z)$.

(ii) The functions $\psi_a(z)$, $\psi_b(z)$, $\psi_c(z)$ are linearly independent over $\mathbb{C}(z)$.

(iii) The admissible expression in the JPP algorithm for ${}^t(\zeta(z),\ \xi(z)) := {}^t(\psi_b(z)/\psi_a(z),\ (\psi_b(z)+\psi_c(z))/\psi_a(z))$ is given by

$$(10)\qquad \begin{pmatrix}\zeta(z)\\ \xi(z)\end{pmatrix} = \begin{bmatrix} 0; & z^{f_{-1}}, & z^{f_0}, & \cdots, & z^{f_{n-2}}, & \cdots \\ 0; & z^{f_0+f_{-1}}, & z^{f_1+f_0}, & z^{f_{n-1}+f_{n-2}}, & \cdots \end{bmatrix}.$$

Remark 3: If in the expression (9) g is replaced by a variable z, then (9) changes into a JPP expression, but this is not admissible in JPP algorithm.

THEOREM 3. *Let $g \geqq 2$ be an integer, and let τ be a morphism*

$$\tau(a_i) \in \{0, 1, \cdots, g-1\}^* - \{\lambda\} \ (1 \leqq i \leqq 3)$$

such that rank $(|\tau(a_i)|_j)_{1\leqq i\leqq 3,\ 0\leqq j\leqq g-1} \geqq 2$,

where a_1 (resp. a_2, a_3) is the symbol a (resp. b, c), and $|u|_j$ denotes the number of times the symbol j appears in the word u. Then the number defined by the g-adic expansion $0.\tau(\omega)$ is transcendental.

4. The proofs of the main results.

We first prove Theorem 1, (i), and Theorem 2, (iii) except for the admissibility of (10), which will be shown in the last paragraph.

Let us denote by B_n and P_n the 3×3 matrices given by

(11)
$$B_n = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & b_n \\ 0 & 1 & c_n \end{pmatrix} \in \mathrm{SL}(3; \mathbb{L}),\ P_n = B_1 B_2 \cdots B_n \ (n \geqq 1),\ P_0 = E,$$

where E is the 3×3 unit matrix. We can set

$$P_n = \begin{pmatrix} p_{n-2} & p_{n-1} & p_n \\ q_{n-2} & q_{n-1} & q_n \\ r_{n-2} & r_{n-1} & r_n \end{pmatrix} \in \mathrm{SL}(3; \mathbb{L}). \tag{12}$$

Then, we have the following well-known formula: (for completeness, we give a short proof of Lemma 1 following Nikišin and Sorokin [14], cf. Bernstein [3], Chap. 1, §3.)

LEMMA 1. *Let $\underline{\theta} \in \mathbb{L}^2$ have the expression (4) (not necessarily admissible) with*

$\underline{\theta}_m = {}^t(\theta_m, \eta_m)$, $\underline{b}_n = {}^t(b_n, c_n) \in \mathbb{L}^2$ $(0 \leqq n \leqq m)$ *such that* $b_0 = c_0 = 0$. *Then*

$$\underline{\theta} = \begin{pmatrix} (\theta_m p_{m-2} + \eta_m p_{m-1} + p_m)/(\theta_m r_{m-2} + \eta_m r_{m-1} + r_m) \\ \\ (\theta_m q_{m-2} + \eta_m q_{m-1} + q_m)/(\theta_m r_{m-2} + \eta_m r_{m-1} + r_m) \end{pmatrix}$$

in particular, $\underline{\pi}_m(\underline{\theta}) := [\underline{0}; \underline{b}_1, \underline{b}_2, \cdots, \underline{b}_m] = {}^t(p_m/r_m, q_m/r_m)$

Proof. Let $P^2(\mathbb{L}) := (\mathbb{L}^3 \setminus \{\underline{0}\})/\sim$ be the 2-dimensional projective space over L, i.e. the set of all equivalence classes of the elements of $\mathbb{L}^3$ with the relation $\sim$ defined by

$$\underline{\Theta} \sim \underline{\Phi}\ (\underline{\Theta},\ \underline{\Phi} \in \mathbb{L}^3) \text{ if and only if } \exists\psi \in \mathbb{L}\ -\ \{0\} \text{ such that } \underline{\Theta} = \psi\underline{\Phi}\ .$$

We denote by $\underline{\Theta}^\wedge$ the element of $P^2(\mathbb{L})$ which contains $\underline{\Theta}$. By $\nu : \mathbb{L}^2 \to P^2(\mathbb{L})$, and $\pi : P^2(\mathbb{L}) \to \mathbb{L}^2$, we denote the inclusion map, and the projection map given by

$$\nu({}^t(\theta\ ,\ \eta))\ :=\ {}^t(1\ ,\ \theta\ ,\ \eta)^\wedge,$$

$$\pi({}^t(\psi\ ,\ \theta\ ,\ \eta)^\wedge)\ :=\ {}^t(\theta/\psi\ ,\ \eta/\psi),$$

respectively. $A\ :\ \mathbb{L}^3\ \longrightarrow\ \mathbb{L}^3$ indicates the linear map over L for a given matrix $A \in GL(3;\mathbb{L})$ as usual. Then we can define the maps ${}^PA$, and ${}^\pi A$, which make the following diagram commutative:

