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The distribution of square-free numbers of the form $[n^c]$

par XIAODONG CAO et WENGUANG ZHAI

RÉSUMÉ. Nous montrons que pour $1 < c < \frac{61}{36} = 1.6944\dots$, la suite $[n^c]$ ($n = 1, 2, \dots$) contient une infinité d'entiers sans facteur carré ; cela améliore un résultat antérieur dû à Rieger qui obtenait l'infinitude de ces entiers pour $1 < c < 1.5$.

ABSTRACT. It is proved that the sequence $[n^c]$ ($n = 1, 2, \dots$) contains infinite squarefree integers whenever $1 < c < \frac{61}{36} = 1.6944\dots$, which improves Rieger's earlier range $1 < c < 1.5$.

1. INTRODUCTION

A positive integer n is called squarefree if it is a product of different primes. Following a paper of Stux [15], Rieger showed in [11] that for all real c with $1 < c < 1.5$, the equation

$$(1.1) \quad S_c(x) = \sum_{\substack{n \leq x \\ [n^c] \text{ squarefree}}} 1 = \frac{6}{\pi^2}x + O(x^{\frac{2c+1}{4}+\varepsilon})$$

holds, which is an immediate consequence of Deshouillers [4]. Here $[t]$ denotes the fractional part of t and ε is a positive constant small enough. It is an easy exercise to prove that

$$(1.2) \quad S_c(x) = \frac{6}{\pi^2}x + o(x)$$

for $0 < c \leq 1$. When $1 < c < 2$, one still expects (1.2) to hold, but if $c = 2$, $[n^c]$ is always a square, so that $S_c(x) = 0$.

It is worth remarking that Stux [15] has shown that $S_c(x)$ tends to infinity for almost all positive real $1 < c < 2$ (in the sense of Lebesgue measure), however this result provides no specific value of c .

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Mots-clés. Square-free number, exponential sum, exponent pair.

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The aim of this paper is to further improve Rieger’s range $1 < c < 1.5$ by the method of exponential sums.

Basic Proposition. *Let $1 < c < 2, \gamma = \frac{1}{c}$, and $x > 1$. Then we have*

$$(1.3) \quad S_c(x) = \frac{6}{\pi^2}x + \Delta_c(x),$$

with

$$(1.4) \quad \Delta_c(x) = \sum_{n \leq x^c} |\mu(n)| (\psi(-(n+1)^\gamma) - \psi(-n^\gamma)) + O(x^{1-c/2}),$$

where $\psi(t) = t - [t] - 1/2$ and $\mu(n)$ is the well-known Möbius function.

Using the simple one-dimensional exponent pair, we can prove immediately from the Basic Proposition that

Corollary. *Let $1 < c < 1.625$, then for $\varepsilon > 0$*

$$(1.5) \quad S_c(x) = \frac{6}{\pi^2}x + O(x^{\frac{8c+3}{16}+\varepsilon}).$$

Combining Fouvry and Iwaniec’s new method in [5] and Heath-Brown’s new idea in [6], we can prove the following better Theorem .

Theorem. *Let c be a real constant such that $1 < c < 61/36$, then*

$$(1.6) \quad S_c(x) = \frac{6}{\pi^2}x + O(x^{\frac{36(c+1)}{97}+\varepsilon}).$$

Notations. $f(x) \ll g(x)$ means that $f(x) = O(g(x))$, $m \sim M$ means $c_1M < m \leq c_2M$ for some constants $c_1, c_2 > 0$. We also use notations $L = \log(x), e(x) = \exp(2\pi ix)$ and $\psi(\theta) = \theta - [\theta] - 1/2$. To simplify writing logarithms, we will assume that all parameters are bounded by a power of x . Throughout the paper we allow the constants implied by ‘ O ’ or ‘ \ll ’ to depend on only arbitrarily small positive number ε and c when it occurs.

