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KEN YAMAMURA

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par KEN YAMAMURA

RÉSUMÉ. Nous déterminons la structure du groupe de Galois $\mathrm{Gal}(K_{ur}/K)$ de l'extension maximale non ramifiée K_{ur} de chaque corps quadratique imaginaire de conducteur $\leqq 420$ ($\leqq 719$ sous GRH). Pour tous ces corps K, l'extension K_{ur} coïncide avec K, ou avec le corps de classes de Hilbert de K, ou avec le second corps de classes de Hilbert de K ou avec le troisième corps de classes de Hilbert de K. Les bornes d'Odlyzko sur les discriminants et les informations sur la structure des groupes de classes obtenues par l'action du groupe de Galois sur les groupes de classes sont ici essentielles. Nous utilisons aussi des relations sur le nombre de classes et un ordinateur pour le calcul du nombre de classes de corps de bas degré pour obtenir le nombre de classes de corps de degré plus élevé. Nous utilisons aussi des résultats sur les tours de corps de classes, ainsi que notre connaissance des 2-groupes d'ordre $\leqq 2^6$ et des groupes linéaires sur des corps finis.

ABSTRACT. We determine the structures of the Galois groups $\mathrm{Gal}(K_{ur}/K)$ of the maximal unramified extensions K_{ur} of imaginary quadratic number fields K of conductors $\leqq 420$ ($\leqq 719$ under the Generalized Riemann Hypothesis). For all such K, K_{ur} is K, the Hilbert class field of K, the second Hilbert class field of K, or the third Hilbert class field of K. The use of Odlyzko's discriminant bounds and information on the structure of class groups obtained by using the action of Galois groups on class groups is essential. We also use class number relations and a computer for calculation of class numbers of fields of low degrees in order to get class numbers of fields of higher degrees. Results on class field towers and the knowledge of the 2-groups of orders $\leqq 2^6$ and linear groups over finite fields are also used.

1. INTRODUCTION

Let K be an algebraic number field (of finite degree) and K_{ur} its maximal unramified extension. Then the Galois group $\mathrm{Gal}(K_{ur}/K)$ can be

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both finite and infinite and in general it is quite difficult to determine the structure of this group. If K has sufficiently small root discriminant, then $K_{ur} = K$, that is, K has no nontrivial unramified extension. This is the case, for example, for the imaginary quadratic number fields with class number one, the cyclotomic number fields with class number one, the real abelian number fields of prime power conductors $\leqq 67$ (see [57, Appendix]). For some fields K with small root discriminant, we can determine $\mathrm{Gal}(K_{ur}/K)$. The purpose of this paper is to determine the structure of $\mathrm{Gal}(K_{ur}/K)$ of imaginary quadratic number fields K of small conductors. For imaginary quadratic number fields K of conductors $\leqq 420$ ($\leqq 719$ under the Generalized Riemann Hypothesis (GRH)) we determine $\mathrm{Gal}(K_{ur}/K)$ and tabulate them for K with $K_{ur} \neq K_1$, where K_1 denotes the Hilbert class field of K. (If $K_{ur} = K_1$, then $\mathrm{Gal}(K_{ur}/K) = \mathrm{Gal}(K_1/K) \cong \mathrm{Cl}(K)$, the class group of K by class field theory.) For all such K, K_{ur} is one of K, K_1, K_2, or K_3, where K_2 (resp. K_3) is the second (resp. third) Hilbert class field of K. In other words, K_{ur} coincides with the top of the class field tower of K and the length of the tower is at most three. If possible, we give also simple expressions of K_1 and K_2. Also for $K = \mathbf{Q}(\sqrt{d})$ with $723 \leqq |d| < 1000$, we tabulate $\mathrm{Gal}(K_{ur}/K)$ except for some d.

From now on let $K = \mathbf{Q}(\sqrt{d})$ be an imaginary quadratic number field with discriminant $d < 0$. J. Martinet stated in [34] that if $|d| < 250$, then $K_{ur} = K_1$ except for 7 fields, for which he gave the structure of $\mathrm{Gal}(K_{ur}/K)$. (We note that $\mathrm{Gal}(K_{ur}/K) \cong H_{24}$ for $K = \mathbf{Q}(\sqrt{-248})$ in [34] is false[1].) He also stated that this fact is proved by using the methods which J. Masley [35] (and later F. J. van der Linden [50]) used for calculation of class numbers of real abelian number fields of small conductors. They used Odlyzko's discriminant bounds and information on the structure of class groups obtained by using the action of Galois groups on class groups. In addition to their methods, we use a computer for calculation of class numbers of fields of low degrees (we use KANT) and then use class number relations to get class numbers of fields of higher degrees (see §3). Results on class field towers [5, 25, 29, 31, 49] and the knowledge of the 2-groups of orders $\leqq 2^6$ [17] and linear groups over finite fields (see §4) are also used.

We know that if $|d| \leqq 499$ ($|d| \leqq 2003$ under GRH), then the degree $[K_{ur} : K]$ is finite (see §2). For these d, we want to determine $\mathrm{Gal}(K_{ur}/K)$. The key fact is that any unramified (finite) extension L of K has the same root discriminant as K: $rd_L = |d_L|^{1/[L:\mathbf{Q}]} = rd_K = \sqrt{|d|}$. Thus, if we have $rd_K < B(2N)$, where $B(2N)$ denotes the lower bound for the root

[1] The referee pointed out that this was corrected by Martinet in a supplement to his paper which he distributed along with the original reprint.

discriminants of the totally imaginary number fields of (finite) degrees $\geqq 2N$, then we get $[K_{ur} : K] < N$. We do not know the real values of $B(2N)$ (except for $N \leqq 4$), however, some lower bounds for $B(2N)$ are known. The best known unconditional lower bounds for $B(2N)$ can be found in the tables due to F. Diaz y Diaz [12]. If we assume the truth of GRH, much better lower bounds can be obtained. The best known conditional (GRH) lower bounds are found in the unpublished tables due to A. M. Odlyzko [38], which are copied in Martinet's expository paper [34]. Let K_l be the top of the class field tower of K: $K = K_0 \subsetneqq K_1 \subsetneqq K_2 \subsetneqq \cdots$ (K_{i+1} is the Hilbert class field of K_i), that is, l is the smallest number with $K_{l+1} = K_l$. If we cannot get $[K_{ur} : K_l] < 60$, which implies $K_{ur} = K_l$, from available lower bounds for $B(2N)$, we need to judge whether K_l has an unramified nonsolvable Galois extension and this is quite difficult. For the fields $\mathbf{Q}(\sqrt{-423})$ and $\mathbf{Q}(\sqrt{-723})$, we have $h(K_1) = 1$, that is, $l = 1$ and we cannot get $[K_{ur} : K_1] < 60$ from available lower bounds for $B(2N)$ (even under GRH for $\mathbf{Q}(\sqrt{-723})$). For $|d| \leqq 420$ ($|d| \leqq 719$ under GRH), we get $[K_{ur} : K_l] < 60$ and our main problem is to determine the degree $[K_l : \mathbf{Q}]$. In general, it is difficult to determine $[K_2 : \mathbf{Q}]$, because it is very hard to calculate the class number $h(K_1)$ of K_1. (Of course, for K with small $\mathrm{Cl}(K)$, we can calculate $h(K_1)$ with the help of a computer.) Now let K_g be the genus field of K, that is, the maximal unramified abelian extension of K which is abelian over $\mathbf{Q}$. If d is the discriminant of K and $d = d_1 d_2 \cdots d_t$ is the factorization of d into the product of fundamental prime discriminants, then $K_g = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2}, \dots, \sqrt{d_t})$, and we have

$$\mathbf{Q} \subset K \subseteqq K_g \subseteqq K_1 \subseteqq (K_g)_1 \subseteqq K_2,$$

which implies

$$[K_2 : \mathbf{Q}] = [K_2 : (K_g)_1][(K_g)_1 : \mathbf{Q}] = [K_2 : (K_g)_1] h(K_g)[K_g : \mathbf{Q}].$$

As K_g is a multi-quadratic number field, $h(K_g)$ can be calculated by the method in [51], and we may expect that $[K_2 : (K_g)_1]$ is small for fields we consider on the ground of the following proposition (§4, Proposition 2).

PROPOSITION. *Let L be the Hilbert class field of the genus field K_g of an imaginary abelian number field K. Then for any prime number p with $p \nmid [L : \mathbf{Q}]$, the p-class group $\mathrm{Cl}^{(p)}(L)$ of L is trivial or noncyclic.*

For all K with $|d| < 1000$ such that $h(K_g) > h(K)/[K_g : K]$, which is equivalent to $(K_g)_1 \supsetneqq K_1$, we have $K_2 = (K_g)_1$. For $h(K_g) > h(K)/[K_g : K]$, $h(K)$ must necessarily be even. Now $h(K)$ is even if and only if d has (at least) two distinct prime factors, however, for most K, this inequality holds. In fact, if a quadratic subfield $\neq K$ of K_g has class number divisible by an odd prime p, then we have $h(K_g) \geqq ph(K)/[K_g : K]$. Thus, the following problem arises naturally:

PROBLEM. *Characterize the imaginary quadratic number fields K with $K_2 = (K_g)_1 \neq K_1$.*

The author expects that this problem can be settled group-theoretically and a similar problem can be posed for real quadratic number fields. We will discuss this problem in the Appendix 2.

For 12 fields K with $|d| < 1000$ for which we have verified $K_{ur} \supsetneqq (K_g)_1$, there exists an S_4-extension M of $\mathbf{Q}$ such that K_gM/K_g is an unramified A_4-extension, where S_4 (resp. A_4) denotes the symmetric (resp. alternating) group of degree four, and therefore $K_{ur} \supseteqq (K_g)_1 M \supsetneqq (K_g)_1$. Except for $\mathbf{Q}(\sqrt{-856})$ and $\mathbf{Q}(\sqrt{-996})$, we can charaterize such K simply as $d_E \mid d$ for some quartic number field E: If the discriminant of d of K is divisible by the discriminant d_E of a quartic number field E, then K_g has an unramified A_4-extension. Then the normal closure of E is an S_4-extension of $\mathbf{Q}$ unramified at all finite primes over its quadratic subfield $\mathbf{Q}(\sqrt{d_E})$. Since $d_E \mid d$, $K\mathbf{Q}(\sqrt{d_E})/K$ is unramified by genus theory and therefore K_gM/K_g is an unramified A_4-extension. Also for the fields $\mathbf{Q}(\sqrt{-856})$ and $\mathbf{Q}(\sqrt{-996})$, we can find S_4-extensions of $\mathbf{Q}$ that give their unramified S_4-extensions by composition. (For details, see §7.) Therefore, data for quartic number fields are useful for our study. Thus, $K = \mathbf{Q}(\sqrt{d})$ with $|d| < 1000$, can be classified simply as follows:

$$
\left\{
\begin{array}{l}
d_E \nmid d \left\{
\begin{array}{l}
d \neq -856, -996 \left\{
\begin{array}{l}
h(K_g) = \frac{h(K)}{[K_g:K]} \left\{
\begin{array}{l}
h(K) = 1 \cdots K_{ur} = K \\
h(K) > 1 \cdots \quad l = 1
\end{array}
\right. \\
h(K_g) > \frac{h(K)}{[K_g:K]} \qquad\qquad\quad \cdots 2 \leqq l \leqq 3
\end{array}
\right. \\
d = -856, -996 \qquad\qquad\qquad\qquad\qquad\qquad\quad \cdots \quad l \geqq 3
\end{array}
\right. \\
d_E \mid d \left\{
\begin{array}{l}
d = d_E \left\{
\begin{array}{l}
\text{prime} \qquad\qquad\qquad\qquad\qquad \cdots K_{ur} = K_3 \\
\text{composite} \qquad\qquad\qquad\qquad \cdots \quad l \geqq 2
\end{array}
\right. \\
d = d' d_E \qquad\qquad\qquad\qquad\qquad\qquad\quad \cdots \quad l \geqq 3
\end{array}
\right.
\end{array}
\right.
$$

where the last four cases ($d = -856, -996$; $d = d_E$ prime; $d = d_E$ composite; $d = d'd_E$) satisfy $K_{ur} \supsetneqq (K_g)_1$.

Here, by "$d_E \mid d$" (resp. "$d_E \nmid d$") we mean "d is divisible by d_E for some quartic number field" (resp. "d is never divisible by the discriminant d_E for any quartic number field"), and in the factorization of $d = d'd_E$, d' denotes a fundamental quadratic discriminant. We expect that $K_{ur} = K_l$ holds for all K with $|d| < 1507$, because we expect that $\mathbf{Q}(\sqrt{-1507})$ is the first K having an unramified nonsolvable Galois extension (see below). This actually holds for all K with $|d| < 1000$ and $l \geqq 2$. We also expect that for all fields K with $d_E \nmid d \neq -856, -996$, $K_{ur} = (K_g)_1$ holds and that all inequalities for l are equalties. If our expectation is true, the classification above can be replaced by the following:

$$
\left\{
\begin{array}{l}
d_E \nmid d \left\{
\begin{array}{l}
d \neq -856, -996 \left\{
\begin{array}{l}
h(K_g) = \frac{h(K)}{[K_g:K]} \left\{
\begin{array}{l}
h(K) = 1 \cdots K_{ur} = K \\
h(K) > 1 \cdots K_{ur} = K_1
\end{array}
\right. \\
h(K_g) > \frac{h(K)}{[K_g:K]} \qquad \cdots K_{ur} = K_2
\end{array}
\right\} K_{ur} = (K_g)_1 \\
d = -856, -996 \qquad \cdots K_{ur} = K_3
\end{array}
\right. \\
d_E \mid d \left\{
\begin{array}{l}
d = d_E \left\{
\begin{array}{l}
\text{prime} \qquad \cdots K_{ur} = K_3 \\
\text{composite} \qquad \cdots K_{ur} = K_2
\end{array}
\right. \\
d = d' d_E \qquad \cdots K_{ur} = K_3
\end{array}
\right.
\end{array}
\right.
\qquad K_{ur} \supsetneqq (K_g)_1
$$

Though, there are possible exceptions, the author think that this classification is meaningful because this is complete for $|d| \leqq 719$ and even if an exception would occur for $|d| > 719$, modification is easy.

For most K we considered, $K_{ur} = K_l$ is verified. Thus, the following natural question arises: What is the first imaginary quadratic number field having an unramified nonsolvable Galois extension ? (What is the first K with $K_{ur} \neq K_l$?) Recent data for quintic number fields [3, 44] enable us to give a partial answer (§8, Proposition 8):

PROPOSITION. *The field* $\mathbf{Q}(\sqrt{-1507})$ *is the first imaginary quadratic number field having an unramified* A_5*-extension which is normal over* $\mathbf{Q}$ *in the sense that none of* $\mathbf{Q}(\sqrt{d})$ *of discriminant* d *with* $0 > d > -1507$ *has such an extension.*

We expect that the field $\mathbf{Q}(\sqrt{-1507})$ gives the answer to the question above.

For the determination of the structure of $\mathrm{Gal}(K_{ur}/K)$, the results on 2-class field towers due to H. Kisilevsky [25], F. Lemmermeyer [29, 31], and E. Benjamin, F. Lemmermeyer, and C. Snyder [5] are very helpful. They give us information on the structure of the Galois group $\mathrm{Gal}(K_2^{(2)}/K)$ of the second Hilbert 2-class field $K_2^{(2)}$ of K over K in many cases.

Now we explain the notations in our table. In the simple expressions of K_1 and K_2, α_i, β_i and γ_i denote any algebraic numbers generating the ith cubic number field of signature $(1,1)$, the ith quartic number field of signature $(2,1)$ with Galois group isomorphic to S_4, and the ith quintic number field of signature $(1,2)$ with Galois group isomorphic to D_5, respectively, where we consider that the number fields of each signature and each type (of Galois group of normal closure) are numbered up to conjugacy by absolute values of discriminants. (Here we do not need to consider nonisomorphic fields with same discriminants.)

G denotes the Galois group $\mathrm{Gal}(K_{ur}/K)$. As usual, C_n is the cyclic group of order n, V_4 is the four group, that is, $V_4 = C_2^2 = C_2 \times C_2$, D_n ($n \geqq 3$) is the dihedral group of order $2n$, Q_{4n} ($n \geqq 2$) is the generalized quaternion group of order $4n$, and SD_{8n} ($n \geqq 2$) is the semi-dihedral group of order $8n$:

$$\begin{aligned} D_n &= \langle\, a, b \mid a^n = b^2 = 1,\ b^{-1}ab = a^{-1}\,\rangle, \\ Q_{4n} &= \langle\, a, b \mid a^{2n} = 1,\ b^2 = a^n,\ b^{-1}ab = a^{-1}\,\rangle, \\ SD_{8n} &= \langle\, a, b \mid a^{4n} = b^2 = 1,\ b^{-1}ab = a^{2n-1}\,\rangle. \end{aligned}$$

I_n^{2m} ($m \geqq 2, n \geqq 3$) denotes the group of order $2mn$ given by

$$\langle\, a, b \mid a^{2m} = b^n = 1,\ a^{-1}ba = b^{-1}\,\rangle.$$

M_{2^n} ($n \geqq 4$) denotes the modular group of order 2^n given by

$$\langle\, a, b \mid a^{2^{n-1}} = b^2 = 1,\ b^{-1}ab = a^{2^{n-2}+1}\,\rangle.$$

$\widetilde{A_4}$ is the double cover of $A_4 : \widetilde{A_4} \cong \mathrm{SL}(2,3)$.