$$\begin{tikzcd}
\mathbb{L}^3 \arrow[r, "\frown"] \arrow[d, "A"'] & P^2(\mathbb{L}) \arrow[r, equal] \arrow[d, "{}^PA"'] & \mathbb{L}^2 \arrow[d, "{}^\pi A"'] \\
\mathbb{L}^3 \arrow[r, "\frown"] & P^2(\mathbb{L}) \arrow[r, equal] & \mathbb{L}^2
\end{tikzcd}$$

where

$$P^2(\mathbb{L}) =\!=\!=\!= \mathbb{L}^2$$

stands for

$$P^2(\mathbb{L}) \underset{\nu}{\overset{\pi}{\rightleftharpoons}} \mathbb{L}^2.$$

Then we get ${}^\pi B(\underline{\theta}) = \pi\ {}^PB\nu(\underline{\theta}) := \pi({}^PB(\nu(\underline{\theta}))) = \underline{b} + T(\underline{\theta})$ ($\underline{\theta} \in \mathbb{L}^2$, $\underline{b} = {}^t(b, c)$) for

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}.$$

Therefore, by the commutativity of the diagram, it follows from (4) that

$\underline{\theta} =\ {}^\pi B_0\ {}^\pi B_1 \cdots\ {}^\pi B_{m-1}(\underline{b}_m + \underline{\theta}_m) =\ {}^\pi(B_0P_{m-1})(\underline{b}_m + \underline{\theta}_m)$, which implies the lemma. ■

Throughout this section, K denotes the set $\{a, b, c\} = \{a_1, a_2, a_3\}$ as in Theorem 3, σ is the substitution (2), $\omega = \omega_1\omega_2\omega_3 \cdots (\omega_n \in K)$ is the fixed point of σ, and f_n ($n \geqq -2$) is the sequence (3).

Since $\sigma^3(a) = abacaba = \sigma^2(a)\ \sigma(a)\ a$,

$$\sigma^n(a) = \sigma^{n-1}(a)\ \sigma^{n-2}(a)\ \sigma^{n-3}(a), \quad n \geqq 3, \tag{13}$$

and so

$$|\sigma^{n-1}(a)| \ = \ f_n, \quad n \geqq 1, \tag{14}$$

where $|w|$ denotes the length of a word w.

LEMMA 2. *Let v_n $(n \geqq 1)$ be the sequence of words in K^* defined by the recurrence relation*

$$\begin{aligned} v_{3m+1} &= v_{3m}cv_{3m-1}bv_{3m-2}, \\ v_{3m+2} &= v_{3m+1}av_{3m}cv_{3m-1}, \\ v_{3m+3} &= v_{3m+2}bv_{3m+1}av_{3m}, \ m \geqq 1, \end{aligned}$$

with initial conditions $v_1 = \lambda$, $v_2 = a$, $v_3 = aba$. Then v_n is a prefix of ω with $|v_n| = f_n - 1$.

Proof

Put $u_{3m+1} = v_{3m+1}a$, $u_{3m+2} = v_{3m+2}b$, $u_{3m+3} = v_{3m+3}c$ $(m \geqq 0)$. Then $u_{n+3} = u_{n+2}u_{n+1}u_n$ with $u_1 = a$, $u_2 = ab$, $u_3 = abac$, which together with (13) and (14) implies the lemma. ■

Let ${}^t(p_n^*, q_n^*, r_n^*)$ $(n \geqq -2)$ be the sequence in $(\mathbb{Z}[z])^3$ defined by

$$p_n^* = p_{n-1}^* + z^{f_{n-1}}p_{n-2}^* + z^{f_{n-1}+f_{n-2}}p_{n-3}^* \ , \ n \geqq 1, \tag{15}$$

and the same recurrences with q_n^*, r_n^* in place of p_n^* with initial conditions

$$\begin{pmatrix} p_{-2}^* & p_{-1}^* & p_0^* \\ q_{-2}^* & q_{-1}^* & q_0^* \\ r_{-2}^* & r_{-1}^* & r_0^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & z \end{pmatrix}. \tag{16}$$

For a given polynomial $p = p(z)$, we denote by ord p, and deg p the highest power of z that divides p, and the degree of p in z, respectively. Then we have the following

LEMMA 3.

$$f_{n-1} + \mathit{ord}\, p^*_{n-2} - \deg\, p^*_{n-1} = \begin{cases} 2 \ (n \equiv 0,\ 1\ (\mathit{mod.}\ 3)) \\ \\ 1\ (n \equiv 2\ (\mathit{mod.}\ 3)) \end{cases} , \ n \geqq 3,$$

$$f_{n-1} + \mathit{ord}\, q^*_{n-2} - \deg\, q^*_{n-1} = \begin{cases} 4\ (n \equiv 1\ (\mathit{mod.}\ 3)) \\ 3\ (n \equiv 2\ (\mathit{mod.}\ 3)) \\ 2\ (n \equiv 0\ (\mathit{mod.}\ 3)) \end{cases} , \ n \geqq 4,$$

$$f_{n-1} + \mathit{ord}\, r^*_{n-2} - \deg\, r^*_{n-1} = \begin{cases} 7\ (n \equiv 2\ (\mathit{mod.}\ 3)) \\ 6\ (n \equiv 0\ (\mathit{mod.}\ 3)) \\ 4\ (n \equiv 1\ (\mathit{mod.}\ 3)) \end{cases} , \ n \geqq 5.$$