2. PROOFS OF BASIC PROPOSITION AND COROLLARY

Proof of Basic Proposition. It is well-known that (see [9])

$$(2.1) \quad \sum_{n \leq x} |\mu(n)| = \frac{6}{\pi^2}x + O(x^{\frac{1}{2}})$$

and

$$(2.2) \quad |\mu(n)| = \sum_{d^2|n} \mu(d).$$

Obviously, $[n^c]$ is square-free if and only if $m^\gamma \leq n < (m+1)^\gamma, m$ square-free. Therefore

$$\begin{aligned}
 (2.3) \quad S_c(x) &= \sum_{\substack{n \leq x \\ [n^c] \text{ squarefree}}} 1 = \sum_{\substack{m \leq x^c \\ m \text{ squarefree}}} ([-m^\gamma] - [-(m+1)^\gamma]) + O(1) \\
 &= \sum_{m \leq x^c} |\mu(m)| ((m+1)^\gamma - m^\gamma) + E_{1c}(x) \\
 &= \gamma \sum_{m \leq x^c} |\mu(m)| m^{\gamma-1} + E_{2c}(x),
 \end{aligned}$$

where

$$E_{jc}(x) = \sum_{m \leq x^c} |\mu(m)| (\psi(-(m+1)^\gamma) - \psi(-m^\gamma)) + O(1), \quad j = 1, 2.$$

From (2.3), (2.1) and partial summation we can get the Basic Proposition at once.

The proof of Corollary will need the following two lemmas. Lemma 1 is well-known (see [1]), Lemma 2 is contained in Theorem 18 of Vaaler [16].

Lemma 1. *Let $|g^{(m)}(x)| \sim YX^{1-m}$ for $1 < X < x \leq 2X$ and $m = 1, 2, \dots$. Then*

$$(2.4) \quad \sum_{X < n \leq 2X} e(g(n)) \ll Y^\kappa X^\lambda + Y^{-1}$$

where (κ, λ) is any exponent pair.

Lemma 2. *Suppose $J > 1$. There is a function $\psi^*(x)$ such that*

- (1) $\psi^*(x) = \sum_{1 \leq |h| \leq J} \gamma(h)e(hx),$
- (2) $\gamma(h) \ll \frac{1}{|h|} \quad \text{and} \quad (\gamma(h))' \ll \frac{1}{h^2},$
- (3) $|\psi^*(x) - \psi(x)| \leq \frac{1}{2(J+1)} \sum_{|h| \leq J} (1 - \frac{|h|}{J})e(hx).$

By Lemma 1 and Lemma 2 we immediately obtain

Lemma 3. *Let $y > 0, X > 1, 0 \leq \sigma < 1, g(n) = (n + \sigma)^\gamma$. Then*

$$(2.5) \quad \sum_{n \sim X} \psi(yg(n)) \ll y^{\frac{\kappa}{1+\kappa}} X^{\frac{\lambda+\gamma\kappa}{1+\kappa}} + y^{-1} X^{1-\gamma}.$$

Proof of Corollary. Taking $M = x^{\frac{8c-5}{16}}$, by (2.2) we have

$$(2.6) \quad \sum_{n \leq x^c} |\mu(n)| (\psi(-(n+1)^\gamma) - \psi(-n^\gamma))$$

$$\begin{aligned}
 &= \sum_{d \leq x^{\frac{5}{2}}} \mu(d) \sum_{n \leq x^c d^{-2}} (\psi(-(d^2 n + 1)^\gamma) - \psi(-(d^2 n)^\gamma)) \\
 &= \sum_{d \leq M} \mu(d) \sum_{n \leq x^c d^{-2}} (\psi(-(d^2 n + 1)^\gamma) - \psi(-(d^2 n)^\gamma)) \\
 &\quad + \sum_{M < d \leq x^{\frac{5}{2}}} \mu(d) \sum_{n \leq x^c d^{-2}} (\psi(-(d^2 n + 1)^\gamma) - \psi(-(d^2 n)^\gamma)).
 \end{aligned}$$

□

By Lemma 3 with $(\kappa, \lambda) = (\frac{2}{7}, \frac{4}{7})$ and simple splitting argument we have

$$\begin{aligned}
 (2.7) \quad &\sum_{d \leq M} \mu(d) \sum_{n \leq x^c d^{-2}} (\psi(-(d^2 n + 1)^\gamma) - \psi(-(d^2 n)^\gamma)) \\
 &\ll L \sum_{d \leq M} \left((d^{2\gamma})^{\frac{\kappa}{1+\kappa}} (x^c d^{-2})^{\frac{\lambda+\gamma\kappa}{1+\kappa}} + (d^{2\gamma})^{-1} (x^c d^{-2})^{1-\gamma} \right) \\
 &\ll L \sum_{d \leq M} x^{\frac{4c+2}{9}} d^{-\frac{8}{9}} + x^{c-1} L \\
 &\ll x^{\frac{4c+2}{9}} M^{\frac{1}{9}} L + x^{c-1} L \\
 &\ll x^{\frac{8c+3}{16} + \varepsilon}.
 \end{aligned}$$