For some 2-groups we use designations given in the table by M. Hall and J. K. Senior [17]. We note that T. W. Sag and J. W. Wamsley give minimal presentations for all 2-groups of orders $\leqq 2^6$ [42]. For simplicity, for some of them we use the following designations used in [5]. $\Gamma^1_{m,t}$ denotes the group of order 2^{m+t+1} given by

$$\langle\, a, b \mid a^4 = b^{2^m} = 1,\ c = [a, b],\ a^2 = c^{2^{t-1}},\ [a, c] = c^2,\ [ab, c] = 1\,\rangle.$$

$\Gamma^2_{m,t}$ denotes the group of order 2^{m+t+1} given by

$$\langle\, a, b \mid a^4 = 1,\ c = [a, b],\ a^2 = b^{2^m} = c^{2^{t-1}},\ [a, c] = c^2,\ [ab, c] = 1\,\rangle.$$

We note that $\Gamma^2_{3,2} = 64\Gamma_3 p, \Gamma^1_{2,3} = 64\Gamma_8 c_2, \Gamma^1_{3,2} = 64\Gamma_3 n_2, \Gamma^2_{2,3} = 64\Gamma_8 e$.

The organization of this paper is as follows: In Section 2, we review how to obtain an upper bound for $[K_{ur} : K]$ by using discriminant bounds. In Section 3, we review some results on class number relations. In Section 4, we describe the information on the structure of class groups that can be obtained by considering the action of Galois groups on class groups. In Section 5, we review some results on class field towers. In Section 6, we describe how to determine $\mathrm{Gal}(K_{ur}/K)$ for selected values of d. In Section 7, we describe fields having an unramified extension not contained in $(K_g)_1$. In Section 8, we describe unramified nonsolvable Galois extensions of imaginary quadratic number fields. In the Appendix 1, we explain how to calculate class numbers of S_4-extensions of $\mathbf{Q}$. In the Appendix 2, we discuss the problem of characterizing quadratic number fields K with $K_2 = (K_g)_1 \neq K_1$.

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Table of imaginary quadratic number fields $K = \mathbf{Q}(\sqrt{d})$, $|d| \leqq 719$ with $K_{ur} \neq K_1$

$-d$	$\mathrm{Cl}(K)$	K_1	K_2	l	G
115	C_2	$K(\sqrt{5})$	$K_1(\alpha_1)$	2	D_3
120	V_4	$K(\sqrt{-3}, \sqrt{5})$	$K_1(\sqrt{(2\sqrt{2}+\sqrt{5})(2+\sqrt{5})})$	2	Q_8
155	C_4	$K(\sqrt{(-1+5\sqrt{5})/2})$	$K_1(\alpha_2)$	2	Q_{12}
184	C_4	$K(\sqrt{-3+4\sqrt{2}})$	$K_1(\alpha_1)$	2	Q_{12}
195	V_4	$K(\sqrt{-3}, \sqrt{5})$		2	Q_{16}
235	C_2	$K(\sqrt{5})$	$K_1(\gamma_1)$	2	D_5
248	C_8		$K_1(\alpha_2)$	2	I_3^8
255	$C_6 \times C_2$	$K(\sqrt{5}, \sqrt[3]{(9+\sqrt{85})/2})$	$K_1(\sqrt{(5+2\sqrt{-3})(2+\sqrt{5})})$	2	$Q_8 \times C_3$
260	$C_4 \times C_2$	$K(\sqrt{5}, \sqrt{8+\sqrt{65}})$		2	M_{16}
276	$C_4 \times C_2$	$K(\sqrt{-1}, \sqrt{13+8\sqrt{3}})$	$K_1(\alpha_1)$	2	$Q_{12} \times C_2$
280	V_4	$K(\sqrt{-7}, \sqrt{5})$		2	Q_{16}
283	C_3	$K(\alpha_{31})$	$K_1(\beta_1)$	3	$\widetilde{A_4}$
295	C_8		$K_1(\alpha_4)$	2	I_3^8
299	C_8		$K_1(\alpha_1)$	2	I_3^8
312	V_4	$K(\sqrt{-3}, \sqrt{2})$		2	Q_{16}
331	C_3	$K(\alpha_{36})$	$K_1(\beta_2)$	3	$\widetilde{A_4}$
340	V_4	$K(\sqrt{-1}, \sqrt{5})$		2	SD_{16}
355	C_4	$K(\sqrt{-3+4\sqrt{5}})$		2	Q_{28}
372	V_4	$K(\sqrt{-1}, \sqrt{-3})$	$K_1(\alpha_2)$	2	D_6
376	C_8		$K_1(\gamma_1)$	2	I_5^8
391	C_{14}		$K_1(\alpha_1)$	2	$D_3 \times C_7$
395	C_8		$K_1(\gamma_2)$	2	I_5^8
403	C_2	$K(\sqrt{13})$	$K_1(\alpha_2)$	2	D_3
408	V_4	$K(\sqrt{-3}, \sqrt{2})$	$K_1(\sqrt{-(5+\sqrt{17})/2})$	2	D_4
415	C_{10}	$K(\sqrt{5}, \gamma_{18})$	$K_1(\alpha_6)$	2	$D_3 \times C_5$
420	C_2^3	$K(\sqrt{-1}, \sqrt{-3}, \sqrt{5})$		2	$32\Gamma_4 c_3$

Continued (under GRH)

$-d$	$\mathrm{Cl}(K)$	K_1	K_2	l	G
435	V_4	$K(\sqrt{-3},\sqrt{5})$		2	$Q_{16} \ltimes C_3$
440	$C_6 \times C_2$	$K(\sqrt{2},\sqrt{5},\alpha_{50})$		2	$Q_{16} \times C_3$
455	$C_{10} \times C_2$	$K(\sqrt{-7},\sqrt{5},\gamma_{21})$		2	$Q_8 \times C_5$
472	C_6	$K(\sqrt{2},\alpha_4)$	$K_1(\alpha_4)$	2	$D_3 \times C_3$
483	V_4	$K(\sqrt{-3},\sqrt{-7})$	$K_1(\alpha_1)$	2	D_6
491	C_9		$K_1(\beta_3)$	3	$Q_8 \rtimes C_9$
515	C_6	$K(\sqrt{5},\alpha_{60})$	$K_1(\gamma_3)$	2	$D_5 \times C_3$
520	V_4	$K(\sqrt{-2},\sqrt{5})$		2	Q_{24}
527	C_{18}		$K_1(\alpha_2)$	2	$D_3 \times C_9$
535	C_{14}		$K_1(\alpha_9)$	2	$D_3 \times C_7$
552	$C_4 \times C_2$	$K(\sqrt{-3},\sqrt{-1+2\sqrt{6}})$	$K_1(\alpha_1)$	2	$Q_{12} \times C_2$
555	V_4	$K(\sqrt{-3},\sqrt{5})$		2	Q_{32}
563	C_9		$K_1(\beta_4)$	3	$Q_8 \rtimes C_9$
564	$C_4 \times C_2$	$K(\sqrt{-1},\sqrt{1+4\sqrt{3}})$	$K_1(\gamma_1)$	2	$Q_{20} \times C_2$
568	C_4	$K(\sqrt{-1+6\sqrt{2}})$		2	Q_{28}
580	$C_4 \times C_2$	$K(\sqrt{5},\sqrt{12+\sqrt{145}})$		2	$32\Gamma_3 f \ltimes C_3$
595	V_4	$K(\sqrt{-7},\sqrt{5})$		2	Q_{40}
611	C_{10}	$K(\sqrt{13},\gamma_{28})$	$K_1(\gamma_1)$	2	$D_5 \times C_5$
632	C_8		$K_1(\gamma_2)$	2	I_5^8
635	C_{10}	$K(\sqrt{5},\gamma_{31})$	$K_1(\gamma_5)$	2	$D_5 \times C_5$
643	C_3	$K(\alpha_{72})$	$K_1(\beta_5)$	3	$\widetilde{A_4}$
644	$C_8 \times C_2$		$K_1(\alpha_1)$	2	$D_3 \times C_8$
651	$C_4 \times C_2$	$K(\sqrt{-7},\sqrt{(13+\sqrt{217})/2})$	$K_1(\alpha_2)$	2	$D_3 \times C_4$
655	C_{12}	$K(\sqrt{7+6\sqrt{5}},\alpha_{75})$	$K_1(\gamma_6)$	2	$Q_{20} \times C_3$
660	C_2^3	$K(\sqrt{-1},\sqrt{-3},\sqrt{5})$		2	$64\Gamma_{15} f_2$
663	$C_8 \times C_2$			2	$\Gamma_{3,2}^2$
664	C_{10}	$K(\sqrt{2},\gamma_{32})$	$K_1(\alpha_6)$	2	$D_3 \times C_5$
667	C_4	$K(\sqrt{(-13+3\sqrt{29})/2})$	$K_1(\alpha_1)$	2	Q_{12}
680	$C_6 \times C_2$	$K(\sqrt{-2},\sqrt{5},\alpha_{80})$		2	$Q_{16} \times C_3$
687	C_{12}	$K(\sqrt{(11+\sqrt{229})/2},\alpha_{81})$		3	$(A_4 \rtimes C_4) \times C_3$
695	C_{24}		$K_1(\alpha_{13})$	2	$I_3^8 \times C_3$
696	$C_6 \times C_2$	$K(\sqrt{2},\sqrt[3]{99+13\sqrt{58}})$		2	$Q_{24} \times C_3$
708	V_4	$K(\sqrt{-1},\sqrt{-3})$	$K_1(\alpha_4)$	2	D_6
715	V_4	$K(\sqrt{-11},\sqrt{5})$		2	$Q_{16} \ltimes C_5$

Table of $\mathrm{Gal}(K_{ur}/K)$ for $723 \leqq |d| < 1000$ (not complete)

$-d$	$\mathrm{Cl}(K)$	K_1	K_2	l	G
723	C_4	$K(\sqrt{(-7+3\sqrt{241})/2})$		1	?
724	C_{10}	$K(\sqrt{-1}, \gamma_{37})$		1	C_{10}
727	C_{13}			1	C_{13}
728	$C_6 \times C_2$	$K(\sqrt{-7}, \sqrt{2}, \alpha_{86})$		2	$Q_{16} \times C_3$
731	C_{12}	$K(\sqrt{-5+2\sqrt{17}}, \alpha_{87})$		$\geqq 2$	?
739	C_5	$K(\gamma_{38})$		1	C_5
740	$C_8 \times C_2$			2	M_{32}
743	C_{21}			1	C_{21}
744	$C_6 \times C_2$	$K(\sqrt{-2}, \sqrt[3]{63+8\sqrt{62}})$	$K_1(\alpha_2)$	2	$D_3 \times C_6$
751	C_{15}	$K(\alpha_{91}, \gamma_{39})$	$K_1(\beta_8)$	3	$\widetilde{A_4} \times C_5$
755	C_{12}	$K(\sqrt{13+8\sqrt{5}}, \alpha_{92})$		2	$Q_{28} \times C_3$
759	$C_{12} \times C_2$	$K(\sqrt{-3}, \sqrt{(-5+\sqrt{69})/2}, \alpha_{94})$	$K_1(\alpha_1)$	2	$D_3 \times C_{12}$
760	V_4	$K(\sqrt{2}, \sqrt{5})$		2	$Q_{32} \ltimes C_3$
763	C_4	$K(\sqrt{(-9+\sqrt{109})/2})$		1	?
767	C_{22}		$K_1(\alpha_4)$	2	$D_3 \times C_{11}$
771	C_6	$K(\sqrt[3]{16+\sqrt{257}})$		$\geqq 3$	?
772	C_4	$K(\sqrt{(-7+\sqrt{193})/2})$		1	?
776	C_{20}	$K(\sqrt{(-5+\sqrt{97})/2}, \gamma_{40})$		1	C_{20}
779	C_{10}	$K(\sqrt{-19}, \gamma_{41})$		1	C_{10}
787	C_5	$K(\gamma_{42})$		1	?
788	C_{10}	$K(\sqrt{-1}, \gamma_{43})$		1	C_{10}
791	C_{32}			1	C_{32}
795	V_4	$K(\sqrt{-3}, \sqrt{5})$		2	Q_{40}
799	C_{16}		$K_1(\gamma_1)$	2	I_5^{16}
803	C_{10}	$K(\sqrt{-11}, \gamma_{44})$		1	C_{10}
804	$C_6 \times C_2$	$K(\sqrt{-1}, \sqrt[3]{22+2\sqrt{67}})$		1	$C_6 \times C_2$
807	C_{14}			1	C_{14}
808	C_6	$K(\sqrt{-2}, \alpha_{98})$		1	?
811	C_7			1	C_7
815	C_{30}	$K(\sqrt{5}, \alpha_{100}, \gamma_{45})$		1	C_{30}
820	$C_4 \times C_2$	$K(\sqrt{-1}, \sqrt{6+\sqrt{41}})$		2	$\Gamma_{2,3}^1$
823	C_9			1	C_9
824	C_{20}	$K(\sqrt{5+8\sqrt{2}}, \gamma_{46})$	$K_1(\gamma_3)$	2	$Q_{20} \times C_5$
827	C_7			1	C_7
831	C_{28}			1	C_{28}
835	C_6	$K(\sqrt{5}, \alpha_{102})$		2	$D_{11} \times C_3$
836	$C_{10} \times C_2$	$K(\sqrt{-1}, \sqrt{-11}, \gamma_{47})$		1	$C_{10} \times C_2$
839	C_{33}			1	C_{33}
840	C_2^3	$K(\sqrt{-2}, \sqrt{-3}, \sqrt{5})$		2	$32\Gamma_4 d$
843	C_6	$K(\sqrt[3]{(5+\sqrt{281})/2})$		1	?

Continued

$-d$	$\mathrm{Cl}(K)$	K_1	K_2	l	G
851	C_{10}	$K(\sqrt{-23}, \gamma_{49})$	$K_1(\alpha_4)$	2	$D_3 \times C_5$
852	$C_4 \times C_2$	$K(\sqrt{-1}, \sqrt{2+5\sqrt{3}})$		2	$Q_{28} \times C_2$
856	C_6	$K(\sqrt{2}, \alpha_{105})$	$K_1(\alpha_9)$	$\geqq 3$	?
859	C_7			1	C_7
863	C_{21}			1	C_{21}
868	$C_4 \times C_2$	$K(\sqrt{-1}, \sqrt{9+4\sqrt{7}})$	$K_1(\alpha_2)$	2	$Q_{12} \times C_2$
871	C_{22}			1	C_{22}
872	C_{10}	$K(\sqrt{-2}, \gamma_{50})$		1	C_{10}
879	C_{22}			1	C_{22}
883	C_3	$K(\alpha_{109})$		1	?
884	$C_8 \times C_2$			2	$\Gamma^1_{3,2}$
887	C_{29}			1	C_{29}
888	$C_6 \times C_2$	$K(\sqrt{2}, \sqrt[3]{43+5\sqrt{74}})$		2	$Q_{32} \times C_3$
895	C_{16}		$K_1(\gamma_9)$	2	I_5^{16}
899	C_{14}		$K_1(\alpha_2)$	2	$D_3 \times C_7$
903	$C_8 \times C_2$			1	$C_8 \times C_2$
904	C_8			1	?
907	C_3	$K(\alpha_{112})$		1	?
911	C_{31}			1	C_{31}
915	$C_4 \times C_2$	$K(\sqrt{-3}, \sqrt{(-1+\sqrt{61})/2})$		2	$\Gamma^2_{2,3}$
916	C_{10}	$K(\sqrt{-1}, \gamma_{52})$		$\geqq 3$	?
919	C_{19}			1	C_{19}
920	$C_{10} \times C_2$	$K(\sqrt{2}, \sqrt{5}, \gamma_{53})$		2	$(Q_{16} \ltimes C_3) \times C_5$
923	C_{10}	$K(\sqrt{13}, \gamma_{54})$		2	$D_7 \times C_5$
932	C_{12}	$K(\sqrt{(13+\sqrt{233})/2}, \alpha_{115})$		1	?
935	$C_{14} \times C_2$			2	$Q_{16} \times C_7$
939	C_8			1	?
943	C_{16}		$K_1(\alpha_4)$	2	I_3^{16}
947	C_5	$K(\gamma_{55})$		1	?
948	$C_6 \times C_2$	$K(\sqrt{-1}, \sqrt[3]{80+9\sqrt{79}})$	$K_1(\gamma_2)$	2	$D_5 \times C_6$
951	C_{26}			1	C_{26}
952	$C_4 \times C_2$	$K(\sqrt{-7}, \sqrt{9+10\sqrt{2}})$		$\geqq 2$	?
955	C_4	$K(\sqrt{(-9+13\sqrt{5})/2})$		2	Q_{52}
959	C_{36}			1	C_{36}
964	C_{12}	$K(\sqrt{(-15+\sqrt{241})/2}, \alpha_{119})$		1	?
967	C_{11}			1	C_{11}
971	C_{15}	$K(\alpha_{120}, \gamma_{57})$		1	?
979	C_8			1	?

Continued

$-d$	$\mathrm{Cl}(K)$	K_1	K_2	l	G
983	C_{27}			1	C_{27}
984	$C_6 \times C_2$	$K(\sqrt{2}, \sqrt[3]{9+\sqrt{82}})$		$\geqq 2$	?
987	$C_4 \times C_2$	$K(\sqrt{-3}, \sqrt{(-1+3\sqrt{21})/2})$		$\geqq 2$	?
991	C_{17}			1	C_{17}
995	C_8			2	I_9^8
996	$C_6 \times C_2$	$K(\sqrt{-1}, \sqrt[3]{82+9\sqrt{83}})$	$K_1(\alpha_2)$	$\geqq 3$	?