Proof. By induction on n, we get

$$\text{ord } p^*_n = 1, \ \deg\ p^*_n = \begin{cases} f_n \quad (n \equiv 1\ (\text{mod. } 3)) \\ \\ f_{n-1}\ (n \equiv 2,\ 0\ (\text{mod. } 3)) \end{cases} , \ n \geqq 1,$$

$$\text{ord } q^*_n = 2, \ \deg\ q^*_n = \begin{cases} f_n - 1\ (n \equiv 1\ (\text{mod. } 3\ )) \\ f_n \qquad (n \equiv 2\ (\text{mod. } 3)) \\ f_n - 2\ (n \equiv 0\ (\text{mod. } 3)) \end{cases} , \ n \geqq 2,$$

$$\text{ord } r^*_n = 4, \ \deg\ r^*_n = \begin{cases} f_n - 3\ (n \equiv 1\ (\text{mod. } 3)) \\ f_n - 2\ (n \equiv 2\ (\text{mod. } 3)) \\ f_n \qquad (n \equiv 0\ (\text{mod. } 3)) \end{cases} , \ n \geqq 3.$$

These equalities imply the lemma. ■

In what follows, $\chi(w\ ;\ x)$ $(w \in K^* \cup K^\infty,\ x \in K)$ denotes the set $\{n \in \mathbb{N};\ w_n = x\}$ for $w = w_1 w_2 \cdots w_n \cdots \in K^*$ $(w_n \in K)$, and $\int_i D$ stands for the sequence $\{i + \sum_{m=1}^{n-1} d_m\}_{n=1,2,3\cdots}$ for a sequence $D = \{d_n\}_{n=1,2,3\cdots}$.

LEMMA 4. *Let τ_i $(1 \leqq i \leqq 3)$ be the coding defined by*

$$\begin{pmatrix} \tau_1(a_1) & \tau_1(a_2) & \tau_1(a_3) \\ \tau_2(a_1) & \tau_2(a_2) & \tau_2(a_3) \\ \tau_3(a_1) & \tau_3(a_2) & \tau_3(a_3) \end{pmatrix} := \begin{pmatrix} 2 & 2 & 1 \\ 4 & 3 & 2 \\ 7 & 6 & 4 \end{pmatrix}$$

Then $\chi\ (\omega\ ;\ a_i) = f_{\tau_i(a_3)}\ \tau_i(\omega)$ *for each* $1 \leqq i \leqq 3$.

Proof. It is clear that $\omega = \sigma(\omega_1)\sigma(\omega_2)\cdots\sigma(\omega_n)\cdots$, here, a_1 appears precisely once in the word $\sigma(\omega_n)$ as its prefix. Hence, noting $|\sigma(a_j)| = \tau_1(a_j)\ (1 \leqq j \leqq 3)$, we have the assertion when $i = 1$.

Let $i = 2$ or 3. It is clear that $\omega = \sigma^i(\omega_1)\sigma^i(\omega_2)\cdots\sigma^i(\omega_n)\cdots$, and $\sigma^i(a_j) = \sigma^{i-1}(a_1)\sigma^{i-1}(a_{j+1})\ (j = 1,\ 2)$, $\sigma^i(a_3) = \sigma^{i-1}(a_1)$, here, a_i occurs precisely once in the word $\sigma^{i-1}(a_1)$ as its suffix, and does not occur in the words $\sigma^{i-1}(a_{j+1})\ (j = 1,\ 2)$. Thus, noticing $|\sigma^i(a_j)| = \tau_i(a_j)\ (1 \leqq j \leqq 3)$, we get the assertion when $i = 2,\ 3$. ■

We denote by $z{}^\wedge D$ the power series (resp. polynomial) $\sum_{n\in D} z^n$ for a given subset (resp. finite subset) of $\mathbb{N}$.

LEMMA 5. *We have, for all* $n \geqq 1$,

$$P_n^* = z{}^\wedge\chi(\sigma^{n-1}(a); a),\ q_n^* = z{}^\wedge\chi(\sigma^{n-1}(a); b),\ r_n^* = z{}^\wedge\chi(\sigma^{n-1}(a); c).$$

Proof. We can write by Lemma 3 together with (15)

$$p_n^* = z{}^\wedge S_n,\ q_n^* = z{}^\wedge T_n,\ r_n^* = z{}^\wedge U_n$$

for suitable subsets $S_n,\ T_n,\ U_n$ of $\mathbb{N}$ such that

$$|S_n| = |S_{n-1}| + |S_{n-2}| + |S_{n-3}|,\ n \geqq 4$$

and the same recurrences with $|T_n|$, $|U_n|$ in place of $|S_n|$ hold. Here $|D|$ denotes the number of elements of a finite set D. Checking their initial terms, we get $|S_n| = f_{n-1},\ |T_n| = f_{n-2},\ |U_n| = f_{n-3}\ (n \geqq 1)$. In view of Lemma 3 together with Lemma 2 and (15), we obtain Lemma 5 by Lemma 4. ■

Proof of Theorem 1, (i), and the identity (10) in Theorem 2.