By Lemma 3 with $(\kappa, \lambda) = (\frac{1}{6}, \frac{2}{3})$ we have

$$\begin{aligned}
 (2.8) \quad &\sum_{M < d \leq x^{\frac{5}{2}}} \mu(d) \sum_{n \leq x^c d^{-2}} (\psi(-(d^2 n + 1)^\gamma) - \psi(-(d^2 n)^\gamma)) \\
 &\ll L \sum_{M < d \leq x^{\frac{5}{2}}} \left((d^{2\gamma})^{\frac{\kappa}{1+\kappa}} (x^c d^{-2})^{\frac{\lambda+\gamma\kappa}{1+\kappa}} + (d^{2\gamma})^{-1} (x^c d^{-2})^{1-\gamma} \right) \\
 &\ll L \sum_{M < d \leq x^{\frac{5}{2}}} x^{\frac{4c+1}{7}} d^{-\frac{8}{7}} + x^{c-1} L \\
 &\ll x^{\frac{4c+1}{7}} M^{-\frac{1}{7}} L + x^{c-1} L \\
 &\ll x^{\frac{8c+3}{16} + \varepsilon}.
 \end{aligned}$$

Now the Corollary follows from (2.6), (2.7), (2.8) and the Basic Proposition.

3. SOME LEMMAS

Lemma 4. Let $0 < a < b \leq 2a$. Let $f(\frac{z}{a})$ be holomorphic on an open convex set \mathbf{I} containing the real line segment $[1, b/a]$. Assume also that (1) $|f''(\frac{z}{a})| \leq M$ on \mathbf{I} , (2) $f(x)$ is real when x is real, (3) $f''(x) \leq -cM$ for some $c > 0$. Let $f'(b) = \alpha$, $f'(a) = \beta$. For each integer v in the range $\alpha \leq v \leq \beta$, define x_v by $f'(x_v) = v$. Then

$$\sum_{a < n \leq b} e(f(n)) = \sum_{\alpha < v \leq \beta} \frac{e(f(x_v) - vx_v - \frac{1}{8})}{\sqrt{|f''(x_v)|}} + O\left(M^{-\frac{1}{2}} + \log(2 + M(b - a))\right).$$

For the proof of Lemma 4, see Heath-Brown [6], Lemma 6.

Lemma 5. Suppose A_i, B_j, a_i and b_j are all positive numbers. If Q_1 and Q_2 are real with $0 < Q_1 \leq Q_2$, then there exists some q such that $Q_1 \leq q \leq Q_2$ and

$$\sum_{i=1}^m A_i q^{a_i} + \sum_{j=1}^n B_j q^{-b_j} \leq 2^{m+n} \left(\sum_{i=1}^m \sum_{j=1}^n (A_i^{b_j} B_j^{a_i})^{\frac{1}{a_i+b_j}} + \sum_{i=1}^m A_i Q_1^{a_i} + \sum_{j=1}^n B_j Q_2^{-b_j} \right).$$

This is Lemma 3 of Srinivasan [14].

Lemma 6. Let $0 < M \leq N < \mu N \leq \lambda M$, and let a_m be complex numbers with $|a_m| \leq 1$. Then we have

$$\sum_{N < m \leq \mu N} a_m = \frac{1}{2\pi} \int_{-M}^M \left(\sum_{M < m \leq \lambda M} a_m m^{-it} \right) N^{it} (\mu^{it} - 1) t^{-1} dt + O(\log(2 + M)).$$

See Lemma 6 of Fouvry and Iwaniec [5].

Lemma 7. Let α, β_1, β_2 be given real numbers with $\alpha(\beta_1 - 1)\beta_2 \neq 0$ and $\alpha \notin N$. Let $|a_m| \leq 1, |b_{m_1 m_2}| \leq 1, y \neq 0$ and

$$S = S(M, M_1, M_2) = \sum_{m \sim M} \sum_{m_1 \sim M_1} \sum_{m_2 \sim M_2} a_m b_{m_1 m_2} e(y m^\alpha m_1^{\beta_1} m_2^{\beta_2}).$$

Let $F = |y| M^\alpha M_1^{\beta_1} M_2^{\beta_2}$. Then we have

$$SL^{-3} \ll F^{\frac{\kappa}{2(1+\kappa)}} M^{\frac{1+\kappa+\lambda}{2(1+\kappa)}} (M_1 M_2)^{\frac{2+\kappa}{2(1+\kappa)}} + M(M_1 M_2)^{\frac{1}{2}} + M^{\frac{1}{2}} M_1 M_2 + F^{-\frac{1}{2}} M M_1 M_2$$

where (κ, λ) is any exponent pair.