Supplements. We expect that all inequalities for l in our table are equalities. For this expectation we give the following supplemental data.

For $K = \mathbf{Q}(\sqrt{-731})$. $\mathrm{Gal}(K_2/K) \cong A_4 \times C_4$. If $K_{ur} \neq K_2$, $\mathrm{Gal}(K_{ur}/K_2) \cong C_2$, or C_4.

For $K = \mathbf{Q}(\sqrt{-771})$. $\mathrm{Gal}(K_3/K) \cong S_4 \times C_3$. If $K_{ur} \neq K_3$, $\mathrm{Gal}(K_{ur}/K_3) \cong C_2$, or C_4.

For $K = \mathbf{Q}(\sqrt{-856})$. $\mathrm{Gal}(K_2/K) \cong D_3 \times C_3$. K_2 has an unramified V_4-extension L which is an $S_4 \times C_3$-extension of K. If $K_{ur} \neq L$, then $[K_{ur} : L] = 2$, or 4.

For $K = \mathbf{Q}(\sqrt{-916})$. $\mathrm{Gal}(K_2/K) \cong D_3 \times C_5$. Put $L = K\mathbf{Q}(\sqrt{229})_{w\text{-}ur}$, where $\mathbf{Q}(\sqrt{229})_{w\text{-}ur}$ denotes the maximal extension of $\mathbf{Q}(\sqrt{229})$ unramified at all finite primes, which is an S_4-extension of $\mathbf{Q}$. (Cf. For $K = \mathbf{Q}(\sqrt{-687})$, $K_{ur} = K_1\mathbf{Q}(\sqrt{229})_{w\text{-}ur}$. See §7. The index "$w$-$ur$" means "weakly-unramified".) L is an unramified V_4-extension of K_2 which is an $S_4 \times C_5$-extension of K. If $K_{ur} \neq L$, then $[K_{ur} : L] = 2$, or 4.

For $K = \mathbf{Q}(\sqrt{-952})$. $\mathrm{Gal}(K_2/K) \cong \Gamma_{2,2}^2 \rtimes C_5$. (Note that $\Gamma_{2,2}^2 \cong C_4 \wr C_2 (= 32\Gamma_3 e.)$ If $K_{ur} \neq K_2$, $\mathrm{Gal}(K_{ur}/K_2) \cong C_2$.

For $K = \mathbf{Q}(\sqrt{-984})$. $\mathrm{Gal}(K_2/K) \cong Q_8 \times C_3$. If $K_{ur} \neq K_2$, $\mathrm{Gal}(K_{ur}/K_2) \cong C_3^2$, or C_5^2.

For $K = \mathbf{Q}(\sqrt{-987})$. $\mathrm{Gal}(K_2/K) \cong Q_{20} \times C_2$. If $K_{ur} \neq K_2$, $\mathrm{Gal}(K_{ur}/K_2) \cong C_2^4$.

For $K = \mathbf{Q}(\sqrt{-996})$. $\mathrm{Gal}(K_2/K) \cong D_3 \times C_6$. K_2 has an unramified V_4-extension L which is an $S_4 \times C_6$-extension of K. We have $[K_{ur} : L] \leqq 32$ and the odd part of $\mathrm{Cl}(L)$ is trivial or isomorphic to C_3^2, C_3^3, or C_5^2.

2. UPPER BOUND FOR $[K_{ur} : K]$

We review here how to obtain an upper bound for $[K_{ur} : K]$ for a given number field K by using discriminant bounds.

In this section K denotes an algebraic number field of finite degree. If we denote by n_K the degree of K, the n_Kth root of the absolute value of the discriminant d_K of K is called *root discriminant* of K and denoted by rd_K:

$$rd_K = |d_K|^{1/n_K}.$$

The following lemma is fundamental for our study.

LEMMA 1. *Let L/K be a finite extension of algebraic number fields. Then $rd_L = rd_K$ if and only if L/K is unramified at all finite primes.*

This can be easily proved by the transitive law of discriminants.

The following estimates for $rd = rd_L$ are known (see [39]): Unconditionally we have

$$\begin{aligned} rd &\geqq (4\pi e^{1+C})^{r_1/n}(4\pi e^{C})^{2r_2/n} - O(n^{-3/2}) \\ &= (60.8395\dots)^{r_1/n}(22.3816\dots)^{2r_2/n} - O(n^{-3/2}), \end{aligned}$$

as $n \to \infty$, where $C = 0.5772\dots$ denotes Euler's constant, and $n = n_L$ and r_1 (resp. r_2) denotes the number of real (resp. imaginary) primes of L. From this, for all imaginary quadratic number fields $K = \mathbf{Q}(\sqrt{d}), d < 0$ with $|d| \leqq 499$, $[K_{ur} : K] < \infty$. Under GRH we have

$$\begin{aligned} rd &\geqq (8\pi e^{C+\pi/2})^{r_1/n}(8\pi e^{C})^{2r_2/n} - O((\log n)^{-2}) \\ &= (215.3325\dots)^{r_1/n}(44.7632\dots)^{2r_2/n} - O((\log n)^{-2}), \end{aligned}$$

as $n \to \infty$. From this, for all $K = \mathbf{Q}(\sqrt{d}), d < 0$ with $|d| \leqq 2003$, $[K_{ur} : K] < \infty$.

By Lemma 1 we immediately obtain the following proposition, which tells us how to get an upper bound for $[K_{ur} : K]$.

PROPOSITION 1. *Let $B(n, r_1, r_2)$ ($\mathbf{N} \ni n = r_1 + 2r_2$, r_1, r_2 nonnegative integers) be the lower bound for the root discriminants of algebraic number fields F of finite degrees ($\geqq n$) such that $r_i(F)/n_L = r_i/n$ for $i = 1, 2$, where $r_1(F)$ (resp. $r_2(F)$) is the number of real (resp. imaginary) primes of F. Suppose that K has an unramified normal extension L of degree m.*

Let H be a positive integer.
(i) *If $rd_K < B(Hmn_K, Hmr_1(K), Hmr_2(K))$, then $[K_{ur} : L] < H$ and therefore $h(L) < H$. In particular, if $rd_K < B(2mn_K, 2mr_1(K), 2mr_2(K))$, then $K_{ur} = L$.*
(ii) *If $h(L) = 1$ and $rd_K < B(60mn_K, 60mr_1(K), 60mr_2(K))$, then $K_{ur} = L$.*

We note that the number 60 in (ii) is the minimal order of the finite nonsolvable groups, that is the order of A_5, the alternating group of degree five.

Thus, the knowledge of good lower bounds for $B(n, r_1, r_2)$ is very important for our study. The best known unconditional lower bounds for $B(n, r_1, r_2)$ can be found in the tables due to F. Diaz y Diaz [12]. If we assume GRH, much better lower bounds can be obtained. The best known conditional (GRH) lower bounds are found in the unpublished tables due to A. M. Odlyzko [38], which are copied in Martinet's expository paper [34].

All the imaginary quadratic number fields $K = \mathbf{Q}(\sqrt{d}), -d = 3, 4, 7, 8, 11, 19, 43, 67, 163$ satisfy the condition in (ii) for $L = K$. In fact by the table in [12] we have $rd_K \leqq \sqrt{163} < B(60 \cdot 2, 0, 60 \cdot 1)$. Therefore none of these fields has any nontrivial unramified extension.

Let K be an imaginary quadratic number field of small conductor. What unramified normal extension can we take as L when we apply Proposition 1 ? Since known lower bounds for $B(n, r_1, r_2)$ grow large as n grows large if r_2/n is fixed, we need to take an extension L of large degree. By the reason described in the Introduction it seems that the best choice is $L = (K_g)_1$, the Hilbert class field of the genus field K_g of K except when we can easily get an unramified extension of larger degree. In most cases we cannot conclude $K_{ur} = L$ only by (i) of Proposition 1, that is, we get only $[K_{ur} : L] < H$ for some $H \geqq 3$. Therefore we need to show $h(L) = 1$ by class number calculation or using criteria for class number divisibility.

3. Class number relations

In this section we review some results on class number relations on normal extensions of algebraic number fields. We will use them to calculate class numbers of fields of large degrees from class numbers of subfields.

Let L/K be a (finite) normal extension of algebraic number fields and G its Galois group. For any subgroup H of G, we denote by 1_H^G the induced character of G from the principal character of H. S. Kuroda [27] and R. Brauer [6] proved independently the following relation among the class numbers $h(M)$, the regulators $R(M)$, and the numbers $w_2(M)$ of roots of unity of 2-power orders of intermediate fields M of L/K: If we have a linear relation among 1_H^G

$$\sum_{H \leq G} a_H 1_H^G = 0,$$

then we have the following relation

$$(\dagger) \qquad \prod_{H \leq G} \left(\frac{h(L^H)R(L^H)}{w_2(L^H)} \right)^{a_H} = 1.$$

Here, L^H denotes the intermediate field of L/K corresponding to H by Galois theory.

We want to get $h(L)$ from the knowledge of $h(K)$ and $h(M)$ for intermediate fields M. When an arbitrary finite group G is given, there does not exist necessarily such a nontrivial (that is, $a_H \neq 0$ for some H) linear relation. Even if there exists a nontrivial linear relation, it is difficult to use ($\dagger$) in this form because it contains regulators. However, in some cases the product $\prod_H R(L^H)^{a_H}$ can be simplified into the form $[E_L : E]/q$ by elementary calculations, where E_L is the group of units in L, E is its subgroup generated by E_M for some subfields M of L, and q is some power of a prime, and moreover $[E_L : E]$ is also some power of the prime dividing q. Such a simplified relation is often very useful for calculation of class numbers of fields of high degrees.

Let $G = \langle a, b \mid a^{2n} = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ be the dihedral group of order $2n$ ($n \geqq 2$) (the four group if $n = 2$). Then we have the following relation:

$$1_{\langle 1 \rangle}^G + 2 \cdot 1_G^G = 1_{\langle b \rangle}^G + 1_{\langle a \rangle}^G + 1_{\langle ab \rangle}^G.$$

From this and some elementary calculations, we get the following lemmas.

LEMMA 2. ([27, 28]) *Let L/K be a V_4-extension of algebraic number fields and F, M and N its three intermediate fields. Then we have the following class number relation:*

$$h(L) = \frac{[E_L : E_F E_M E_N]}{2^{2+r+v-t}} \cdot \frac{h(F)h(M)h(N)}{h(K)^2},$$

where r is the $\mathbf{Z}$-rank of E_K, t is the number of infinite primes ramified in L/K, and $v \in \{0, 1\}$ (for the definition of v, see [28]). Moreover, the index $[E_L : E_F E_M E_N]$ is a nonnegative power of 2.

The special case $K = \mathbf{Q}$ and $L \supset \mathbf{Q}(\sqrt{-1})$ of this is a classical result due to Dirichlet, which we can consider as a prototype. By this lemma we can calculate class numbers of some fields of degree 24.

LEMMA 3. ([18, 19, 36]) *Let K be an imaginary quadratic number field or the rational number field $\mathbf{Q}$, and p an odd prime number. Let L be a D_p-extension of K. Let M and M' be any two intermediate fields of L/K with $[M : K] = [M' : K] = p$ and N the unique quadratic subextension of L/K. Then we have the following class number relation:*

$$h(L) = \frac{[E_L : E_M E_{M'} E_N]}{p^b} \cdot \frac{h(M)^2 h(N)}{h(K)^2},$$

where $b = 1$ if $K = \mathbf{Q}$ and L is imaginary, and $b = 2$ otherwise. Moreover, the index $[E_L : E_M E_{M'} E_N] = p^a$ with $0 \leqq a \leqq b$. Furthermore, if L/N is unramified, then $a = b - 1$, that is,

$$h(L) = \frac{1}{p} \cdot \frac{h(M)^2 h(N)}{h(K)^2}.$$

Let K be an imaginary quadratic number field whose p-class group is cyclic of order p. Then the Hilbert p-class field of K is a D_p-extension of $\mathbf{Q}$. If $p = 3$, or 5, by Lemma 3 we can compute its class number by calculating the class number of its subfield of degree p. For this purpose, we use data for cubic fields and quintic fields. Data for number fields of degrees n with $3 \leqq n \leqq 7$ and small discriminant (in absolute value) are available by anonymous ftp from megrez.math.u-bordeaux.fr (147.210.16.17).

LEMMA 4. ([9]) *Let L be an imaginary D_4-extension of $\mathbf{Q}$. Let M and N be the two nonisomorphic nonnormal quartic subfields and K its quadratic subfield such that L/K is cyclic. Then we have the following class number relation:*

$$h(L) = \frac{[E_L : E_M E_N E_K]}{2^v} \cdot h(M)h(N)h(K).$$

Here, $v = 3$ if L is a CM-field and $v = 2$ otherwise. Moreover, the index $[E_L : E_M E_N E_K]$ is a nonnegative power of 2.

If K is an imaginary quadratic number field with cyclic 2-class group of order $\geqq 4$, then by this lemma the class number of the unique unramified cyclic quartic extension L of K can be computed from the class numbers of two nonnormal quartic subfields of L. In this case, L is a D_4-extension of $\mathbf{Q}$

and all of its nonnormal quartic subfields have odd class number. Therefore we have

$$h(L) = h(M)h(N)h(K)/4,$$

because the exact 2-power dividing $h(L)$ is $h^{(2)}(K)/4$, where $h^{(2)}(K)$ is the 2-class number of K.

4. The action of Galois groups on class groups

The action of Galois groups on class groups can often be used to obtain useful information on the structure of class groups.

We first review the following, often called p-rank theorem.

LEMMA 5. (See [53, Theorem 10.8]) *Let L/K be a finite cyclic extension of degree n of algebraic number fields. Let p be a prime number with $p \nmid n$ and assume that all fields E with $K \subseteq E \subsetneqq L$ have trivial p-class group. Then for each positive integer a, the p^a-rank of the p-class group of L is a multiple of the order f of p modulo n. In particlar, if $p \mid h(L)$, then $p^f \mid h(L)$, and if $h(L) < p^f$, then $p \nmid h(L)$.*

The following proposition gives us more information in a sense.

PROPOSITION 2. *Let K be an imaginary abelian number field and K_g its genus field, that is, the maximal unramified abelian extension of K which is abelian over $\mathbf{Q}$. Let L be the Hilbert class field of K_g. Then for any prime number p with $p \nmid [L:\mathbf{Q}]$, the p-class group $\mathrm{Cl}^{(p)}(L)$ of L is trivial or noncyclic.*

Proof. We first note that both L and its Hilbert p-class field M are normal over $\mathbf{Q}$. The Galois group $\Gamma = \mathrm{Gal}(L/\mathbf{Q})$ acts by conjugation on $\mathrm{Gal}(M/L)$, which is isomorphic to $\mathrm{Cl}^{(p)}(L)$ by class field theory. This action induces a group homomorphism

$$\rho \;:\; \Gamma \to \mathrm{Aut}(\mathrm{Cl}^{(p)}(L)).$$

Now we assume that $\mathrm{Cl}^{(p)}(L)$ is cyclic. Then $\mathrm{Aut}(\mathrm{Cl}^{(p)}(L))$ is abelian and therefore the kernel of ρ contains the derived group of Γ. Hence if we let F be the field corresponding to $\mathrm{Ker}(\rho)$, F is abelian over $\mathbf{Q}$ and therefore F is contained in K_g. Since $\mathrm{Ker}(\rho) = \mathrm{Gal}(L/F)$ acts on $\mathrm{Gal}(M/L)$ trivially, by the assumption $p \nmid [L:\mathbf{Q}] = |\Gamma|$ we conclude that $\mathrm{Gal}(M/F)$ is factored as

$$\mathrm{Gal}(M/F) = \mathrm{Gal}(M/L) \times H, \quad \text{with } H \cong \mathrm{Ker}(\rho).$$

This implies that the class number of F is divisible by $|\mathrm{Gal}(M/L)| = |\mathrm{Cl}^{(p)}(L)|$ and therefore $|\mathrm{Cl}^{(p)}(L)| \mid h(K_g)$. Hence $\mathrm{Cl}^{(p)}(L)$ must be trivial.

Remark. As described in the Introduction, if K is an imaginary quadratic number field, we can easily calculate the class number of K_g. Thus, this will be a powerful tool.

The group homomorphism ρ above often gives a useful information on $\mathrm{Cl}^{(l)}(L)$ also for prime divisors l of $[L:\mathbf{Q}]$. The same argument works in more general situation. The referee pointed out that this proposition is a special case of a result due to O. Grün with proofs (essentially the same as above) by L. Holzer and A. Scholz (Jaresber. DMV **44** (1934), 74–75).

Let L/K be a finite Galois extension of algebraic number fields with Galois group Γ. Let p be a prime number and A the p-class group of L. The action of Γ on $V = A^{p^{a-1}}/A^{p^a}$ induces a group homomorphism

$$\Gamma \to \mathrm{Aut}(V) \cong \mathrm{GL}(r,p),$$

where r is the p^a-rank of A. Thus, the knowledge of the structure of linear groups over finite fields is often useful for the study of the structure of class groups. Later we use the following facts.

LEMMA 6. ([22, II, Satz 7.3a]) *Let p be a prime and f and n positive integers. $\mathrm{GL}(f,p^n)$ has a cyclic subgroup S of order $p^{fn}-1$. (S is called Singer cycle.) For any subgroup T of S whose order is not a divisor of $p^{en}-1$ for any $e<f$, its normalizer in $\mathrm{GL}(f,p^n)$ is the semi-direct product of S with a cyclic group C of order f such that the action of one generator of C on S is p^nth powering.*

Remark. Let q be a prime number $\neq p$ and f the order of p modulo q. Then any Sylow q-subgroup of $\mathrm{GL}(f,p)$ is a subgroup of a Singer cycle and its normalizer in $\mathrm{GL}(f,p)$ has order $f(p^f-1)$.