We set

$$\begin{aligned}
p_n &= p_n(z) := z^{f_n} p_n^*(z^{-1}),\\
q_n &= q_n(z) := z^{f_n}(p_n^*(z^{-1}) + q_n^*(z^{-1})),\\
r_n &= r_n(z) := z^{f_n}(p_n^*(z^{-1}) + q_n^*(z^{-1}) + r_n^*(z^{-1})).
\end{aligned}$$

Then we get by (15)

$$p_n = z^{f_{n-2}+f_{n-3}} p_{n-1} + z^{f_{n-3}} p_{n-2} + p_{n-3}, \ n \geqq 1, \tag{17}$$

and the same recurrences with q_n and r_n in place of p_n. Let P_n $(n \geqq 0)$ be matrices defined by (12). Then, in view of (16) and (17), we see that P_n $(n \geqq 0)$ satisfy the relations in (11) with

$$b_n = z^{f_{n-3}}, \ c_n = z^{f_{n-2}+f_{n-3}}. \tag{18}$$

In what follows, $\underline{b}_n = \underline{b}_n(z)$ indicates the vector ${}^t(b_n, c_n)$ of polynomials given by (18), and $\chi(w;\ x,\ y,\ \cdots)$ $(w \in K^* \cup K^\infty\ ,\ x, y, \cdots \in K)$ denotes the set $\chi(w; x) \cup \chi(w; y) \cup \cdots$.

Noting that $\chi(\sigma^{n-1}(a); a, b, c) = \{n \in \mathbb{N};\ 1 \leqq n \leqq f_n\}$ by (14), we obtain the following identity by Lemma 1 together with Lemma 5:

$$\begin{aligned} &[0; \underline{b}_1, \underline{b}_2, \cdots, \underline{b}_n] = {}^t(p_n/r_n,\ q_n/r_n) \\ &= \frac{z-1}{1-z^{-f_n}} . {}^t\Big(\sum_{m \in \chi(\sigma^{n-1}(a); a)} z^{-m},\ \sum_{m \in \chi(\sigma^{n-1}(a); a, b)} z^{-m}\Big), \end{aligned} \tag{19}$$

which converges to $(z-1).{}^t((z^{-1})^\wedge\chi(\omega; a),\ (z^{-1})^\wedge\chi(\omega\ ;\ a, b))$ in $\mathbb{L}$ as $n \to \infty$.

Therefore, setting $z = g$ $(2 \leqq g \in \mathbb{N})$, we obtain Theorem 1, (i). By taking T^{-1}, and then subtracting $\underline{b}_1 = {}^t(1,1)$ from both sides of (19), we get the identity (10). ■

Since $\| (z-1)\ {}^t((z^{-1})^\wedge\chi(\omega\ ;\ a) \| = 1$, $[0; \underline{b}_1, \underline{b}_2, \cdots, \underline{b}_n, \cdots]$ is not admissible in the JPP algorithm, see Remark 3.

Proof of Theorem 1, (ii). As a special case of Fraenkel [10], any nonnegative integer has precisely one representation as a sum of distinct numbers in $F := \{f_n;\ n \geqq 1\}$ such that the sum contains no three consecutive terms. If

$$n = \sum_{j=1}^{K} f_{i_j} \ (1 \leqq i_1 < i_2 < \cdots < i_k) \tag{20}$$

contains no three consecutive terms of F in the sum, then (20) will be called the canonical representation of n (in base F).

LEMMA 6. *Let $n \geqq 1$ have canonical representation (20). Then $\omega_n = \omega_{f_{i_1}}$.*

Proof. The canonical representation (20) of $n \geqq 1$ is given by the greedy algorithm, namely,

(i) $f_{i_1} = \max\{f \in F;\ f \leqq n\}$,

(ii) if $f_{i_1} + f_{i_2} + \cdots + f_{i_j} < n$ $(j \geqq 1)$, then $f_{i_{j+1}} = \max\{f \in F; f_{i_1} + f_{i_2} + \cdots + f_{i_j} + f \leqq n\}$, see Shallit [19], Fraenkel [10], Knuth [12]. Then, noting (13) and (14), we have

$$\omega_n = \omega_{f_{i_1}+f_{i_2}+\cdots+f_{i_k}} = \omega_{f_{i_1}+f_{i_2}+\cdots+f_{i_{k-1}}} = \cdots = \omega_{f_{i_1}}. \blacksquare$$

LEMMA 7. *We have*

(i) $\omega_1\,\omega_2\,\cdots\,\omega_{f_n} = \omega_{f_n+1}\,\omega_{f_n+2}\cdots\omega_{2f_n}$, $n \geqq 5$,

(ii) $\omega_1\,\omega_2\,\cdots\,\omega_{f_n-3} = \omega_{2f_n+1}\,\omega_{2f_n+2}\cdots\omega_{2f_n+f_{n-3}}$, $n \geqq 8$.