Lemma 7 can be proved in the same way as the proof of Theorem 2 of Baker [1]. The idea of the proof is due to Heath-Brown [6].

Lemma 8. *Under the conditions of Lemma 7, if we further suppose that $F \gg M$, then*

$$SL^{-3} \ll \sqrt[22]{(M_1 M_2)^{19} M^{13} F^3} + M_1 M_2 M^{\frac{5}{8}} \sqrt[16]{1 + M^7 F^{-4}} + \sqrt[4]{(M_1 M_2)^3 M^4 (1 + FM^{-2})}.$$

Proof. This is Theorem 3 of Liu [8], which is proved essentially by the large sieve inequality developed by Bombieri and Iwaniec [3]. But the term

$$\sqrt[32]{(M_1 M_2)^{29} M^{28} F^{-2} \max(1, F^5 M^{-10})}$$

in Liu’s result is superfluous, since

$$\begin{aligned} \sqrt[32]{(M_1 M_2)^{29} M^{28} F^{-2}} &= \left(\sqrt[16]{F^{-4} M^{17} M_1 M_2} \right)^{1/4} \left(M^{5/8} M_1 M_2 \right)^{3/8} \\ &\quad \times \left(M^4 \sqrt[4]{M_1^3 M_2^3} \right)^{3/8}, \\ \sqrt[32]{(M_1 M_2)^{29} M^{18} F^3} &= \left(\sqrt[22]{F^3 M^{13} M_1^{19} M_2^{19}} \right)^{11/16} \\ &\quad \times \left(M^{5/8} M_1 M_2 \right)^{5/16} M^{-5/128}. \end{aligned}$$

□

Lemma 9. *Let $\frac{1}{2} < \gamma < 1, M > 1, H > 1, |c(h)| \leq 1, |b(d)| \leq 1$. Let $N = N_j = x^c M^{-2} 2^{-j}, (j = 1, 2, \dots), F = H(M^2 N)^\gamma$ and*

$$S(H, M, N) = \sum_{h \sim H} \sum_{d \sim M} \sum_{x^c d^{-2} 2^{-j} < n \leq x^c d^{-2} 2^{-j+1}} c(h) b(d) e(h d^2 n^\gamma),$$

then

$$(3.1) \quad S(H, M, N) L^{-4} \ll F^{\frac{1}{2}} \left(M^{1+\lambda+\kappa} N^\kappa H^{2+\kappa} \right)^{\frac{1}{2(1+\kappa)}} + (FM)^{\frac{1}{2}} H + M(NH)^{\frac{1}{2}} + MH + F^{-\frac{1}{2}} MNH.$$

Proof. By Lemma 4 and partial summation we get

$$(3.2) \quad S(H, M, N) \ll F^{-\frac{1}{2}} N \left| \sum_{h \sim H} \sum_{d \sim M} \sum_{\tilde{n} \in I} a(d) b(h, \tilde{n}) e \left((hd^{2\gamma})^{\frac{1}{1-\gamma}} \tilde{n}^{\frac{\gamma}{\gamma-1}} \right) \right| + F^{-\frac{1}{2}} MNH + LMH,$$

where $|a(d)| \leq 1, |b(h, \tilde{n})| \leq 1$ and

$$I = (\gamma 2^{(j-1)(1-\gamma)} x^{1-c} h d^2, \gamma 2^{j(1-\gamma)} x^{1-c} h d^2] \subset [c_1 \frac{F}{N}, c_2 \frac{F}{N}]$$

for some $c_1, c_2 > 0$.