LEMMA 7. ([8]) *Let q be a power of an odd prime and W a Sylow 2-subgroup of $\mathrm{GL}(2,q)$. If $q \equiv 1 \pmod 4$, then W is isomorphic to $C_{2^s} \wr C_2$, the wreath product of C_{2^s} by C_2, where 2^s is the exact power of 2 dividing $q-1$, while if $q \equiv 3 \pmod 4$, then W is a semi-dihedral group of order 2^{t+2}, where 2^t is the exact power of 2 dividing $q+1$.*

Remark. By this lemma Sylow 2-subgroups of $\mathrm{GL}(2,3)$ and $\mathrm{GL}(2,5)$ are isomorphic to SD_{16} and $C_4 \wr C_2$ $(= 32\Gamma_3 e)$, respectively. We note that

$C_4 \wr C_2$ has three maximal subgroups isomorphic to $C_4^2, D_4 \curlyvee C_4$, and SD_{16}. Here, $D_4 \curlyvee C_4$ denotes the central product of D_4 and C_4.

Later for an unramified Galois extension L of an imaginary quadratic number field which is normal over $\mathbf{Q}$, we consider the action of the Galois group $\mathrm{Gal}(L/\mathbf{Q})$ on the class group $\mathrm{Cl}(L)$ of L. Then an important problem is to determine the image of the induced group homomorphism $\mathrm{Gal}(L/\mathbf{Q}) \to \mathrm{Aut}(\mathrm{Cl}(L))$. Thus, we prepare the following.

LEMMA 8. *Let K be a quadratic number field and L its finite extension which is normal over $\mathbf{Q}$. Suppose that L/K is unramified at all finite primes. Then the Galois group of $L/\mathbf{Q}$ is generated by elements of order two not contained in the Galois group of L/K. In particular,* $\mathrm{Gal}(L/\mathbf{Q})$ *is isomorphic to none of the 2-groups C_4, Q_8, SD_{16}, and $C_4 \wr C_2$.*

The former assertion of this is a special case of [11, Corollary 16.31], which is easily deduced from Čebotarev monodromy theorem. None of the 2-groups C_4, Q_8, SD_{16}, and $C_4 \wr C_2$ is generated by elements of order two. We note that if a quadratic number field has an unramified quaternionic extension which is normal over $\mathbf{Q}$, then its Galois group over $\mathbf{Q}$ is isomorphic to $D_4 \curlyvee C_4$ ([30]).

5. RESULTS ON CLASS FIELD TOWER

We also use results on class field tower. We review here some results. The following is well known.

LEMMA 9. (See [49, Theorem I].) *Let K be an algebraic number field of finite degree and p any prime number. If the p-class group, i.e., the p-part of the class group of K is cyclic, then the p-class group of the Hilbert p-class field of K is trivial. Moreover, if $p = 2$ and the 2-class group of K is isomorphic to V_4, then the 2-class group of the second Hilbert 2-class field, that is, the Hilbert 2-class field of the Hilbert 2-class field, of K is trivial.*

This can be easily proved by using group theory.

All fields we consider have cyclic p-class group for any odd prime p. In fact, a noncyclic p-class group for an odd prime p first occurs as $\mathrm{Cl}(\mathbf{Q}(\sqrt{-3299}))$, which is isomorphic to $C_9 \times C_3$ ([7]). On the other hand, noncyclic 2-class groups often occur as the 2-class group of $\mathbf{Q}(\sqrt{d}), d < 0$ with small $|d|$. But for all fields we consider, the 4-ranks of the class groups are at most one. The first example for bigger 4-rank is $d = -2379$.

In the rest of this section, we review some results on the 2-class field towers of imaginary quadratic number fields K:

$$K = K_0^{(2)} \subsetneqq K_1^{(2)} \subsetneqq K_2^{(2)} \subsetneqq \cdots \ (K_{i+1}^{(2)} \text{ is the Hilbert 2-class field of } K_i^{(2)}.)$$

By the lemma above, if the 2-class group $\mathrm{Cl}^{(2)}(K)$ of K is cyclic, then the 2-class field tower of K terminates with $K_1^{(2)}$, that is, $K_2^{(2)} = K_1^{(2)}$. When does $K_2^{(2)} = K_1^{(2)}$ hold? F. Lemmermeyer [29] gave a necessary condition: If $K_2^{(2)} = K_1^{(2)}$, then $\mathrm{Cl}^{(2)}(K)$ is cyclic, or isomorphic to $C_2 \times C_{2^m} (m \geqq 1)$. He also obtained sufficient criteria in terms of the factorization of d.

If $\mathrm{Cl}^{(2)}(K) \cong V_4$, then $K_3^{(2)} = K_2^{(2)}$ by the lemma above. We know that any nonabelian finite groups of order 2^n with $n > 2$ whose abelianization is isomorphic to V_4 is isomorphic to $D_{2^{n-1}}, Q_{2^n}$, or SD_{2^n}. Thus, if $\mathrm{Cl}^{(2)}(K) \cong V_4$, then $\mathrm{Gal}(K_2^{(2)}/K) \cong V_4, D_{2^{n-1}}, Q_{2^n}$, or SD_{2^n}, where n is determined by $2^{n-2} \parallel h(K_1^{(2)})$ (and therefore n can be easily calculated, because $K_1^{(2)} = K_g$). H. Kisilevsky [25] characterized the occurrence of these groups in terms of the factorization of the discriminant d of K.

When does $K_3^{(2)} = K_2^{(2)}$ hold? The complete answer has not been given yet. However, Lemmermeyer [29] determined when $\mathrm{Gal}(K_2^{(2)}/K)$ is metacyclic. (Of course, if $\mathrm{Gal}(K_2^{(2)}/K)$ is metacyclic, $K_3^{(2)} = K_2^{(2)}$ by Lemma 9.) He proved that if $\mathrm{Gal}(K_2^{(2)}/K)$ is nonabelian but metacyclic, then $\mathrm{Cl}^{(2)}(K) \cong V_4$, or d is of the form $-4pq$, where p and q are primes congruent to 5 modulo 8:

(i) If $(p/q) = 1$, then $\mathrm{Gal}(K_2^{(2)}/K)$ is isomorphic to the group given by

$$\langle \sigma, \tau \mid \tau^{2^{n+1}} = 1,\ \sigma^4 = \tau^{2^n},\ \sigma^{-1}\tau\sigma = \tau^{-1} \rangle,$$

or

$$\langle \rho, \sigma \mid \rho^{2^{n+1}} = \sigma^{2^m} = 1,\ \sigma^{-1}\rho\sigma = \rho^{-1} \rangle,$$

according as the norm of the fundamental unit of the real quadratic number field $\mathbf{Q}(\sqrt{pq})$ is -1 or 1. Here, n is determined by $2^n \parallel h(\mathbf{Q}(\sqrt{pq}))$ (in both cases), and m is determined by $2^{m+n} \parallel h(\mathbf{Q}(\sqrt{-1}, \sqrt{pq}))$. In [5], these groups are denoted by MC_n^- and $MC_{n,m}^+$, respectively. We use these notations in the Appendix 2.

(ii) If $(p/q) = -1$, then $\mathrm{Gal}(K_2^{(2)}/K) \cong M_{2^{m+2}}$, where m is determined by $2^{m+1} \parallel h(K)$. Note that the presentation of $\mathrm{Gal}(K_2^{(2)}/K)$ given in [29] is

$$\langle \rho, \sigma \mid \rho^2 = \sigma^{2^m},\ \rho^4 = 1,\ \sigma^{-1}\rho\sigma = \rho^{-1} \rangle,$$

and that if we put $\tau = \rho\sigma^{2^{m-1}}$, then $\sigma^{2^{m+1}} = \tau^2 = 1$ and $\tau^{-1}\sigma\tau = \sigma^{2^m+1}$.

Recently E. Benjamin, F. Lemmermeyer, and C. Snyder characterized K whose 2-class fields have cyclic 2-class group [5]. They proved that $\mathrm{Gal}(K_2^{(2)}/K)$ is nonmetacyclic and its derived group is cyclic, if and only if d is of one of the following forms: $d = -4pp'$, where p and p' are primes with $p \equiv 1, p' \equiv 5 \pmod 8$ and $(p/p') = 1$, $(p/p')_4(p'/p)_4 = -1$; $d = -rpp'$, where $-r$ is a negative discriminant $\neq -4$, and p and p' are positive prime discriminants with $(p/p') = (r/p) = 1$ and $(r/p') = (p/p')_4(p'/p)_4 = -1$. Moreover, $\mathrm{Gal}(K_2^{(2)}/K)$ is isomorphic to $\Gamma^1_{m,t}$ or $\Gamma^2_{m,t}$, according as $d = -4pp'$ (in this case $2^t = h^{(2)}(\mathbf{Q}(\sqrt{-4p}))$, or $d = -rpp'$ (in this case $2^t = h^{(2)}(\mathbf{Q}(\sqrt{-rp}))$, and in both cases $2^m = h^{(2)}(K)/2$.

6. DETERMINATION OF $\mathrm{Gal}(K_{ur}/K)$

In this section, we determine $\mathrm{Gal}(K_{ur}/K)$. Our procedure for each $K = \mathbf{Q}(\sqrt{d})$ is as follows:

(1) We take a normal unramified extension L of K of large degree for which we can obtain an upper bound for $[K_{ur} : L]$ less than 60 by discriminant bounds (unconditional for $|d| \leqq 420$ and conditional for $|d| > 420$).

(2) We show $h(L) = 1$ and conclude $K_{ur} = L$ by Proposition 1. (For fields with cyclic class group of odd order prime to 15, we can conclude $K_{ur} = K_1$ from $[K_{ur} : K_1] < 168$ (see below).)

(3) We determine the structure of $\mathrm{Gal}(K_{ur}/K)$. (This is not difficult for most fields.)

As described in §2, for most K, we take $L = (K_g)_1$. The exceptional cases are the following two cases:

(E1) d is divisible by the discriminant d_E of a quartic number field E and the quotient d/d_E is a fundamental quadratic discriminant or 1. In this case the normal closure M of E is an unramified A_4-extension of the quadratic number field $\mathbf{Q}(\sqrt{d_E})$ and the composite field KM is not contained in $(K_g)_1$. This case is divided into the following three subcases:

(a) $d = d_E = -p$, where p is a prime $\equiv 3 \pmod 4$: $d = -283, -331, -491, -563, -643, -751$. In this case, K has an unramified $\widetilde{A_4}$-extension, which yields a quaternionic extension of $K_1 = (K_g)_1$ by composition.

(b) $d = d_E$ is a composite: $d = -731 = 17 \cdot (-43)$. In this case, K has an unramified A_4-extension, which yields a V_4-extension of $K_1 = (K_g)_1$ by composition.

(c) $d = d'd_E$, where d' is a fundamental quadratic prime discriminant:

$d = -687 = (-3) \cdot 229, d = -771 = (-3) \cdot 257, d = -916 = (-4) \cdot 229$. In this case, K has an unramified S_4-extension, which yields a V_4-extension of $(K_g)_1$ by composition.
(E2) Though K does not satisfy the condition in (E1), we can check that K has an unramified S_4-extension: $d = -856, -996$.

These exceptional cases will be treated in the next section. In the rest of this section, we treat the other fields. Let $L = (K_g)_1$. Then we will show $K_{ur} = L = (K_g)_1$.

We can treat fields with same class group similarly. (Of course, for some fields, additional consideration is needed.) None of K with $h(K) = 1$ has any nontrivial unramified extension. For the fields K with $h(K) = 2$, we have determined $\mathrm{Gal}(K_{ur}/K)$ in [58].

We first treat fields with odd class number $\geqq 3$. Note that $K_g = K$ if and only if $h(K)$ is odd. We note that except for fields listed in (a) no fields with odd class number appear in our table as fields with $l \geqq 2$. For all K with odd class number and $|d| < 1000$ except for $d = -787, -827, -859, -883, -947, -967, -971, -991$, we get $[K_{ur} : K_1] < 60$ by discriminant bounds. Therefore our task is to prove $h(K_1) = 1$. We have the following.

PROPOSITION 3. *Let K be an imaginary quadratic number field with cyclic class group of odd order $n \geqq 7$. Assume that for all intermediate fields E of K_1/K, we have $E_1 = K_1$. Then if $h(K_1) > 1$ (this is equivalent to $K_2 \neq K_1$), we have $h(K_1) \geqq 2^6 = 64$. (This is valid also for real quadratic number fields. In the case where K is imaginary and $h(K)$ is a prime ($\geqq 7$), $h(K_1) \geqq 64$ can be improved to $h(K_1) \geqq 13^2 = 169$. See the remark after the proof.)*

Proof. Let l be a prime factor of $h(K) = n$. Then by the assumption there exists a subextension E of K_1/K such that $h(E) = l$ and $E_1 = K_1$. Therefore $l \nmid h(K_1)$ by Lemma 9.

Let p be a prime factor of $h(K_1)$ and r the p-rank of the p-class group of K_1. We will show $p^r \geqq 2^6 = 64$. Since by Lemma 5 and Proposition 2 we have $r \geqq 2$ and $p^r \equiv 1 \pmod{n}$, it suffices to eliminate the following four possibilities: (a) $p = 2, r = 3$ if $n = 7$; (b) $p = 2, r = 4$ if $n = 15$; (c) $p = 2, r = 5$ if $n = 31$; (d) $p = 3, r = 3$ if $n = 13$. Note that for all these possibilities, r is the order of p mod n.

Put $\Gamma = \mathrm{Gal}(K_1/\mathbf{Q}) \cong D_n$ and $\Lambda = \mathrm{Gal}(K_1/K) \cong C_n$. The action of Γ on $\mathrm{Cl}(K_1)/\mathrm{Cl}(K_1)^p \cong \mathbf{F}_p^r$ induces a group homomorphism

$$\Gamma \to \mathrm{GL}(r, p).$$

If $\Lambda \cap \mathrm{Ker}(\rho) \neq \{1\}$, then we would have $p^r \mid h(F)$ for the field F corresponding to $\Lambda \cap \mathrm{Ker}(\rho)$, which contradicts the assumption. (Cf. the proof of Proposition 2.) Therefore $\Lambda \cap \mathrm{Ker}(\rho) = \{1\}$ and $\rho(\Lambda) \cong \Lambda \cong C_n$. Since $\mathrm{Im}(\rho)$ is a factor group of $\Gamma \cong D_n$, $\mathrm{Im}(\rho) \cong D_n$. Since Λ is a normal subgroup of Γ, $\mathrm{Im}(\rho)$ is contained in the normalizer N of $\rho(\Lambda)$ in $\mathrm{GL}(r,p)$. Therefore N must have a subgroup of isomorphic to D_n.

Assume that n is a prime and that r is the order of p mod n. Then $\rho(\Lambda)$ is a subgroup of a Singer cycle of $\mathrm{GL}(r,p)$ satisfying the condition of Lemma 6. Therefore $N \cong C_{p^r-1} \rtimes C_r$. Since N has a subgorup of isomorphic to D_n, r must be even. This eliminates the posiibilities (a), (c), and (d).

We know $\mathrm{GL}(4,2) = \mathrm{PSL}(4,2) \cong A_8$ [22, II, Satz 6.14, (5)]. Since this group does not have a subgroup isomorphic to D_{15}, $r \neq 4$ if $p = 2$ and $n = 15$. This completes the proof.

Remark. Regrettably the same argument does not work for $h(K) = 5$: The Singer cycles of $\mathrm{GL}(4,2) \cong A_8$ have a subgroup isomorphic to D_5. However, the class number relation (Lemma 3) enables us to compute $h(K_1)$ by calculating class numbers of quintic subfields of K_1. For all K with $h(K) = 5$ (all such K have discriminant $\leqq 2683$ [2, 52]), we have $h(K_1) = 1$ and therefore $K_2 = K_1$. We note that we can find in [20] the class numbers of the unramified cyclic quintic extensions F of K with $5 \mid h(K)$ and $|d| < 1000$. For all such K, $h(F) = h(K)/5$. By Lemma 3 we can check that for all K with $3 \mid h(K)$ and $|d| < 1000$ except for d listed in (a), we have $h(F) = h(K)/3$, where F is the unique unramified cyclic cubic extension of K.

F. Hajir checked the parities of $h(K_1)$ of K with $\mathrm{Cl}(K) \cong C_q$ (q an odd prime $\leqq 19$) and $|d| < 15000$ by using elliptic units [16]: For all such K except for $d = -283$ ($q = 3$), -331 ($q = 3$), -643 ($q = 3$), and -14947 ($q = 17$), $h(K_1)$ is odd ($2^8 = 256 \mid h(K_1)$ for $d = -14947$). All K with $h(K) = 7$ have discriminant $\leqq 5923$ [2, 52] and therefore $h(K_1)$ is odd for all such K. By this reason in the case where (K is imaginary and) $h(K) = q$ (q an odd prime $\geqq 7$), $h(K_1) \geqq 64$ can be improved to $h(K_1) \geqq 13^2 = 169$. (Note that $\mathrm{GL}(2,13)$ has a subgroup isomorphic to D_7.) Moreover, in the case where $h(K) \geqq 11$ and $h(K) \neq 21, 63$, $h(K_1) \geqq 64$ can be improved to $h(K_1) \geqq 2^8 = 256$ (without assuming the imaginarity of K.) (Note that $\mathrm{GL}(8,2)$ has a subgroup isomorphic to D_{17}.)