Proof. In view of (2) and (14), we have

$$\omega_{f_j} = \begin{cases} a & (j \equiv 1 \ (\mathrm{mod.}3)) \\ b & (j \equiv 2 \ (\mathrm{mod.}3)) \\ c & (j \equiv 0 \ (\mathrm{mod.}3)) \end{cases}, \quad j \geqq 1. \tag{21}$$

Lemma 6 implies

$$\omega_m = \omega_{f_n+m}, \quad 1 \leqq m \leqq f_{n-1} + f_{n-2} - 1 \ (n \geqq 2),$$

and so

$$\omega_1\,\omega_2\,\cdots\,\omega_{f_{n-1}+f_{n-2}-1} = \omega_{f_n+1}\,\omega_{f_n+2}\,\cdots\,\omega_{f_n+f_{n-1}+f_{n-2}-1}. \tag{22}$$

It follows from Lemma 6 and (21) that $\omega_{f_{n-1}+f_{n-2}} = \omega_{f_{n+1}}$, which together with (22) implies

$$\omega_1\,\omega_2\,\cdots\,\omega_{f_{n-1}+f_{n-2}} = \omega_{f_n+1}\omega_{f_n+2}\,\cdots\,\omega_{f_{n+1}}, \quad n \geqq 3.$$

We have

$$\begin{aligned} &\omega_{f_{n-1}+f_{n-2}+1}\,\omega_{f_{n-1}+f_{n-2}+2}\,\cdots\,\omega_{f_n-1} \\ &= \omega_{f_{n-1}+f_{n-2}+1}\,\omega_{f_{n-1}+f_{n-2}+2}\,\cdots\,\omega_{f_{n-1}+f_{n-2}+(f_{n-3}-1)} \\ &= \omega_{f_{n+1}+1}\,\omega_{f_{n+1}+2}\,\cdots\,\omega_{f_{n+1}+(f_{n-3}-1)}, \quad n \geqq 5, \end{aligned}$$

and hence, we get

$$\omega_1\ \omega_2\ \cdots\ \omega_{f_n-1}\ = \omega_{f_n+1}\ \omega_{f_n+2}\ \cdots\ \omega_{f_{n+1}+(f_{n-3}-1)}\ ,$$

and so

$$\omega_1\ \omega_2\ \cdots\ \omega_{f_n}\ = \omega_{f_n+1}\ \omega_{f_n+2}\ \cdots\ \omega_{f_{n+1}+f_{n-3}}\ ,\ n \geqq 5.$$

Noticing $f_{n+1} + f_{n-3} = 2f_n$, we obtain (i).

Let $n \geqq 7$. By the same argument as above, we have

$$\begin{aligned}\omega_1\ \omega_2\ \cdots\ \omega_{f_{n-4}+f_{n-5}-1}\ &= \omega_{f_{n+1}+f_{n-3}+1}\ \omega_{f_{n+1}+f_{n-3}+2}\ \cdots\ \omega_{f_{n+1}+f_{n-3}+(f_{n-4}+f_{n-5}-1)}\\ &= \omega_{f_{n+1}+f_{n-3}+1}\ \omega_{f_{n+1}+f_{n-3}+2}\ \cdots\ \omega_{f_{n+1}+f_{n-2}-1}\ ,\end{aligned}$$

and $\omega_{f_{n-4}+f_{n-5}} = \omega_{f_{n+1}+f_{n-2}}$. Hence we get

$$\omega_1\ \omega_2\ \cdots\ \omega_{f_{n-4}+f_{n-5}}\ = \omega_{2f_n+1}\ \omega_{2f_n+2}\ \cdots\ \omega_{f_{n+1}+f_{n-2}}\ ,$$

and together with $\omega_{f_{n-4}+f_{n-5}+m}\ = \omega_{f_{n+1}+f_{n-2}+m}$ $(1 \leqq m \leqq f_{n-6} - 1,\ n \geqq 8)$ and $\omega_{f_{n-3}}\ = \omega_{f_{n+1}+f_{n-2}+f_{n-6}}\ = \omega_{2f_n+f_{n-3}}$ we obtain (ii). ■

By Lemma 7, we can show the following lemma, which together with Roth's theorem (see, for example, [14], pp. 40-45) leads to the statement Theorem 1, (ii).

LEMMA 8. *Let $\theta(g)$ and $\eta(g)$ $(g \geqq 2)$ be the numbers defined in Theorem 1, and let ε be an arbitrarily fixed positive number. Then the simultaneous rational approximations*

$$|\theta(g) - p/r| < r^{-\rho+\varepsilon}\ ,\ |\eta(g) - q/r| < r^{-\rho+\varepsilon}$$

are valid for infinitely many rational numbers p/r, q/r $(p,\ q,\ r \in \mathbb{Z})$, where $\rho = 2 + \alpha^{-3}$ with $\alpha^3 - \alpha^2 - \alpha - 1 = 0,\ \alpha > 1$.