Applying Lemma 6 to the variable \tilde{n} , we obtain that for some $|b_1(h, \tilde{n})| \leq 1$

(3.3)

$$S(H, M, N)L^{-1} \ll F^{-\frac{1}{2}}N \left| \sum_{h \sim H} \sum_{d \sim M} \sum_{\tilde{n} \sim F/N} a(d)b_1(h, \tilde{n})e\left((hd^{2\gamma})^{\frac{1}{1-\gamma}} \tilde{n}^{\frac{\gamma}{\gamma-1}}\right) \right| + F^{-\frac{1}{2}}MNH + LMH.$$

Now we use Lemma 7 to estimate the above sum, with $M, H, F/N$ in place of M, M_1, M_2 . This completes the proof of Lemma 9. \square

Lemma 10. *Under the conditions of Lemma 9, we have*

$$(3.4) \quad S(H, M, N)L^{-4} \ll \sqrt[22]{F^{11}M^{13}N^3H^{19}} + \sqrt[8]{F^4M^5H^8} + \sqrt[16]{F^4M^{17}H^{16}} + \sqrt[4]{FM^4NH^3} + \sqrt[4]{F^2M^2NH^3} + F^{-\frac{1}{2}}MNH.$$

Proof. Applying Lemma 8 to estimate the sum in (3.3), with $M, H, F/N$ in place of M, M_1, M_2 , we can obtain the bound (3.4) if we notice

$$MH = \left(\sqrt[16]{F^4M^{17}H^{16}}\right)^{2/3} \left(F^{-1/2}MNH\right)^{1/3} (MN^8)^{-1/24}.$$

\square

4. PROOF OF THEOREM

Taking $Y = x^{\frac{c}{3}}$, we have

$$\begin{aligned} & \sum_{n \leq x^c} |\mu(n)| (\psi(-(n+1)^\gamma) - \psi(-n^\gamma)) \\ &= \sum_{d \leq x^{\frac{c}{2}}} \mu(d) \sum_{n \leq x^c d^{-2}} (\psi(-(d^2n+1)^\gamma) - \psi(-(d^2n)^\gamma)) \\ (4.1) \quad &= \sum_{d \leq Y} \mu(d) \sum_{n \leq x^c d^{-2}} (\psi(-(d^2n+1)^\gamma) - \psi(-(d^2n)^\gamma)) \\ &+ \sum_{n \leq Y} \sum_{Y < d \leq x^{\frac{c}{2}} n^{-\frac{1}{2}}} \mu(d) (\psi(-(d^2n+1)^\gamma) - \psi(-(d^2n)^\gamma)) \\ &= \sum_1 + \sum_2, \text{ say.} \end{aligned}$$

Let $M > 1$ and $N = N_j = x^c M^{-2} 2^{-j}$ for $j = 1, 2, \dots$. Define

(4.2)

$$T(M, N) = \sum_{M < d \leq 2M} \mu(d) \sum_{\frac{x^c}{d^2 2^j} < n \leq \frac{x^c}{d^2 2^{j-1}}} (\psi(-(d^2n+1)^\gamma) - \psi(-(d^2n)^\gamma)).$$

Lemma 11. *We have*

$$(4.3) \quad T(M, N)L^{-5} \ll \left((M^2N)^{(1+\gamma)(1+\kappa)} M^{\lambda-\kappa} \right)^{\frac{1}{3+2\kappa}} + (M^2N)^{\frac{1+\gamma}{3}} \\ + MN^{\frac{1}{2}} + M^{1-\gamma}N^{1-\frac{\gamma}{2}} + ((M^2N)^{9\gamma}M^2N^4)^{\frac{1}{16}},$$

where (κ, λ) is any exponent pair.

Proof. By Lemma 2 we get for any $J > 0$

$$(4.4) \quad T(M, N) \ll MNJ^{-1} + \sum_H \frac{1}{H} |S(H, M, N)| + \sum_H \frac{1}{H} |S_1(H, M, N)|,$$

where $H = J, \frac{J}{2}, \frac{J}{2^2}, \dots$ and

$$S_1(H, M, N) = \sum_{h \sim H} c(h) \sum_{M < d \leq 2M} b_1(d) \sum_{\frac{x^c}{d^{2^{2^j}}} < n \leq \frac{x^c}{d^{2^{2^j-1}}}} e(-h(d^2n + 1)^\gamma),$$