Suppose $\mathrm{Cl}(K) \cong C_n$, where n is an odd integer prime to 15. The estimate $h(K_1) \geqq 169$ under the assumption $h(K_1) > 1$ is important. By this, if $[K_{ur} : K_1] < 168$, then we can conclude $K_{ur} = K_1$ under the assumption that K does not have an unramified A_5-extension which is normal over $\mathbf{Q}$. (Though there exist (probably infinitely) many imaginary quadratic number fields with cyclic class group of odd order prime to 15 having such an extension, the maximal discriminant of such fields is -2083. ($h(\mathbf{Q}(\sqrt{-2083}) = 7$. See §8.)) For this conclusion, we need only to show that K_1 does not have an unramified A_5-extension. (Note that the second minimal order of nonabelian simple groups is 168.) Suppose that K_1 has such an extension M. Then the Galois group $\mathrm{Gal}(M/K)$ is an extension of A_5 by C_n. Since any such extension is the direct product $A_5 \times C_n$, K has an unramified A_5-extension and this extension must be normal over $\mathbf{Q}$, for otherwise its normal closure has degree $2 \cdot 60^2 = 7200$. This contradicts the assumption. Thus, $[K_{ur} : K_1] < 168$ implies $K_{ur} = K_1$. For $K = \mathbf{Q}(\sqrt{-827}), \mathbf{Q}(\sqrt{-859}), \mathbf{Q}(\sqrt{-967}), \mathbf{Q}(\sqrt{-991})$, we do not get $[K_{ur} : K_1] < 60$ but get $[K_{ur} : K_1] < 168$ by Odlyzko's (conditional) discriminant bounds and therefore $K_{ur} = K_1$ (under GRH).

Now we treat fields K with even class number. For such K, $K_g \neq K$. First we calculate $h(K_g)$. We have two cases:

(i) $h(K_g) = h(K)/[K_g : K]$.
(ii) $h(K_g) > h(K)/[K_g : K]$.

The equality $h(K_g) = h(K)/[K_g : K]$ is equivalent to $(K_g)_1 = K_1$. (This holds trivially in the case of odd class number.) If $K_{ur} = K_1$, then we have $(K_g)_1 = K_1$. We expect the converse, that is, if $(K_g)_1 = K_1$, then $K_{ur} = K_1$ holds for fields considered. In fact, we can show this for most K.

We first treat fields with nontrivial cyclic 2-class group. For such K, its discriminant d is the product of two prime discriminants: $d = d_1 d_2$ and $K_g = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2})$. Since $\frac{1}{2}h(K) \mid h(K_g)$ but $h(K) \nmid h(K_g)$ by Lemma 9, we have $h(K_g) = h(K)h(\mathbf{Q}(\sqrt{d_1}))h(\mathbf{Q}(\sqrt{d_2}))/2$ by Lemma 2. (Note that since d_1 and d_2 are prime discriminants, both $h(\mathbf{Q}(\sqrt{d_1}))$ and $h(\mathbf{Q}(\sqrt{d_2}))$ are odd. More precisely, we have

$$\mathrm{Cl}(K_g) \cong \mathrm{Cl}(K)^2 \times \mathrm{Cl}(\mathbf{Q}(\sqrt{d_1})) \times \mathrm{Cl}(\mathbf{Q}(\sqrt{d_2})),$$

where $\mathrm{Cl}(K)^2 \neq \mathrm{Cl}(K) \times \mathrm{Cl}(K)$ but $\mathrm{Cl}(K)^2 = \{c^2 \mid c \in \mathrm{Cl}(K)\}$, because the odd part of the class group of a biquadratic bicyclic number field is isomorphic to the direct product of those of its quadratic subfields (see [28]). Hence in this case $h(K_g) = h(K)/[K_g : K]$ is equivalent to $h(\mathbf{Q}(\sqrt{d_1})) = h(\mathbf{Q}(\sqrt{d_2})) = 1$. We take d_2 as $d_2 > 0$. How often does $h(\mathbf{Q}(\sqrt{d_2})) = 1$ occur? H. Cohen and H. W. Lenstra, Jr. [10] posed heuristic conjectures (the so-called Cohen-Lenstra heuristics) on the distribution of class groups of algebraic number fields. Their conjectures state that the probability that a real quadratic number field with prime discriminant has class number one is about 0.75446 and this has been numerically supported by A. G. Stephens and H. C. Williams [48]. The first two discriminants of real quadratic number fields with prime discriminant and class number > 1 are 229 and 257. These are also discriminants of quartic number fields! The Cohen-Lenstra heuristics state also that the probability that an imaginary quadratic number field has class group whose odd part is cyclic is about 0.97757. Thus, for simplicity, we consider here fields with $h(\mathrm{Q}(\sqrt{d_2})) = 1$ and $\mathrm{Cl}(\mathrm{Q}(\sqrt{d_1})) \cong C_n$ (n an odd integer). Then $h(K_g) = h(K)/[K_g : K]$ is equivalent to $n = 1$. When $n = 1$, under some condition by considering Galois action as in Proposition 3, we get an estimate for $h(K_1)$. When $n > 1$, we can easily determine the structure of $\mathrm{Gal}((K_g)_1/K)$. We first give an estimate for $h(K_1)$ if $n = 1$ and then determine $\mathrm{Gal}((K_g)_1/K)$ if $n > 1$.

PROPOSITION 4. *Let K be an imaginary quadratic number field with non-trivial cyclic 2-class group.*
(i) *Assume* $\mathrm{Cl}(K) \cong C_{2^m}$ $(m \geqq 3)$ *and that the unique unramified cyclic quartic extension E of K has class number 2^{m-2}. (The latter assumption implies $h(K_g) = h(K)/2$.) If $h(K_1) > 1$, then we have $h(K_1) \geqq 7^2 = 49$.*
(ii) *Assume $h(K) = 2q$ (q an odd prime $\geqq 7$) and $h(K_g) = q$ $(= h(K)/2)$. If $h(K_1) > 1$, then we have $h(K_1) \geqq 2^6 = 64$.*
(These are valid also for real quadratic number fields if K_g is the unique unramified quadratic extension of K.)

Proof. We give only a sketch.
(i) Put $\Gamma = \mathrm{Gal}(K_1/\mathrm{Q})$. Then $\Gamma \cong D_{2^m}$. By Lemmas 7 and 8 any group homomorphisms $\Gamma \to \mathrm{GL}(2,3)$ and $\Gamma \to \mathrm{GL}(2,5)$ have image of order at most eight and this implies that neither $\mathrm{Cl}(K_1) \cong C_3^2$ nor $\mathrm{Cl}(K_1) \cong C_5^2$ can occur by the assumption. (Cf. the proof of Proposition 2. Note that the Sylow 2-subgroups of $\mathrm{GL}(2,7)$ are isomorphic to SD_{32} and they have a subgroup isomorphic to D_8.) Hence we get the conclusion.
(ii) Put $F = \mathrm{Q}(\sqrt{d_1})$. Then $\mathrm{Gal}(K_1/F) \cong D_q$. Thus, the same argument as in the proof of Proposition 3 works.

Remark. Note that when K has an unramified cyclic quartic extension E, we can calculate $h(E)$ by using Lemma 4. The fields with $\mathrm{Cl}(K) \cong C_{2^m}$ $(m \geqq 2)$ and $|d| < 1000$ are divided into two types: fields K with $h(E) = 2^{m-2}$ and fields K with $h(K_g) = 2^{m-1}q$ (q an odd prime). For most of the fields of the former type, we can easily check $K_{ur} = K_1$ by using the proposition above. For the fields of the latter type, we can check $K_{ur} = (K_g)_1$. Then we consider the Galois group.

Also when K has an unramified cyclic octic extension F, we can calculate $h(F)$ by using Lemma 2, however we must calculate the factorization of a rational prime in its nonnormal quartic subfield and find a suitable generator of a prime factor. (For the construction of F, see [54].) For example, let $K = \mathbf{Q}(\sqrt{-904})$. Then $\mathrm{Cl}(K) \cong C_8$. First we calculate the class number of the unique unramified cyclic quartic extension E of K. Its nonisomorphic nonnormal subfields are $\mathbf{Q}(\sqrt{9+4\sqrt{-2}})$ and $\mathbf{Q}(\sqrt{(9+\sqrt{113})/2})$. Both of them have class number one and therefore $h(E) = 2$. Put $\alpha = \sqrt{9+4\sqrt{-2}}$ and $N = \mathbf{Q}(\alpha)$. Then K_1 is a V_4-extension of N. To obtain $h(K_1)$, we calculate the class number of an intermediate field ($\neq E$) of K_1/N. By using KANT we get the factorization of the rational prime 113 in the ring of integers of N and calculate a generator of each prime factor. Then as such an intermediate field we can take $N(\sqrt{\beta})$, where $\beta = 104 + 48\sqrt{-2} - (235 + 108\sqrt{-2})\alpha$. KANT gives $h(N(\sqrt{\beta})) = 1$ and therefore $h(K_1) = 1$ by Lemma 2.

PROPOSITION 5. *Let K be an imaginary quadratic number field with cyclic class group of order 2^m ($m \geqq 2$). Assume that the class group of its genus field K_g is cyclic of order $2^{m-1}n$, where n is an odd integer $\geqq 3$. Then the Galois group of the Hilbert class field L of K_g over K is isomorphic to $I_n^{2^m}$:* $\mathrm{Gal}(L/K) \cong I_n^{2^m}$.

Proof. In this case, L is a cyclic extension of K_1 of degree n and therefore $\mathrm{Gal}(L/K)$ is isomorphic to a semi-direct product of C_n by C_{2^m}. Thus, it suffices to show that for each prime factor p of n, the action of $\mathrm{Gal}(K_1/K) \cong C_{2^m}$ on the Sylow p-subgroup of $\mathrm{Gal}(L/K_1)$, which is isomorphic to $\mathrm{Gal}(K_1F/K_1) \cong C_q$, is given by inversion, that is, $\mathrm{Gal}(K_1F/K) \cong I_q^{2^m}$, where F is the Hilbert p-class field of K_g and $q = [F : K_g]$. Since $\mathrm{Gal}(K_1F/K) \cong C_q \rtimes C_{2^m}$, we can present it as

$$\langle\, a, b \mid a^{2^m} = b^q = 1,\ a^{-1}ba = b^i \,\rangle$$

for some i with $q \nmid i$. By the assumption $\mathrm{Cl}(K_g) \cong C_{2^{m-1}n}$, a^2 must commute with b and therefore $i = \pm 1$. Since $\mathrm{Gal}(K_1F/K)$ is nonabelian, $i = -1$, that is, $\mathrm{Gal}(K_1F/K) \cong I_q^{2^m}$.

Remark. All K with $\mathrm{Cl}(K) \cong C_4$ satisfy $h(K_1) = 1$ or $h(K_g) = 2q$ (q an odd prime). (S. Arno [1] verified that the known list of imaginary quadratic number fields with class number four is complete. Therefore we can easily check this.) Note that $I_q^4 \cong Q_{4q}$.

Now we consider fields $K = \mathbf{Q}(\sqrt{d_1 d_2})$ with $2 \parallel h(K)$, $h(\mathbf{Q}(\sqrt{d_2})) = 1$ and $\mathrm{Cl}(F) \cong C_n$ (n an odd integer $\geqq 3$), where $F = \mathbf{Q}(\sqrt{d_1})$. For such K,

$\mathrm{Cl}(K_g) \cong \mathrm{Cl}_{\mathrm{odd}}(K) \times C_n$, where $\mathrm{Cl}_{\mathrm{odd}}(K)$ is the odd part of $\mathrm{Cl}(K)$ and $\mathrm{Gal}(KF_1/K) \cong \mathrm{Gal}(F_1/\mathbf{Q}) \cong D_n$. Hence $\mathrm{Gal}((K_g)_1/K) \cong \mathrm{Gal}(KF_1/K) \times \mathrm{Cl}_{\mathrm{odd}}(K) \cong D_n \times \mathrm{Cl}_{\mathrm{odd}}(K)$. Thus, groups of this form occur as G.

For the other fields, the 2-rank of the 2-class group is larger than or equal to 2, and the length of the 2-class field tower often becomes two. Therefore an important problem is to determine the Galois group $\mathrm{Gal}(K_2^{(2)}/K)$. For this purpose we do not have sufficient knowledge of the 2-class field tower. In fact, we cannot get the structure of $\mathrm{Gal}(K_2^{(2)}/K)$ for $K = \mathbf{Q}(\sqrt{-660})$ and $K = \mathbf{Q}(\sqrt{-840})$ without additional consideration. For some fields many computer calculations of class numbers are needed. In the rest of this section, we will describe details for selected values of d. For simplicity, we denote by $B(2N)$ the lower bound for the root discriminants of the totally imaginary number fields of (finite) degrees $\geqq 2N$ instead of $B(2N, 0, N)$.

$d = -420$. J. Martinet writes in [34] that R. Schoof communicated to him an unconditional proof for $[K_{ur} : \mathbf{Q}] = 64$, but that proof is not given there. The author was informed by Schoof with a complete proof that there had been a gap in the proof he had communicated to Martinet and that this gap was filled by F. Lemmermeyer a few years ago [43]. Recently in [31] he gives an unconditional proof for $[K_{ur} : \mathbf{Q}] = 64$. He also determines the structure of $\mathrm{Gal}(K_2^{(2)}/K)$ for some fields K with $\mathrm{Cl}(K) \cong C_2^3$ from which we get $\mathrm{Gal}(K_{ur}/K) \cong \Gamma_1 \cong 32\Gamma_4 c_3$ for $K = \mathbf{Q}(\sqrt{-420})$, where Γ_n is the designation used in [31] denoting the group of order 2^{n+4} given by

$$\begin{aligned} \langle\, \rho, \sigma, \tau \mid \rho^4 = \sigma^{2^{n+1}} = \tau^4 = 1,\ \rho^2 = \sigma^{2^n}\tau^2, \\ [\sigma,\tau] = 1,\ [\rho,\sigma] = \sigma^2,\ [\rho,\tau] = \rho^2 \,\rangle. \end{aligned}$$

We can give another proof for $\mathrm{Gal}(K_{ur}/K) \cong 32\Gamma_4 c_3$, but we do not give it here because the situation is similar to the case $K = \mathbf{Q}(\sqrt{-840})$.

$d = -660$. F. Lemmermeyer told the author the structure of $\mathrm{Gal}(K_{ur}/K)$ and sketch of his proof [32]. We give here another more computational proof.

We have $\mathrm{Cl}(K) \cong C_2^3$, $K_1 = \mathbf{Q}(\sqrt{-1}, \sqrt{-3}, \sqrt{-11}, \sqrt{5})$, and $h(K_1) = 8$. We will show $K_{ur} = K_2$. Since $rd_K = \sqrt{660} = 25.6904\ldots$ and $rd_K < B(2\cdot 8\cdot 8\cdot 4)$ ([38]), we have $[K_{ur} : K_2] \leqq 3$. By Proposition 2, $h(K_2) \neq 3$. Before proving $h(K_2) \neq 2$, we determine the structure of $\mathrm{Cl}(K_1)$ and $\mathrm{Gal}(K_2/K)$.

We first show that $\mathrm{Cl}(K_1) \cong C_4 \times C_2$. We have $\mathrm{Cl}(\mathbf{Q}(\sqrt{3}, \sqrt{-55})) \cong C_8 \times C_2$. Therefore $\mathrm{Cl}(K_1)$ contains an element of order at least four. Hence the 2-rank of $\mathrm{Cl}(K_1)$ is at most two, that is, $\mathrm{Cl}(K_1) \cong C_8$ or $C_4 \times C_2$. Assume $\mathrm{Cl}(K_1) \cong C_8$. Put $G = \mathrm{Gal}(K_2/K)$. Then G is a group of order

64 such that

$$G/G' \cong C_2^3,\ G' \cong C_8.$$

Therefore G belongs to the family Γ_8 or Γ_{19}. We note that any quadratic subextensions of K_1/K has class number eight or sixteen. In fact, we have

$$\begin{aligned} &\mathrm{Cl}(\mathbf{Q}(\sqrt{-1},\sqrt{165})) \cong \mathrm{Cl}(\mathbf{Q}(\sqrt{-3},\sqrt{55})) \\ \cong &\mathrm{Cl}(\mathbf{Q}(\sqrt{-5},\sqrt{33})) \cong \mathrm{Cl}(\mathbf{Q}(\sqrt{11},\sqrt{-15})) \cong C_4 \times C_2, \\ &\mathrm{Cl}(\mathbf{Q}(\sqrt{5},\sqrt{-33})) \cong C_4 \times V_4,\ \mathrm{Cl}(\mathbf{Q}(\sqrt{-11},\sqrt{15})) \cong C_2^3, \\ &\mathrm{Cl}(\mathbf{Q}(\sqrt{3},\sqrt{-55})) \cong C_8 \times C_2. \end{aligned}$$

Therefore G does not have an abelian subgroup of index two. Hence G must be isomorphic to $64\Gamma_{19}a_1$ or $64\Gamma_{19}a_2$. Now we put $H_1 = \mathrm{Gal}(K_2/\mathbf{Q}(\sqrt{3}, \sqrt{-55}))$. Then H_1 is a group of order 32 such that

$$H_1/H_1' \cong C_8 \times C_2,\ H_1' \cong C_2.$$

Therefore H_1 is isomorphic to $32\Gamma_2 j_1, 32\Gamma_2 j_2$, or $32\Gamma_2 k$. However, neither $64\Gamma_{19}a_1$ nor $64\Gamma_{19}a_2$ has a subgroup isomorphic to any one of these groups. This is a contradiction. Hence $\mathrm{Cl}(K_1) \cong C_4 \times C_2$.