Proof. We denote by $\overset{*}{\omega}_1\ \omega_2 \cdots \overset{*}{\omega}_k$ the infinite periodic word

$$\omega_1\omega_2\cdots\omega_k\omega_1\omega_2\cdots\omega_k\cdots$$

Let τ and υ be codings as in Theorem 1. Then we have by (19)

$$p_n(g)/r_n(g) = 0.\tau(\overset{*}{\omega}_1\ \omega_2\ \cdots\ \overset{*}{\omega}_{f_n}),\ q_n(g)/r_n(g) = 0.\upsilon(\overset{*}{\omega}_1\ \omega_2\ \cdots\ \overset{*}{\omega}_{f_n}).$$

Thus we get by Lemma 7 that

$$|\theta(g) - p_n(g)/r_n(g)| < g^{-2f_n - f_{n-3}}\ ,\ |\eta(g) - q_n(g)/r_n(g)| < g^{-2f_n - f_{n-3}}$$

hold for all sufficiently large n. It follows from the proof of Lemma 5 that

$$\lim_{n\to\infty} \chi(\sigma^{n-1}(a); a_i)/f_n = \alpha^{-i}\ (1 \leqq i \leqq 3),$$

which implies that ω is not an ultimately periodic word, since α is an irrational number of degree 3. Hence, $\theta(g)$ and $\eta(g)$ are irrational, and so the inequalities

$$|\theta(g) - \tilde{p}_n(g)/\tilde{r}_n(g)| < \tilde{r}_n^{-2-f_{n-3}/f_n}\ ,\ |\eta(g) - \tilde{q}_n(g)/\tilde{r}_n(g)| < \tilde{r}^{-2-f_{n-3}/f_n}$$

hold for infinitely many rational numbers $\tilde{p}_n(g)/\tilde{r}_n(g)$, and $\tilde{q}_n(g)/\tilde{r}_n(g)$, where $\tilde{p}_n(g)/\tilde{r}_n(g) = p_n(g)/r_n(g)$ and $\tilde{q}_n(g)/\tilde{r}_n(g) = q_n(g)/r_n(g)$, with $(\tilde{p}_n(g), \tilde{r}_n(g)) = 1$ and $(\tilde{q}_n(g), \tilde{r}_n(g)) = 1$. ■

Proof of Theorem 3. Using Lemma 7 and the following lemma, we get Theorem 3 in the same manner as the proof of Theorem 1, (ii). ■

LEMMA 9. *Let τ be as in Theorem 3. Then $\tau(\omega)$ is not an ultimately periodic word.*

Proof. Put $\tau(\omega) = t_1 t_2 \cdots t_j\ \cdots\ (t_j \in \{0, 1, \cdots, g-1\})$, and

$$N = N(n) := \sum_{i=1}^{3} f_{n-i}\ |\tau(a_i)|.$$

Then we have

$$|t_1 t_2 \cdots t_N|_j = \sum_{i=1}^{3} f_{n-i}\ |\tau(a_i)|_j\ \ (j \in \{0, 1, \cdots, g-1\}).$$

Hence we get

$$\lim_{n\to\infty} \frac{|t_1 t_2 \cdots t_{N(n)}|_j}{N(n)} = \sum_{i=1}^{3} \alpha^{-i}\ |\tau(a_i)|_j\ /\ \sum_{i=1}^{3} \alpha^{-i} |\tau(a_i)|,$$

which is irrational for some $0 \leqq j \leqq g-1$ when rank $(|\tau(a_i)|_j)_{1\leqq i\leqq 3,\ 1\leqq j\leqq g-1} \geqq 2$. Thus the lemma follows. ■

Proof of Theorem 2, (i). Since $\psi_x(z)$ $(x \in \{a, b, c\})$ are not rational functions, the assertion follows from the following

LEMMA 10. *(Fatou)*

If a power series with integral coefficients represents an algebraic function that is not a rational function, then its radius of convergence is smaller than one. (Cf. Fatou [9], p. p. 368-371, or Polya and Szegö [17], p.139, ♯167.)

Proof of Theorem 2, (ii).

LEMMA 11. *Let* $\underline{\theta} = {}^t(\theta, \eta) \in \mathbb{L}^2$ *and let* $\underline{\theta} = [\underline{0};\ \underline{b}_1, \underline{b}_2, \cdots]$ *be the JPP expression with* $\underline{b}_n \in (\mathbb{C}[z])^2$ $(n \geqq 1)$ *not necessarily admissible. Let* ${}^t(p_n/r_n, q_n/r_n)$ *be its nth convergent given in Lemma 1. Assume that the inequalities* $\| r_n\theta - p_n \| = o(1)$, *and* $\| r_n\eta - q_n \| = o(1)$ *hold. Then* 1, θ *and* η *are linearly independent over* $\mathbb{C}(z)$.

Proof. Suppose that $s\theta + t\eta + u = 0$ with $0 \neq (s, t, u) \in \mathbb{C}[z]$. Then we get $\| sp_n + tq_n + ur_n \| = o(1)$ by the assumption. Hence, $sp_n + tq_n + ur_n = 0$ $(\forall n \gg 1)$, so that $\det P_n = 0$ $(\forall n \gg 1)$. This contradicts the fact $P_n \in SL(3; \mathbb{C}[z])$. ■

Hence, it follows from (19) and Lemma 7 that 1, $\psi_a(z)$ and $\psi_a(z) + \psi_b(z)$ are linearly independent over $\mathbb{C}(z)$, which implies Theorem 2, (ii). ■

Proof of Theorem 1, (iii).