$|c(h)| \ll 1, b_1(d) \ll 1, S(H, M, N)$ is defined in Lemma 9. □

Write $\Phi_h(y) = e(h(y^\gamma - (y + 1)^\gamma)) - 1$. By partial summation and Lemma 1, we have

$$(4.5) \quad \sum_{X < n \leq 2X} e(-h(d^2n + 1)^\gamma) - e(-h(d^2n)^\gamma) \\ = \sum_{X < n \leq 2X} \Phi_h(d^2n)e(-h(d^2n)^\gamma) \\ \ll \max_{X \leq t \leq 2X} |\Phi_h(d^2t)| \left| \sum_{X < n \leq t} e(-h(d^2n)^\gamma) \right| \\ + \int_X^{2X} \left| \frac{\partial \Phi_h(d^2t)}{\partial t} \right| \left| \sum_{X < n \leq t} e(-h(d^2n)^\gamma) \right| dt \\ \ll h(d^2X)^{\gamma-1} \max_{X \leq t \leq 2X} \left| \sum_{X < n \leq t} e(h(d^2n)^\gamma) \right| \\ \ll h(d^2X)^{\gamma-1} \{ (hd^{2\gamma}X^{\gamma-1})^\kappa X^\lambda + (hd^{2\gamma}X^{\gamma-1})^{-1} \} \\ \ll h^{1+\kappa}(d^2X)^{\gamma-1+\kappa\gamma} X^{\lambda-\kappa} + d^{-2}.$$

Here we used $\Phi_h(d^2t) \ll h(d^2X)^{\gamma-1}$ and $\frac{\partial \Phi_h(d^2t)}{\partial t} \ll hd^2(d^2X)^{\gamma-2}$ for $X \leq t \leq 2X$.

Now take $(\kappa, \lambda) = (2/7, 4/7)$ in (4.5) we have

$$\begin{aligned}
 (4.6) \quad S_1(H, M, N) &= \sum_{h \sim H} c(h) \sum_{M < d \leq 2M} b_1(d) \sum_{\frac{x^c}{d^{2j}} < n \leq \frac{x^c}{d^{2j-1}}} e(-hd^{2\gamma}n^\gamma) \\
 &+ \sum_{h \sim H} c(h) \sum_{M < d \leq 2M} b_1(d) \sum_{\frac{x^c}{d^{2j}} < n \leq \frac{x^c}{d^{2j-1}}} e(-h(d^2n + 1)^\gamma) - e(-hd^{2\gamma}n^\gamma) \\
 &= \sum_{h \sim H} c(h) \sum_{M < d \leq 2M} b_1(d) \sum_{\frac{x^c}{d^{2j}} < n \leq \frac{x^c}{d^{2j-1}}} e(-hd^{2\gamma}n^\gamma) \\
 &\quad + \sum_h \sum_d O(h^{9/7}(d^2N)^{9\gamma/7-1}N^{2/7} + d^{-2}) \\
 &= \sum_{h \sim H} c(h) \sum_{M < d \leq 2M} b_1(d) \sum_{\frac{x^c}{d^{2j}} < n \leq \frac{x^c}{d^{2j-1}}} e(-hd^{2\gamma}n^\gamma) \\
 &\quad + O\left(H^{\frac{16}{7}}(M^2N)^{\frac{9\gamma}{7}}M^{-1}N^{-\frac{5}{7}}\right).
 \end{aligned}$$

From (4.4)–(4.6) we get

$$(4.7) \quad T(M, N) \ll \frac{MN}{J} + J^{\frac{9}{7}}(M^2N)^{\frac{9\gamma}{7}}M^{-1}N^{-\frac{5}{7}} + \sum_H \frac{1}{H} |S(H, M, N)|,$$

where in $S(H, M, N)$ the coefficient of d is $b(d)$ or $b_1(d)$.

Now use Lemma 9 to estimate the above sum and then choose a best $J \in (0, +\infty)$ by Lemma 5, we obtain the bound (4.3).

Lemma 12. *We have*

$$\begin{aligned}
 (4.8) \quad T(M, N)L^{-5} &\ll \sqrt[30]{(M^2N)^{11\gamma}M^{21}N^{11}} + \sqrt[12]{(M^2N)^{4\gamma}M^9N^4} \\
 &+ \sqrt[20]{(M^2N)^{4\gamma}M^{21}N^4} + \sqrt[5]{(M^2N)^{2\gamma}M^3N^2} \\
 &+ \sqrt[4]{(M^2N)^\gamma M^4N} + \sqrt[16]{(M^2N)^{9\gamma}M^2N^4} + M^{1-\gamma}N^{1-\frac{7}{2}}.
 \end{aligned}$$

Proof. In the proof of Lemma 11, using Lemma 10 in place of Lemma 9 to estimate the sum in (4.8), we can get Lemma 12. □

Lemma 13. *We have*

$$(4.9) \quad \sum_1 \ll x^{\frac{36(c+1)}{97}}L^8.$$

Proof. In the proof of Lemma 13, we will use $M^2N \ll x^c$.