Next, we determine $G = \mathrm{Gal}(K_2/K)$. G is a group of order 64 such that

$$G/G' \cong C_2^3,\ G' \cong C_4 \times C_2.$$

Therefore G belongs to the family $\Gamma_{14}, \Gamma_{15}, \Gamma_{16}$, or Γ_{24}. Since G has a subgroup isomorphic to $32\Gamma_2 j_1, 32\Gamma_2 j_2$, or $32\Gamma_2 k$, G belongs to the family Γ_{15}.

In order to know subgroups of G of order 32, we determine three such groups H_1, $H_2 = \mathrm{Gal}(K_2/\mathbf{Q}(\sqrt{5},\sqrt{-33}))$, and $H_3 = \mathrm{Gal}(K_2/\mathbf{Q}(\sqrt{-11}, \sqrt{15}))$.

First, we determine H_1 exactly. For this, we determine the class groups of the three intermediate fields of $K_1 = \mathbf{Q}(\sqrt{-1},\sqrt{-3},\sqrt{5},\sqrt{-11})/\mathbf{Q}(\sqrt{3}, \sqrt{-55})$, that is, of $\mathbf{Q}(\sqrt{-1},\sqrt{-3},\sqrt{-55})$, $\mathbf{Q}(\sqrt{3},\sqrt{-5},\sqrt{11})$, and $\mathbf{Q}(\sqrt{3},\sqrt{5}, \sqrt{-11})$. We can execute this by computer. We have

$$\mathrm{Cl}(\mathbf{Q}(\sqrt{-1},\sqrt{-3},\sqrt{-55})) \cong \mathrm{Cl}(\mathbf{Q}(\sqrt{3},\sqrt{-5},\sqrt{11})) \cong C_8 \times C_2,$$

and

$$\mathrm{Cl}(\mathbf{Q}(\sqrt{3},\sqrt{5},\sqrt{-11})) \cong C_4 \times V_4.$$

Thus, H_1 contains two subgroups isomorphic to $C_8 \times C_2$ and one subgroup isomorphic to $C_4 \times V_4$. Hence $H_1 \cong 32\Gamma_2 j_1$.

Next, we determine H_2. For this, we calculate the class groups of some quadratic subextensions of $K_1/\mathbf{Q}(\sqrt{5}, \sqrt{-33})$, that is, of $\mathbf{Q}(\sqrt{3 \pm 2\sqrt{5}}, \sqrt{-33})$. (Note that the Hilbert class field of $\mathbf{Q}(\sqrt{-55})$ is $\mathbf{Q}(\sqrt{-11}, \sqrt{3+2\sqrt{5}}) = \mathbf{Q}(\sqrt{-11}, \sqrt{3-2\sqrt{5}})$.) By computer calculation we have

$$\mathrm{Cl}(\mathbf{Q}(\sqrt{3+2\sqrt{5}}, \sqrt{-33})) \cong \mathrm{Cl}(\mathbf{Q}(\sqrt{3-2\sqrt{5}}, \sqrt{-33})) \cong C_4^2.$$

Therefore H_2 contains two subgroups isomorphic to C_4^2 and one subgroup isomorphic to $C_4 \times V_4$. Since H_2 is a group of order 32 such that

$$H_2/H_2' \cong C_4 \times V_4, \qquad H_2' \cong C_2,$$

we have $H_2 \cong 32\Gamma_2 f$.

Thus, G contains a subgroup isomorphic to $32\Gamma_2 j_1$ and one isomorphic to $32\Gamma_2 f$, and therefore $G \cong 64\Gamma_{15} f_1$ or $64\Gamma_{15} f_2$.

Now, we determine H_3. H_3 is a group of order 32 such that

$$H_3/H_3' \cong C_2^3, \ |H_3| = 4.$$

Since H_3 is a subgroup of $64\Gamma_{15} f_1$ or $64\Gamma_{15} f_2$, $H_3 \cong 32\Gamma_4 b_1$ or $32\Gamma_4 b_2$, according as $G \cong 64\Gamma_{15} f_1$ or $64\Gamma_{15} f_2$. To judge the structure of H_3, we consider its maximal subgroups. Among the seven maximal subgroups of $32\Gamma_4 b_1$ (resp. $32\Gamma_4 b_2$), three are (isomorphic to) $16\Gamma_2 a_1$, two are $16\Gamma_2 c_1$, and one is $16\Gamma_2 c_1$ (resp. one is $16\Gamma_2 a_2$, two are $16\Gamma_2 c_1$, and three are $16\Gamma_2 c_2$). They correspond to the quadratic subextensions of $K_2/\mathbf{Q}(\sqrt{-11}, \sqrt{15})$, that is, $\mathbf{Q}(\sqrt{3}, \sqrt{5}, \sqrt{-11}), \mathbf{Q}(\sqrt{-1}, \sqrt{-11}, \sqrt{-15})$, $\mathbf{Q}(\sqrt{-3}, \sqrt{-5}, \sqrt{-11}), \mathbf{Q}(\sqrt{(3 \pm \sqrt{-11})/2}, \sqrt{15}), \mathbf{Q}(\sqrt{-(3 \pm \sqrt{-11})/2}, \sqrt{15})$. (Note that the Hilbert class field of $\mathbf{Q}(\sqrt{-55})$ can be expressed as $\mathbf{Q}(\sqrt{5}, \sqrt{(3 \pm \sqrt{-11})/2}) = \mathbf{Q}(\sqrt{5}, \sqrt{-(3 \pm \sqrt{-11})/2})$.) By computer calculation we have

$$\mathrm{Cl}(\mathbf{Q}(\sqrt{(3 \pm \sqrt{-11})/2}, \sqrt{15})) \cong \mathrm{Cl}(\mathbf{Q}(\sqrt{-(3 \pm \sqrt{-11})/2}, \sqrt{15})) \cong C_4 \times C_2.$$

Noting that both of the abelianizations of $16\Gamma_2 a_1$ and $16\Gamma_2 a_2$ (resp. $16\Gamma_2 c_1$ and $16\Gamma_2 c_2$) are isomorphic to C_2^3 (resp. $C_4 \times C_2$), we conclude that H_3 contains at least four subgroups isomorphic to $16\Gamma_2 c_1$ or $16\Gamma_2 c_2$. Therefore $H_3 \cong 32\Gamma_4 b_2$. Hence $G \cong 64\Gamma_{15} f_2$.

Now we prove $h(K_2) \neq 2$. Assume $h(K_2) = 2$. If we denote by Γ the Galois group $\mathrm{Gal}(K_3/\mathbf{Q}(\sqrt{-1}, \sqrt{-3}, \sqrt{-55}))$. Then Γ is a group of order 32 such that

$$\Gamma/\Gamma' \cong C_8 \times C_2, \ \Gamma' \cong C_2.$$

Therefore Γ is isomorphic to $32\Gamma_2 j_1, 32\Gamma_2 j_2$, or $32\Gamma_2 k$. All of these groups have only abelian maximal subgroups. This contradicts that the maximal subgroup $\mathrm{Gal}(K_3/K_1)$ of Γ is nonabelian. Hence $h(K_2) \neq 2$. Thus, $K_{ur} = K_2$.

$d = -759$. We have $\mathrm{Cl}(K) \cong C_{12} \times C_2$, $K_g = \mathbf{Q}(\sqrt{-3}, \sqrt{-11}, \sqrt{-23})$, and $h(K_g) = 18$. Put $L = (K_g)_1$. We will show $K_{ur} = L$. Since $rd_K = \sqrt{759} = 27.5499\ldots$ and $rd_K < B(2 \cdot 4 \cdot 18 \cdot 5)$ ([38]), we have $[K_{ur} : L] \leqq 4$. We will show $h(L) = 1$.

First we show $h(L) \neq 3$. Put $F = \mathbf{Q}(\sqrt{-23})$. Then $h(F) = 3$ and $L = K_1 F_1$. Since $K_1^{(2)}/KF$ is cyclic (quartic), $K_1/K_1^{(3)}F$ is cyclic and therefore $L/K_1^{(3)}F_1$ is cyclic. Hence if the class number of the intermediate field $K_g K_1^{(3)} F_1$ of $L/K_1^{(3)}F_1$ is not divisible by 3, then the 3-rank of $\mathrm{Cl}(L)$ is a multiple of 2 by Lemma 5 and then we can conclude $h(L) \neq 3$. In order to show $3 \nmid h(K_g K_1^{(3)} F_1)$, it suffices to show $3 \parallel h(K_g K_1^{(3)})$. So we calculate $h(K_g K_1^{(3)})$. For this we calculate the class numbers of the intermediate fields $K_1^{(3)}F, K_1^{(3)}(\sqrt{-3}), K_1^{(3)}(\sqrt{-11})$ of the V_4-extension $K_g K_1^{(3)}/K_1^{(3)}$. $K_1^{(3)}F/F$ is a D_3-extension and its quadratic subextension is $KF = \mathbf{Q}(\sqrt{-23}, \sqrt{33})$. We have $h(KF) = 36$. We calculate the class number of a cubic subextension $F(\alpha_{94})$ of $K_1^{(3)}F/F$, where $\alpha_{94}^3 - \alpha_{94}^2 + 6\alpha_{94} - 3 = 0$. Put $\lambda = \alpha_{94} + \sqrt{-23}$. Then $F(\alpha_{94}) = \mathbf{Q}(\lambda)$ and $\lambda^6 - 2\lambda^5 + 82\lambda^4 - 110\lambda^3 + 1676\lambda^2 - 956\lambda^2 + 7047 = 0$. The function 'polred' of pari-gp gives a simpler generating polynomial: $\mathbf{Q}(\lambda) = \mathbf{Q}(\theta)$, where $\theta^6 - 3\theta^5 + \theta^4 + 3\theta^3 - 11\theta^2 + 9\theta + 27 = 0$. Then KANT gives $h(\mathbf{Q}(\theta)) = 3$. Hence by Lemma 3 we get $h(K_1^{(3)}F) = 12$ and therefore $\mathrm{Cl}(K_1^{(3)}F) \cong C_{12}$. Similarly we get $h(K_1^{(3)}(\sqrt{-3})) = h(K_1^{(3)}(\sqrt{-11})) = 4$. Thus, by Lemma 4 we conclude that $h(K_g K_1^{(3)})$ is 3 up to a 2-power. Hence $h(L) \neq 3$.

Next, we show $2 \nmid h(L)$. Assume that $2 \mid h(L)$. Since $\mathrm{Cl}(K_1^{(3)}F) \cong C_{12}$ and $K_1/K_1^{(3)}F$ is cyclic quartic, by Lemma 9 we have $2 \nmid h(K_1)$. Therefore since L/K_1 is cyclic cubic, $\mathrm{Cl}(L) \cong V_4$ by Lemma 6. Now we consider the action of $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(KF_1/K) \times C_{12} \cong D_3 \times C_{12}$ on $\mathrm{Cl}(L)$. This induces a group homomorphism

$$\rho \ : \ D_3 \times C_{12} \to \mathrm{Aut}(\mathrm{Cl}(L)) \cong \mathrm{Aut}(V_4) \cong D_3.$$

Obviously $3 \mid |\mathrm{Ker}(\rho)|$ and therefore $\mathrm{Gal}(L/K_1)$ or $\mathrm{Gal}(L/K_1^{(2)}F)$ acts trivially on $\mathrm{Cl}(L)$. Hence $4 \mid h(K_1)$ or $4 \mid h(K_1^{(2)}F)$. We already see $2 \nmid h(K_1)$. Thus we calculate $h(KF_1)$. KF_1/K is a D_3-extension and $K(\alpha_1)$ is its cubic subextension. As above, we get $h(K(\alpha_1)) = 24$ and $h(KF_1) = 12$ by Lemma 3. Since $K_1^{(2)}F_1/KF_1$ is cyclic quartic, $2 \nmid h(K_1^{(2)}F)$ by Lemma 9. This is a contradiction. Hence $h(L) = 1$. Thus, $K_{ur} = L$.

$d = -840$. We have $\mathrm{Cl}(K) \cong C_2^3$, $K_1 = \mathbf{Q}(\sqrt{-2}, \sqrt{-3}, \sqrt{5}, \sqrt{-7})$, and $h(K_1) = 4$. We will show $K_{ur} = K_2$. Since $rd_K = \sqrt{840} = 28.9827\ldots$ and $rd_K < B(2 \cdot 8 \cdot 4 \cdot 11)$ ([38]), we have $[K_{ur} : K_2] < 16$. We first show that $h(K_2)$ is odd. Since $\mathrm{Cl}(\mathbf{Q}(\sqrt{5}, \sqrt{-42})) \cong C_4^2$, we have $\mathrm{Cl}(K_1) \cong V_4$. Hence by Lemma 9, the 2-class group of K_2 is cyclic. Assume that $h(K_2)$ is even. Let M be the unique quadratic subextension of K_3/K_2. Then obviously M is normal over K. We put $\Lambda = \mathrm{Gal}(M/K)$. Then Λ is a group of order 64 such that

$$\Lambda/\Lambda' \cong C_2^3, \ \Lambda'/\Lambda'' \cong V_4, \ \Lambda''/\Lambda''' \cong C_2.$$

However, any group of order 64 has abelian derived group (see [17]). This is a contradiction. Thus, $h(K_2)$ is odd and therefore $\mathrm{Cl}(K_2)$ is trivial or $\mathrm{Cl}(K_2) \cong C_3^2$ by Proposition 2. (Note that $K_g = K_1$ in this case.)

Assume $\mathrm{Cl}(K_2) \cong C_3^2$. Put $\Gamma = \mathrm{Gal}(K_2/\mathbf{Q})$. The action of Γ on $\mathrm{Cl}(K_2)$ induces a group homomorphism

$$\rho \ : \ \Gamma \to \mathrm{Aut}(\mathrm{Cl}(K_2)) \cong \mathrm{Aut}(C_3^2) \cong \mathrm{GL}(2,3).$$

By Lemma 8 the image of ρ is isomorphic to D_4, because if $|\mathrm{Im}(\rho)| \leqq 4$, then ρ has an abelian image and the same argument as in the proof of Proposition 2 shows that $h(K_1)$ must be divisible by 9. Let F be the field corresponding to $\mathrm{Ker}(\rho)$. Then F is normal over $\mathbf{Q}$ and $\mathrm{Gal}(F/\mathbf{Q}) \cong D_4$. Let E be the quadratic subfield of F such that F/E is cyclic and E' the intermediate field of F/E. If E'/E is ramified, then F/E is totally ramified. Therefore E'/E must be unramified because all the inertia groups of primes of F have order at most two. We take the quadratic subfield $\mathbf{Q}(\sqrt{d'})$ of K_1 such that the primes ramified in this field coincide with those ramified in F. Then $F(\sqrt{d'})/\mathbf{Q}(\sqrt{d'})$ is unramified at all finite primes. Thus, if $\mathbf{Q}(\sqrt{d'})$ is contained in F, then F/E' is unramified at all finite primes, but this is impossible, because any quadratic subfield of K_1 has narrow class group of exponent at most two. Hence $F(\sqrt{d'})$ is a $D_4 \times C_2$-extension of $\mathbf{Q}$. Then $F(\sqrt{d'})/E'$ is a V_4-extension. Let F' be the third quadratic subextension of it. Then F' is a D_4-extension of $\mathbf{Q}$. There exists a prime p that is ramified in $F/\mathbf{Q}$ but not ramified in $E'/\mathbf{Q}$. Then the prime divisors of p

in E' must be ramified in both F/E' and F'/E'. Since $F(\sqrt{d'})/E'(\sqrt{d'})$ is unramified, these divisors are ramified also in $E'(\sqrt{d'})/E'$. Thus, the inertia group of any prime divisor of p in $F(\sqrt{d'})$ is isomorphic to V_4. This is a contradiction. Hence $\mathrm{Cl}(K_2)$ is trivial and $K_{ur} = K_2$.