LEMMA 12. *Let* $\theta = {}^t(\theta, \eta) \in \mathbb{R}^2$ *and let* $\underline{\theta} = [\underline{0}; \underline{b}_1, \underline{b}_2, \cdots]$ *with* $\underline{b}_n = {}^t(b_n, c_n) \in \mathbb{Z}^2$ $(n \geqq 1)$ *not necessarily admissible in the JP algorithm. Let* ${}^t(p_n/r_n, q_n/r_n)$ *be its nth convergent determined by (8), (9). Assume that the inequalities* $r_n\theta - p_n = o(1)$, *and* $r_n\eta - q_n = o(1)$ *hold. Then* 1, θ, *and* η *are linearly independent over* $\mathbb{Q}$.

Proof. The lemma is shown in a manner parallel to that of the proof of Lemma 11. ■

In view of the proof of Lemma 8 and Lemma 12, we get Theorem 1, (iii) when $g \geqq 2$. The assertion is clear when $g = 1$. ■

It remains to prove only the admissibility results. We use the following lemma for showing the admissibility result in the JPP algorithm.

LEMMA 13. *(Parusnikov [16]) Let* $P_n \in SL(3; \mathbb{C}[z])$ *be the matrix (12) defined by (11) under conditions (7) with* $b_1 \neq 0$. *Then, for* $n \geqq 1$,

$$\deg p_n = \sum_{k=2}^{n} \deg c_k, \ \deg q_n = \deg b_1 + \sum_{k=2}^{n} \deg c_k, \ \deg r_n = \sum_{k=1}^{n} \deg c_k.$$

Proof. It is easily seen by induction on n. ■

Proof of Theorem 2, (iii). We have already shown the identity (10). We put

$$\begin{aligned} {}^t(\zeta_n^{(m)}, \xi_n^{(m)}) &:= [0; \underline{b}_m, \underline{b}_{m+1}, \underline{b}_{m+2}, \cdots, \underline{b}_n], \\ {}^t(\zeta^{(m)}, \xi^{(m)}) &:= [0; \underline{b}_m, \underline{b}_{m+1}, \underline{b}_{m+2}, \cdots], \\ \underline{b}_j = {}^t(b_j, c_j) &:= {}^t(z^{f_{j-3}}, z^{f_{j-2}+f_{j-3}}). \end{aligned}$$

Then we get $\| \zeta_n^{(m)} \| < e^{-1}$, $\| \xi_n^{(m)} \| < e^{-1}$ $(2 \leqq m \leqq n)$ by Lemma 13 and Lemma 1, since $0 \leqq \deg b_j < \deg c_j$ holds for all $j \geqq 2$. Letting $n \to \infty$, we obtain $\| \zeta^{(m)} \| < 1$ and $\| \xi^{(m)} \| < 1$ for all $m \geqq 2$, from which the admissibility of (10) in the JPP algorithm follows. ∎

The admissibility of (9) in the JP algorithm can be shown by using the following lemma, which corresponds to Lemma 13.

LEMMA 14. *Let $P_n \in SL(3;\mathbb{Z})$ be the matrix (12) defined by (11) under the conditions (8) with $b_n \geqq 1$. Then the following 12 inequalities hold:*

$$\begin{array}{ccccc} p_{n-1} & \leqq & p_n & \leqq & p_{n+1} \\ \wedge \| & & \wedge \| & & \wedge \| \\ q_{n-1} & \leqq & q_n & \leqq & q_{n+1} \\ \wedge \| & & \wedge \| & & \wedge \| \\ r_{n-1} & \leqq & r_n & \leqq & r_{n+1} \end{array} \qquad (n \geqq 1).$$

Proof. By induction on n. ∎

Proof of Remark 1. We use the same notation as in the proof of Theorem 2, (iii). It follows from Lemma 14 that $0 \leqq \zeta^{(m)}(g) \leqq 1$ and $0 \leqq \xi^{m)}(g) \leqq 1$ for all $m \geqq 1$. We suppose that $\zeta^{(m)}(g) = 0$ or 1; or $\xi^{(m)}(g) = 0$ or 1 for some $m \geqq 1$. Then the admissible expression of ${}^t(\zeta^{(m)}(g), \xi^{(m)}(g))$ in the JP algorithm terminates, which contradicts the fact that 1, $\zeta^{(m)}(g)$, $\xi^{(m)}(g)$ are linearly independent over $\mathbb{Q}$. ∎

5. Automata and The Fixed Point ω.

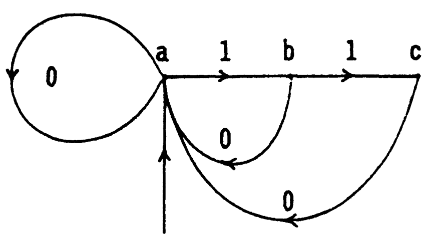
The g-adic expansions $\theta(g)$ and $\eta(g)$ given in Theorem 1, (i) can be considered as explicit ones, since their digits are described in terms of finite automata in the unified framework of Shallit [19] on automata, CDOL-systems, representations of numbers, and locally catenative formulae.