Now we determine $G = \mathrm{Gal}(K_2/K)$. G is a group of order 32 such that

$$G/G' \cong C_2^3, \ G' \cong V_4.$$

There are 9 (nonisomorphic) such groups, which belong to the family Γ_4 (see [17]). Among them, the groups having an abelian subgroup isomorphic to C_4^2 (note that $\mathrm{Gal}(K_2/\mathbf{Q}(\sqrt{5}, \sqrt{-42})) \cong C_4^2$) are $32\Gamma_4 a_2, 32\Gamma_4 a_3, 32\Gamma_4 c_2$, $32\Gamma_4 c_3$, and $32\Gamma_4 d$. Let N be any quadratic subextension of K_1/K other than $\mathbf{Q}(\sqrt{5}, \sqrt{-42})$, that is, any of the fields $\mathbf{Q}(\sqrt{-2}, \sqrt{105})$, $\mathbf{Q}(\sqrt{-3}, \sqrt{70})$, $\mathbf{Q}(\sqrt{-7}, \sqrt{30})$, $\mathbf{Q}(\sqrt{6}, \sqrt{-35})$, $\mathbf{Q}(\sqrt{-10}, \sqrt{21})$, and $\mathbf{Q}(\sqrt{14}, \sqrt{-15})$, and put $H = \mathrm{Gal}(K_2/N)$. We have $\mathrm{Cl}(N) \cong C_4 \times C_2$. Hence H is a group of order 16 such that

$$H/H' \cong C_4 \times C_2, \ H' \cong C_2.$$

Therefore H is isomorphic to $16\Gamma_2 c_1, 16\Gamma_2 c_2$, or $16\Gamma_2 d$. Thus, G is isomorphic to $32\Gamma_4 c_3$, or $32\Gamma_4 d$, according as G has a quotient group isomorphic to $16\Gamma_2 a_2 (\cong Q_8 \times C_2)$, or not. (All these arguments work also for $K = \mathbf{Q}(\sqrt{-420})$. We have $h(\mathbf{Q}(\sqrt{-420})_1) = 4$ and $\mathrm{Gal}(K_2/\mathbf{Q}(\sqrt{5}, \sqrt{-21})) \cong C_4^2$.) If K has an unramified Q_8-extension, then the composite field of it with K_1 is a $Q_8 \times C_2$-extension of K. Since $32\Gamma_4 c_3$ does not have two normal subgroups with quotient isomorphic to $Q_8 \times C_2$, we need only to judge whether K has an unramified Q_8-extension which is normal over $\mathbf{Q}$. F. Lemmermeyer gives a criterion for this [30]. By his result K does not have such an extension. Hence $G \cong 32\Gamma_4 d$. (Since $\mathbf{Q}(\sqrt{-420})$ has an unramified Q_8-extension which is normal over $\mathbf{Q}$, $\mathrm{Gal}(\mathbf{Q}(\sqrt{-420})_{ur}/\mathbf{Q}(\sqrt{-420})) \cong 32\Gamma_4 c_3$.)

$d = -920$. We have $\mathrm{Cl}(K) \cong C_{10} \times C_2$, $K_g = \mathbf{Q}(\sqrt{2}, \sqrt{5}, \sqrt{-23})$, and $h(K_g) = 60$. Put $L = (K_g)_1$. We will show $K_{ur} = L$. Since $rd_K = \sqrt{920} = 30.3315\ldots$ and $rd_K < B(2 \cdot 4 \cdot 60 \cdot 4)$ ([38]), we have $[K_{ur} : L] < 4$. Since $\mathrm{Cl}^{(2)}(K) \cong V_4$, the 2-class field tower of K terminates with $K_2^{(2)}$ and therefore $2 \nmid h(K_2^{(2)})$. Since $K_1 K_2^{(2)}/K_2^{(2)}$ is cyclic quintic, $2 \nmid h(K_1 K_2^{(2)})$ by Lemma 5 and since $L/K_1 K_2^{(2)}$ is cyclic cubic, $2 \nmid h(L)$ by Lemma 5 again.

Next, we show $3 \nmid h(L)$. Put $F = \mathbf{Q}(\sqrt{5}, \sqrt{-46})$. This is one of the intermediate fields of K_g/K. We have $\mathrm{Cl}(F) \cong C_{40}$. Hence $K_2^{(2)}/F$ is cyclic octic and therefore also $K_2^{(2)}(\alpha_1)/F(\alpha_1)$ is cyclic octic. Note that $h(\mathbf{Q}(\sqrt{-23})) = 3$ and $\mathbf{Q}(\sqrt{-23})_1 = \mathbf{Q}(\sqrt{-23}, \alpha_1)$, where $\alpha_1^3 - \alpha_1 - 1 = 0$ and that $(K_g)_1^{(3)} = K_g(\alpha_1)$. Therefore $3 \nmid h((K_g)_1^{(3)})$ by Lemma 9 and then $3 \nmid h(K_2^{(2)}(\alpha_1))$ by Lemma 5. Since $L/K_2^{(2)}(\alpha_1)$ is cyclic quintic, $3 \nmid h(L)$ by Lemma 5. Hence $h(L) = 1$ and therefore $K_{ur} = L = K_2$.

We easily check that $\mathrm{Gal}(K_{ur}/K) \cong \mathrm{Gal}(K_2^{(2)}(\alpha_1)/K) \times \mathrm{Gal}(K_1^{(5)}/K)$. By the result of H. Kisilevsky [25], we have $\mathrm{Gal}(K_2^{(2)}/K) \cong Q_{16}$. Therefore $\mathrm{Gal}(K_{ur}/K) \cong (Q_{16} \ltimes C_3) \times C_5$.

Finally, we note that the maximal unramified extension of $\mathbf{Q}(\sqrt{-920})$ has degree 480 and is the field with class number one of the largest degree we know (under GRH). (J. Martinet gave an example of a field with class number one and degree 116 [33]. (This is an unconditional result.))

7. Quadratic number fields having an unramified extension not contained in $(K_g)_1$

We explain here that from some quartic number fields of type S_4 we can obtain quadratic number fields whose genus fields have an unramified A_4-extension.

First, we review the following general fact.

PROPOSITION 6. ([26]. See also [56].) *Let E be an algebraic number field of degree $n \geqq 3$. If the discriminant d_E of E is a fundamental quadratic discriminant, that is, a discriminant of a quadratic number field, then the normal closure M of E is an S_n-extension of $\mathbf{Q}$ and is unramified at all finite primes over its quadratic subfield $\mathbf{Q}(\sqrt{d_E})$.*

From this, any number field E of degree $n \geqq 3$ whose discriminant is fundamental quadratic yields a quadratic number field $\mathbf{Q}(\sqrt{d_E})$ having an unramified A_n-extension. (If $d_E > 0$ and E is not totally real, then the infinite primes are ramified.) Moreover, E yields infinitely many quadratic number fields having an unramified S_n-extension. In fact, for any quadratic number field $K \neq \mathbf{Q}(\sqrt{d_E})$ with $d_K = d_E d'$, where d' is a fundamental quadratic discriminant, the composite field KM is an unramified S_n-extension of K. (The unramifiedness of $K\mathbf{Q}(\sqrt{d_E})/K$ follows by genus theory.) In this case, since KM and K_g are linearly disjoint over $K\mathbf{Q}(\sqrt{d_E})$, K_g has an unramified A_n-extension. (This is the case also when $K = \mathbf{Q}(\sqrt{d_E})$.) Therefore, if $n \geqq 4$, then $K_g M$ is an unramified extension not contained in $(K_g)_1$. In particular, if $n = 4$, then the (narrow) class field tower of the field $\mathbf{Q}(\sqrt{d_E})$ has length at least two and the (narrow) class field tower of K has length at least three if $d \neq d_E$.

Now we describe the field $K = \mathbf{Q}(\sqrt{-687})$. The factorization of -687 is $-687 = (-3) \cdot 229$ and 229 is the first prime discriminant of totally imaginary quartic number fields. Let E be a quartic number field with discriminant 229 (there exists only one such field up to isomorphism [14]) and M its normal closure. Then by the proposition above, M is an S_4-extension of $\mathbf{Q}$ which is an A_4-extension of the real quadratic number field $F := \mathbf{Q}(\sqrt{229})$ unramified at all finite primes (abbrev. weakly-unramified). We can show that the class number of M is one by the method described in §8 plus some additional consideration. Since by the table in [12], $B(24 \cdot 60, 0, 12 \cdot 60) > rd_F = \sqrt{229}$, M has no nontrivial weakly-unramified extension. (This is unconditional.) Hence M is the maximal weakly-unramified extension of F. (While $F_{ur} = F_1, [F_1 : F] = 3$.) Since $\mathrm{Cl}(K) \cong C_{12}$, the composite field $K_1 M$ has degree 288. By the table in [38], $B(576, 0, 288) > rd_K = \sqrt{687}$ under GRH. Therefore $K_{ur} = K_1 M$ under GRH. Now we determine the structure of the Galois group $G = \mathrm{Gal}(K_{ur}/K)$. Obviously $K_1^{(2)} M$ and $K_1^{(3)}$ are linearly disjoint over K and therefore G is isomorphic to the direct product $H = \mathrm{Gal}(K_1^{(2)} M/K)$ with $\mathrm{Gal}(K_1^{(3)}/K) \cong C_3$. We consider the structure of H. Let C be a cubic subextension of KM/K. Then C corresponds to a Sylow 2-subgroup of H and $\mathrm{Gal}(KM/E)$ is isomorphic to a Sylow 2-subgroup of $\mathrm{Gal}(KM/K) \cong S_4$, that is, isomorphic to D_4. Thus, if we put $T = \mathrm{Gal}(K_1^{(2)} M/C)$, then we have

$$T/T' \cong C_4 \times C_2, \ T' \cong C_2.$$

Since $\mathrm{Gal}(K_1^{(2)} M/K_g F_1) \cong C_2^3$, we have $T \cong 16\Gamma_2 c_1 \cong D_4 \curlywedge C_4$. Hence T contains a cyclic subgroup W of order four containing $\mathrm{Gal}(K_1^{(2)} M/KM)$ with $T \cap W = \{1\}$, and this group and $\mathrm{Gal}(K_1^{(2)} M/K_1^{(2)})$ generate H. Therefore $H \cong A_4 \rtimes C_4$. (We note that H is a nonsplit extension of $\mathrm{Gal}(KM/K) \cong S_4$ by C_2. But H is not a double cover of S_4.) Thus, $G \cong (A_4 \rtimes C_4) \times C_3$.

Some quartic number fields whose discriminants are fundamental quadratic yield infinitely many quadratic number fields with class field tower of length at least four.

PROPOSITION 7. (See [45]) *Let E be a quartic number field with $d_E = -p$ ($p \equiv 3 \pmod 4$) a prime). Then E is embedded into an octic number field $\widetilde{E}$ whose normal closure is an $\widetilde{S_4}$-extension of $\mathbf{Q}$ and is unramified over its quadratic subfield $\mathbf{Q}(\sqrt{-p})$, where $\widetilde{S_4}$ is the double cover of S_4 which is* *characterized by the property that a transposition in S_4 lifts to an element of order two, while a product of two disjoint transpositions lifts to an element of order four:* $\widetilde{S_4} \cong \mathrm{GL}(2,3)$.

From this, from a quartic number field E with $d_E = -p \equiv 1 \pmod 4$, we get an unramified $\widetilde{A_4}$-extension of the imaginary quadratic number field $\mathbf{Q}(\sqrt{-p})$. We note that from the data for quartic number fields of signature $(2,1)$ [13], the quartic prime discriminants $-p \equiv 1 \pmod 4$ with $p \leqq 2003$ are $-283, -331, -491, -563, -643, -751, -1399, -1423, -1823, -1879$, and -1931. All such fields give two dimensional complex projective linear representation

$$\rho \;:\; \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{PGL}(2, \mathbf{C})$$

of the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ of $\mathbf{Q}$ of type S_4 which has a lifting

$$\tilde{\rho} \;:\; \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}(2, \mathbf{C})$$

with conductor p and odd determinant (see [45, §8]). (A. Jehanne [23] has studied the general embedding problem $\widetilde{S_4} \to S_4$, especially for the case where the base field is $\mathbf{Q}$. We find in [23] a generating polynomial of $\widetilde{E}$ for $p = 283, 331, 491, 563, 643, 751$. Field E in Proposition 7 yields infinitely many quadratic number fields with class field tower of length at least four. Let $\widetilde{M}$ be the normal closure of $\widetilde{E}$. Then $\widetilde{M}$ is an $\widetilde{S_4}$-extension of $\mathbf{Q}$. For any quadratic number field K such that d_K/d_E is fundamental quadratic, the composite field $K\widetilde{M}$ is an unramified $\widetilde{S_4}$-extension of K. Then K has (narrow) class field tower of length at least four. For example, let $K = \mathbf{Q}(\sqrt{-5 \cdot 283})$. Since $\mathrm{Cl}(K) \cong C_{34} \cong C_2 \times C_{17}$, $\mathrm{Gal}(K_1\mathbf{Q}(\sqrt{-283})_{ur}/K) \cong \widetilde{S_4} \times C_{17}$ and therefore $[K_1\mathbf{Q}(\sqrt{-283})_{ur} : \mathbf{Q}] = 1632$. Even under GRH, we get only $[K_{ur} : K_1\mathbf{Q}(\sqrt{-283})_{ur}] < 44$ from Odlyzko's bound, however, it is thought likely that $K_{ur} = K_1\mathbf{Q}(\sqrt{-283})_{ur}$ (this implies $K_{ur} = K_4 = K_1\mathbf{Q}(\sqrt{-283})_{ur}$), but no proof has yet been obtained. (We can prove $K_3 = K_1\mathbf{Q}(\sqrt{-283})_2$.)

There exist many quartic number fields of type S_4 whose discriminants are not fundamental quadratic which yield infinitely many quadratic number fields having an unramified S_4-extension.

We will not consider here general such fields. We decribe only such quartic number fields giving unramified S_4-extensions of $\mathbf{Q}(\sqrt{-856})$ and $\mathbf{Q}(\sqrt{-996})$.

Let E be a quartic number field defined by the (irreducible) polynomial $f(X) = X^4 - 2X^3 + 5X^2 - 2X - 1$ and M its normal closure. We easily see $\mathrm{Gal}(M/\mathbf{Q}) \cong S_4$ and $d_E = -6848 = -2^6 \cdot 107$. Let $K = \mathbf{Q}(\sqrt{-856})$. Note that $-856 = 8 \cdot (-107)$. We see that KM/K is an unramified S_4-extension. For this, it suffices to show that KE/K is unramified. We can easily get a generating polynomial of KE. For example, we can easily calculate the minimal polynomial of $\theta + \sqrt{-214}$, where θ is a root of $f(X)$ with $E = \mathbf{Q}(\theta)$. From this we get as a generating polynomial of KE, $X^8 - 2X^7 + 15X^6 + 20X^5 + 25X^4 + 228X^3 + 573X^2 + 170X + 25$ by using the function 'polred' of pari-gp. Then we get $d_{KE} = 2^{12} \cdot 107^4 = d_E^4$ by using 'discf' of pari-gp. Hence KM/K is unramified. For any quadratic number field F such that $856 \mid d_F$ and that the localization of F at the prime divisor of 2 in F is (isomorphic to) the one of K at the prime divisor of 2 in K, FM/F is an S_4-extension unramified at all finite primes.

Let E' be a quartic number field defined by the (irreducible) polynomial $X^4 - 3X^2 - 2X + 1$ and M' its normal closure. We easily see $\mathrm{Gal}(M'/\mathbf{Q}) \cong S_4$ and $d_{E'} = -1328 = -2^4 \cdot 83$. Let $K' = \mathbf{Q}(\sqrt{-996})$. Note that $-996 = (-4) \cdot (-3) \cdot (-83)$. As above we can show that $K'M'/K'$ is an unramified S_4-extension.

We note that the discriminants of the fields E and E' above are of the form $f^2 d'$, where d' is a fundamental discriminant with $h(\mathbf{Q}(\sqrt{d'})) = 3$ and $f = 8 = d_{\mathbf{Q}(\sqrt{2})}$, or $f = -4 = d_{\mathbf{Q}(\sqrt{-1})}$. We also note that the ramification of the prime 2 in $M/\mathbf{Q}$ (resp. $M'/\mathbf{Q}$) occurs over the subfield of M (resp. M') which is an S_3-extension of $\mathbf{Q}$.

8. Unramified nonsolvable Galois extensions of imaginary quadratic number fields

For imaginary quadratic number fields K of small conductors we considered, the maximal unramified extension K_{ur} is the top of the class field tower of K. Then the following natural question arises: What is the first imaginary quadratic number field having an unramified nonsolvable Galois extension ? Recent data for quintic number fields enable us to give a partial answer:

PROPOSITION 8. *The field $\mathbf{Q}(\sqrt{-1507})$ is the first imaginary quadratic number field having an unramified A_5-extension which is normal over $\mathbf{Q}$ in the sense that none of $\mathbf{Q}(\sqrt{d})$ of discriminant d with $0 > d > -1507$ has such an extension. Moreover, such an extension of $K = \mathbf{Q}(\sqrt{-1507})$ is given by the composite field of K with the splitting field of the quintic polynomial $X^5 - 5X^3 + 5X^2 + 24X + 4$, which is an A_5-extension of $\mathbf{Q}$.*

Proof. From the table in [2], the splitting field of the quintic polynomial

$X^5 - 5X^3 + 5X^2 + 24X + 4$ is an A_5-extension of $\mathbf{Q}$ and the only two primes 11 and 137 are ramified in this field with ramification index two. Therefore the composite field of $\mathbf{Q}(\sqrt{-1507})$ with it is an unramified A_5-extension of $\mathbf{Q}(\sqrt{-1507})$ (note that the factorization of 1507 is $1507 = 11 \cdot 137$), which is an $A_5 \times C_2$-extension of $\mathbf{Q}$. Now we show that none of $\mathbf{Q}(\sqrt{d})$ of discriminant d with $0 > d > -1507$ has such an extension. Let L be an unramified A_5-extension of a quadratic number field F which is normal over $\mathbf{Q}$. Then the Galois group $\mathrm{Gal}(L/\mathbf{Q})$ is isomorphic to S_5 or $A_5 \times C_2$. If $\mathrm{Gal}(L/\mathbf{Q}) \cong S_5$, then L is the normal closure of its quintic subfield E and the discriminant d_E of E coincides with the one d_F of F (see [26]). Since the maximal discriminant of quintic number fields of signature $(3,1)$ (whose normal closures are S_5-extensions of $\mathbf{Q}$) is -4511 and the minimal discriminant of quintic number fields of signature $(1,2)$ is 1609 ([44]), $\mathbf{Q}(\sqrt{-4511})$ is the first imaginary quadratic number field having an unramified A_5-extension which is an S_5-extension of $\mathbf{Q}$. Now, we assume $\mathrm{Gal}(L/\mathbf{Q}) \cong A_5 \times C_2$ and let M be the unique subfield of L which is an A_5-extension of $\mathbf{Q}$. Since $L = MF/F$ is unramified, the primes ramified in M are prime divisors of d_F and their ramification indices are all two. Hence if we let E be a quintic subfield of M, we get $d_F^2 | d_E$. From the data for quintic number fields in [44] and the table in [3], $\mathbf{Q}(\sqrt{-1507})$ is the first imaginary quadratic number field having an unramified A_5-extension which is an $A_5 \times C_2$-extension of $\mathbf{Q}$.

Remark. Probably, the condition "which is normal over $\mathbf{Q}$" is unnecessary. It is reasonable to expect this. In fact, if we assume that GRH is true, Odlyzko's discriminant bounds enable us to show that most of $\mathbf{Q}(\sqrt{d})$ with $0 > d > -1507$ do not have an unramified $A_5 \times A_5$-extension. (We can show that under GRH, except for $d = -1387, -1451, -1459, -1480, -1483, -1492$, and -1499, none of $\mathbf{Q}(\sqrt{d})$ with $0 > d > -1507$ has any unramified A_5-extension.) We expect that in future data for number fields of degree ten will enable us to eliminate the condition above; If an A_5-extension L of a quadratic number field K is not normal over $\mathbf{Q}$, its normal closure $\widetilde{L}$ has Galois group isomorphic to $A_5 \times A_5$ over K. Let E be a quintic subextension of L/K. Then E has degree ten. The five conjugate fields of E which are conjugate over K generate L and the other five conjugate fields generate another A_5-extension of K. Thus $\mathrm{Gal}(\widetilde{L}/K) \cong A_5 \times A_5$.

We expect stronger assertion that $\mathbf{Q}(\sqrt{-1507})$ is the first imaginary quadratic number field having an unramified nonsolvable Galois extension. We exhibit here two other examples of an imaginary quadratic number field having an unramified nonsolvable Galois extension. $\mathbf{Q}(\sqrt{-14731})$ has an unramified A_6-extension which is an S_6-extension of $\mathbf{Q}$. Such an extension is given as the splitting field of the sextic polynomial $X^6 + 3X^5 + 5X^4 + 4X^3 + 3X^2 + 2X + 1$. $\mathbf{Q}(\sqrt{-30759})$ has an unramified PSL(2, 7)-extension which is a PSL(2, 7) $\times$ C_2-extension of $\mathbf{Q}$. Such an extension is given as the composite field of $\mathbf{Q}(\sqrt{-30759})$ with the splitting field of the septic polynomial $X^7 + 2X^6 - 3X^4 - X^3 - X^2 - X + 2$. The fact that this splitting field is a PSL(2, 7)-extension of $\mathbf{Q}$ was found by K. Yamazaki [61].

As explained in §2, under GRH, if $|d| \leqq 2003$, then $[K_{ur} : K] < \infty$ for $K = \mathbf{Q}(\sqrt{d}), d = d_K < 0$. In this range, we only know one other field having an unramified nonsolvable Galois extension, namely, the field with $d = -1959 = (-3) \cdot 653$. This is the second imaginary quadratic number field having an unramified A_5-extension which is normal over $\mathbf{Q}$. Let L be the splitting field of the quitic polynomial $X^5 + X^3 + 62X^2 + 104X + 169$. Then L is an imaginary A_5-extension of $\mathbf{Q}$, in which (except for the infinite prime) only the finite prime 653 is ramified and its ramification index is two ([3]). Thus, for any quadratic number field K whose discriminant is a multiple of 653, the compositum KL is an A_5-extension of K unramified at all finite primes. (If K is real, the infinite primes are ramified in KL/K.) We note that $F = \mathbf{Q}(\sqrt{653})$ is the first real quadratic number field having an weakly-unramified A_5-extension which is normal over $\mathbf{Q}$. F has narrow class number one. Therefore F has no nontrivial weakly-unramified abelian extension. Under GRH from Odlyzko's bound we get $[F_{w\text{-}ur} : FL] \leqq 3$ and therefore $\mathrm{Gal}(F_{w\text{-}ur}/F)$ is isomorphic to A_5, or a nonsplit extension of A_5 by C_2 or C_3.

The third imaginary quadratic number field having an unramified A_5-extension which is normal over $\mathbf{Q}$ is $\mathbf{Q}(\sqrt{-2083})$. Moreover, the A_5-extension of $K = \mathbf{Q}(\sqrt{-2083})$ is given by the composite field of K with the splitting field of the quintic polynomial $X^5 + 8X^3 + 7X^2 + 172X + 53$, which is an A_5-extension of $\mathbf{Q}$.

Now we return to the field $\mathbf{Q}(\sqrt{-1507})$. We have $\mathrm{Cl}(K) \cong C_4$. The Hilbert class field K_1 of K is $K(\sqrt{(-23 + 3\sqrt{137})/2})$ and its class number is one. Thus, K_1 is an imaginary D_4-extension of $\mathbf{Q}$ with class number one having an unramified A_5-extension. This field is only example we know of an imaginary normal extension of $\mathbf{Q}$ with class number one having an unramified nonsolvable Galois extension. (On the other hand, there exist many (probably infinitely many) real quadratic number fields with class number one having an unramified A_n-extension for each $n \geqq 5$ [55, 60].)

APPENDIX 1. CALCULATION OF CLASS NUMBERS OF S_4-EXTENSIONS OF $\mathbf{Q}$

Let M be an S_4-extension of $\mathbf{Q}$. We consider here how to calculate its class number. If we know the class numbers of a cubic subfield and sextic subfields containing it, we get the class number $h(M)$ of M up to a power of two by using class number relation for V_4-extension repeatedly. Let L be the unique normal subfield of M which is an S_3-extension of $\mathbf{Q}$. Then M/L is a V_4-extension and three intermediate fields are conjugate. Therefore, if we know the class numbers of L and any one of the intermediate fields of M/L, then we get $h(M)$ up to a 2-power. Let E be a cubic subfield of L. Then M/E is a D_4-extension and the V_4-subextension N of M/E is an intermediate field of M/L. Thus, if the class numbers of three intermediate fields of N/E, which are sextic, are known, we get $h(N)$ up to a 2-power.

Now, we give the sextic fields more explicitly by starting with a quartic subfield of M. Let F be a quartic subfield of M and $f(X) = X^4 + a_1X^3 + a_2X^2 + a_3X + a_4 \in \mathbf{Z}[X]$ its generating polynomial, that is, $F = \mathbf{Q}(\beta)$, where β is a root of $f(X)$. Then the resolvent cubic $f_r(X) = X^3 - a_2X^2 + (a_1a_3 - 4a_4)X - a_4(a_1^2 - 4a_2) - a_3^2$ of $f(X)$ is irreducible over $\mathbf{Q}$ and its roots are $\alpha = \beta_1\beta_2 + \beta_3\beta_4, \beta_1\beta_3 + \beta_2\beta_4\ \beta_1\beta_4 + \beta_2\beta_3$, where $\beta_1, \beta_2, \beta_3$, and β_4 are the roots of $f(X)$. Let $E = \mathbf{Q}(\alpha)$ and N the V_4-subextension of M/E. We want to know the three intermediate fields of N/E and their class numbers. One is $L = KE = E(\sqrt{d})$, which is the splitting field of $f_r(X)$, where K is the unique quadratic subfield of M and d its discriminant. Other one is $\mathbf{Q}(\beta_1\beta_2) = \mathbf{Q}(\beta_1\beta_2 - \beta_3\beta_4)$. Note that $(\beta_1\beta_2 - \beta_3\beta_4)^2 = (\beta_1\beta_2 + \beta_3\beta_4)^2 - 4\beta_1\beta_2\beta_3\beta_4 = \alpha^2 - 4a_4$ and we can write this field as $E(\sqrt{\alpha^2 - 4a_4})$. Thus, the other one is $E(\sqrt{d(\alpha^2 - 4a_4)}) = \mathbf{Q}(\sqrt{d(\alpha^2 - 4a_4)}) = \mathbf{Q}(\sqrt{d}(\beta_1\beta_2 - \beta_3\beta_4))$. The minimal polynomial $g(X)$ of $\beta_1\beta_2$ is

$$X^6 - a_2X^5 + (a_1a_3 - a_4)X^4 + (-a_1^2a_4 + 2a_2a_4 - a_3^2)X^3 + (a_1a_3a_4 - a_4^2)X^2 - a_2a_4^2X + a_4^3$$

and its discriminant is $d_f^2a_4^6(-2a_1^2a_3^2a_4 + a_1^4a_4^2 + a_3^4)$, where d_f is the discriminant of $f(X)$. The minimal polynomial $u(X)$ of $\sqrt{d}(\beta_1\beta_2 - \beta_3\beta_4)$ is

$$X^6 - (a_2^2 - 2a_1a_3 - 4a_4)dX^4 + (-2a_1^2a_2a_4 + a_1^2a_3^2 + 8a_1a_3a_4 - 2a_2a_3^2)d^2X^2 - (-2a_1^2a_3^2a_4 + a_1^2a_4^2 + a_3^4)d^3$$

and its discriminant is $64d_f^{11}(a_1a_2a_3 - a_1^2a_4^2 - a_3^2)(-2a_1^2a_3^2a_4 + a_1^4a_4^2 + a_3^4)$.

We describe ramification. H. J. Godwin [15] proved that the discriminant of $\mathbf{Q}(\beta_1\beta_2)$ equals $d_E d_F$ (see also [24]). In order that the extension M/K is unramified at all finite primes, $d_F = d$ is necessary and sufficient (see [26]). In that case, we have also $d_E = d$ and therefore $d_{\mathbf{Q}(\beta_1\beta_2)} = d^2 = d_E^2$. This implies that both $N/\mathbf{Q}(\sqrt{d(\alpha^2 - 4a_4)})$ and $\mathbf{Q}(\beta_1\beta_2)/E$ are unramified at all finite primes.

Now we explain how we get $h(K_2)$ for $K = \mathbf{Q}(\sqrt{-643})$. It is a classical result that K_2 for $K = \mathbf{Q}(\sqrt{-643})$ is an S_4-extension of $\mathbf{Q}$. So, put $M = K_2$ and we use the notations above. Then we can take $f(X) = X^4+X^3+2X+1$ and $d_f = d = -643$. (From the irreducibility and the squarefreeness of d_f, we can deduce the fact that the splitting field of $f(X)$ is an S_4-extension of $\mathbf{Q}$ and is unramified over the unique quadratic subfield $K = \mathbf{Q}(\sqrt{-643})$. (See [56].)) Then $f_r(X) = X^3 - 2X - 5$ and we can check $h(E) = 2$ by computer. Since $h(K) = 3$, by using the class number relation in Lemma 3, we get $h(L) = 4$. (More precisely, by Lemma 5, $\mathrm{Cl}(L) \cong V_4$.) Now $g(X) = X^6+X^4-5X^3+X^2+1, u(X) = X^6-8{\cdot}643X^4+20{\cdot}643^2X^2+9{\cdot}643^3$ and KANT gives $h(\mathbf{Q}(\beta_1\beta_2)) = 1$. Hence in this case, the field $\mathbf{Q}(\beta_1\beta_2)$ is the Hilbert class field of E and $N/\mathbf{Q}(\sqrt{d(\alpha^2 - 4a_4)})$ is unramified. The function 'polred' of pari-gp reduces $u(X)$ to $X^6 - 3X^5 + 11X^4 - 9X^3 + 23X^2 + 8X + 57$ and KANT gives $\mathrm{Cl}(\mathbf{Q}(\sqrt{d(\alpha^2 - 4a_4)})) \cong C_8$. We also note that the field $\mathbf{Q}(\sqrt{d(\alpha^2 - 4a_4)})$ is totally imaginary and therefore the narrow class group of it concides with its class group. Then the class number relation for V_4-extension shows that $h(N)$ is a power of 2, and since N is the unramified quadratic extension of $\mathbf{Q}(\sqrt{d(\alpha^2 - 4a_4)})$, we get $\mathrm{Cl}(N) \cong C_4$. Using the class number relation for V_4-extension again, $h(M)$ is a power of 2, and since M is the unramified quartic extension of $\mathbf{Q}(\sqrt{d(\alpha^2 - 4a_4)})$, we get $h(M) = 2$. By the same method, we get $h(K_2) = 2$ also for $K = \mathbf{Q}(\sqrt{-283})$ and $\mathbf{Q}(\sqrt{-331})$. Moreover, if we take M as the normal closure of the third (resp. fourth, and eighth) quartic number field of type S_4, which is an unramified A_4-extension of $\mathbf{Q}(\sqrt{-491})$ (resp. $\mathbf{Q}(\sqrt{-563})$, and $\mathbf{Q}(\sqrt{-751})$), we get $h(M) = 6$ (resp. $h(M) = 6$, and $h(M) = 10$).

APPENDIX 2. IMAGINARY QUADRATIC NUMBER FIELDS K WITH $K_2 = (K_g)_1 \neq K_1$

We give here a note on the following.

PROBLEM 1. *Characterize the imaginary quadratic number fields K with*

$$(*) \qquad K_2 = (K_g)_1 \neq K_1.$$

Let K be an imaginary quadratic number field. The inequality $(K_g)_1 \neq K_1$ is equivalent to $h(K_g) > h(K)/[K_g : K]$ and we can easily calculate $h(K), [K_g : K]$, and $h(K_g)$. Since $K_g = K$ if and only if $h(K)$ is odd, for $(K_g)_1 \neq K_1$, $h(K)$ must be even and therefore the discriminant d of K must have two disctinct prime factors.

On the other hand, it is hard to calculate $h(K_1)$ in general. However, by considering the p-class field tower, we can get some necessary condition for $K_2 = (K_g)_1$. Let p be an odd prime. If the class number $h(K_1^{(p)})$ of the Hilbert p-class field $K_1^{(p)}$ of K is divisible by p, then we have

$$(K_g)_1 = (K_g)_1 K_1^{(p)} \subsetneqq (K_g)_1 K_2^{(p)} \subseteqq K_2,$$

because the odd part of the class group of K_g is the direct product of the odd parts of the class groups of the quadratic subfields of K_g. If the p-rank of the p-class group of K is larger than one for some odd prime p, then $h(K_1^{(p)})$ is divisible by p [37, Theorem 2] and therefore in this case $K_2 \supsetneqq (K_g)_1$. For example, let $K = \mathbf{Q}(\sqrt{-3896})$. Then we have $\mathrm{Cl}(K) \cong C_{12} \times C_3, K_g = \mathbf{Q}(\sqrt{2}, \sqrt{-487})$, and $\mathrm{Cl}(K_g) \cong C_{42} \times C_3$. Therefore $K_2 \supsetneqq K_1^{(3)}(K_g)_1 = (K_g)_1$. Thus, for $K_2 = (K_g)_1$, the p-class field tower must terminate with $K_1^{(p)}$ for all odd primes, in other words, the odd part of the class group of K must be cyclic.

Problem 1 seems to be very difficult. Thus, it is reasonable to consider the following before treating Problem 1.

PROBLEM 2. *Characterize the imaginary quadratic number fields K with*

$$(**) \qquad K_2^{(2)} = (K_g)_1^{(2)} \neq K_1^{(2)}.$$

Put $\Lambda = \mathrm{Gal}(K_2^{(2)}/K)$. Then $\mathrm{Gal}(K_2^{(2)}/K_1^{(2)}) = \Lambda', \mathrm{Gal}(K_2^{(2)}/K_g) = \Phi(\Lambda)$, and $\mathrm{Gal}(K_2^{(2)}/(K_g)_1) = \Phi(\Lambda)'$. Problem 2 consists in characterizing K such that

$$(***) \qquad \Lambda' \neq \Phi(\Lambda)' = (\Lambda'' =)\{1\}.$$

If $\mathrm{Cl}^{(2)}(K)$ is elementary abelian, then $K_g = K_1^{(2)}$ and therefore if moreover $h(K_g)$ is divisible by 2, then $(**)$ trivially holds. We can characterize such K: $\mathrm{Cl}^{(2)}(K)$ is elementary abelian if and only if d does not have a factorization of the second kind (in the terminology of Rédei-Reichardt). F. Lemmermeyer [29] characterized K for which Λ is abelian. By known results on the 2-class field tower, if the 2-class group of $K_1^{(2)}$ is nontrivial and cyclic, (**) holds. (Refer to §5.) In fact, for $\Lambda = MC_n^-, MC_{n,m}^+, \Gamma_{m,t}^1$ or $\Gamma_{m,t}^2$, $(***)$ holds. Therefore the remaining case is where $\mathrm{Cl}^{(2)}(K)$ is not elementary abelian and $\mathrm{Cl}^{(2)}(K_1^{(2)})$ is noncyclic. The author knows no example of K with $K_2^{(2)} \supsetneqq (K_g)_1^{(2)} \supsetneqq K_1^{(2)}$.

Finally, we note that there exist 2-groups Λ satisfying $\Lambda' \neq \Phi(\Lambda)' \neq \Lambda'' = \{1\}$. The groups $64\Gamma_{22}a_1, 64\Gamma_{22}a_2, 64\Gamma_{23}a_1, 64\Gamma_{23}a_2, 64\Gamma_{23}a_3$, and $64\Gamma_{23}a_4$ are so. This is the complete list of the nonabelian 2-groups of orders $\leqq 64$ having nonabelian Frattini subgroup. (See [17].)

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Ken YAMAMURA
Department of Mathematics
National Defence Academy
Hashirimizu Yokosuka 239
JAPAN
e-mail : yamamura@cc.nda.ac.jp