Let $\Sigma := \{0, 1\}$, and let $L \subset \Sigma^*$ be the set defined by

$$L := \{1w;\ w \in \Sigma^*\} \cup \{\lambda\} \ - \{v1^3w;\ v,\ w \in \Sigma^*\}.$$

We write $L = \{l_0, l_1, \cdots, l_n, \cdots\}$ in increasing order $\lambda = l_0 \prec l_1 \prec \cdots \prec l_n \prec \cdots$, where by $\prec$ we mean the lexicographic order with $0 \prec 1$ preceded by the length order. l_n can be considered as the canonical representation of n in base F in the sense that $\sharp\, l_n = n$, where $\sharp w$ $(w \in \Sigma^*)$ denotes the number defined by $\sum_{j=1}^{k} w_j f_j$ for $w = w_k w_{k-1} \cdots w_1$ $(w_j \in \Sigma)$, and f_n is the sequence (3).

Let $M = M_H := (K,\ \Sigma,\ \delta,\ a,\ H)$ be the finite automaton with set of states $K = \{a, b, c\}$, input alphabet $\Sigma = \{0, 1\}$, initial state a, set of final states $H \subset K$, and transition function δ defined by



(For precise definitions, see, for example, Hopcroft and Ullman [11]).

Then we have $L = L(M_K) \cap (1\Sigma^* \cup \{\lambda\}) \subset \Sigma^*$, where $L(M_H)$ denotes the language consisting of the words accepted by M_H, and

$$\omega = \delta(l_0)\delta(l_1)\cdots\delta(l_n)\cdots .$$

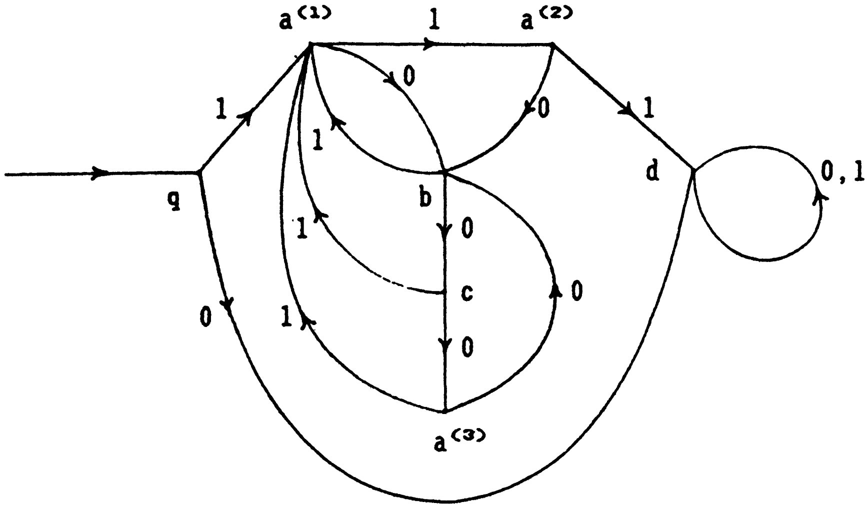
We have also

$$\omega = \mu(\delta_+(h_1))\mu(\delta_+(h_2))\cdots\mu(\delta_+(h_n))\cdots ,$$

where h_n is the binary expression of n, μ is the morphism defined by

$$\mu(a^{(j)}) = a\ (1 \leqq j \leqq 3),\ \mu(b) = b,\ \mu(c) = c,\ \mu(d) = \lambda,$$

and δ_+ is the transition function of a deterministic finite automaton $M_+ := (K_+,\ \Sigma,\ \delta_+,\ q,\ H_+)$ with $K_+ := \{q, a^{(1)}, a^{(2)}, a^{(3)}, b, c, d\} \supset H_+$ defined by



6. Conjectures.

Let $\psi_x(z)$ be as in Theorem 2. We can show that the irrationality measure $\rho(\psi_x(g))$ of $\psi_x(g)$ ($2 \leqq g \in \mathbb{Z}$) is not less than $2 + 1/(\alpha^3 - 1)$, where $\alpha > 1$ is the number satisfying $\alpha^3 - \alpha^2 - \alpha - 1 = 0$, cf. Lemma 8.

(i) $\rho(\psi_x(g)) = 2 + 1/(\alpha^3 - 1)$ for all $2 \leqq g \in \mathbb{Z}$, $x \in \{a, b, c\}$?

(ii) $\psi_x(\beta)$ ($x \in \{a, b, c\}$) is transcendental for all $\beta \in \overline{\mathbb{Q}}$ with $|\beta| > 1$.

(iii) $\psi_a(g)$ and $\psi_b(g)$ are algebraically independent for all $2 \leqq g \in \mathbb{Z}$.

(iv) $\psi_x(\beta_j)$ ($1 \leqq j \leqq n$) are algebraically independent for each $x \in \{a, b, c\}$ when $1, \beta_1, \cdots, \beta_n$ are linearly independent over $\mathbb{Q}$ with $|\beta_j| > 1$ ($1 \leqq j \leqq n$).

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