To estimate \sum_1 , we consider the following cases:

Case (i) $M \leq x^{\frac{32c+323}{1164}}$.

Similar to (2.6), by Lemma 3 with $(\kappa, \lambda) = (\frac{13}{31}, \frac{16}{31})$ we get

$$\begin{aligned}
 (4.10) \quad & \sum_{M < d \leq 2M} \mu(d) \sum_{n \leq \frac{x^c}{d^2}} \psi(-(d^2n + 1)^\gamma) - \psi(-(d^2n)^\gamma) \\
 & \ll Lx^{\frac{16c+13}{44}} \sum_{M < d \leq 2M} d^{-\frac{32}{44}} + x^{c-1}L \\
 & \ll x^{\frac{16c+13}{44}} \left(x^{\frac{32c+323}{1164}}\right)^{\frac{12}{44}} L \\
 & \ll x^{\frac{36(c+1)}{97}} L.
 \end{aligned}$$

Case (ii) $Z_1 = x^{\frac{32c+323}{1164}} < M \leq x^{\frac{14(c+1)}{97}} = Z$.

By Lemma 11 with $(\kappa, \lambda) = (\frac{11}{53}, \frac{33}{53})$ (see [6, pp. 265]) we obtain

$$\begin{aligned}
 (4.11) \quad & \sum_{M < d \leq 2M} \mu(d) \sum_{n \leq \frac{x^c}{d^2}} (\psi(-(d^2n + 1)^\gamma) - \psi(-(d^2n)^\gamma)) \\
 & \ll \left\{ \left(x^c\right)^{\frac{64(1+\gamma)}{53}} M^{\frac{22}{53}} \right\}^{\frac{53}{181}} + (x^c)^{\frac{1+\gamma}{3}} \\
 & \quad + x^{\frac{c}{2}} + (x^c)^{1-\frac{7}{2}} M^{-1} + \left\{ (x^c)^{9\gamma+4} M^{-6} \right\}^{\frac{1}{16}} \} L^7 \\
 & \ll \left\{ \left(x^{64(1+c)} Z^{22}\right)^{\frac{1}{181}} + x^{\frac{1+c}{3}} + x^{\frac{c}{2}} + x^{c-\frac{1}{2}} Z_1^{-1} + \left(x^{9+4c} Z_1^{-6}\right)^{\frac{1}{16}} \right\} L^7 \\
 & \ll x^{\frac{36(c+1)}{97}} L^7.
 \end{aligned}$$

Case (iii) $Z = x^{\frac{14(c+1)}{97}} < M \leq Y$.

By Lemma 12 we have

$$\begin{aligned}
 (4.12) \quad & \sum_{M < d \leq 2M} \mu(d) \sum_{n \leq \frac{x^c}{d^2}} (\psi(-(d^2n + 1)^\gamma) - \psi(-(d^2n)^\gamma)) \\
 & \ll \left\{ \sqrt[30]{(x^c)^{11+11\gamma} M^{-1}} + \sqrt[12]{(x^c)^{4+4\gamma} M} + \sqrt[20]{(x^c)^{4+4\gamma} M^{13}} \right. \\
 & \quad + \sqrt[5]{(x^c)^{2+2\gamma} M^{-1}} + \sqrt[4]{(x^c)^{1+\gamma} M^2} + \sqrt[16]{(x^c)^{4+9\gamma} M^{-6}} \\
 & \quad \left. + (x^c)^{1-\frac{7}{2}} M^{-1} \right\} L^7 \\
 & \ll \left\{ \sqrt[30]{x^{11(1+c)} Z^{-1}} + \sqrt[12]{x^{4(1+c)} Y} + \sqrt[20]{x^{4(1+c)} Y^{13}} \right. \\
 & \quad + \sqrt[5]{x^{2(1+c)} Z^{-1}} + \sqrt[4]{x^{1+c} Y^2} + \sqrt[16]{x^{4c+9} Z^{-6}} + x^{c-\frac{1}{2}} Z^{-1} \left. \right\} L^7 \\
 & \ll x^{\frac{36(c+1)}{97}} L^7.
 \end{aligned}$$

Combining (4.10)—(4.12) completes the proof of Lemma 13. □

Lemma 14. *We have*

$$(4.13) \quad \sum_2 \ll x^{\max(\frac{7c}{12}, \frac{32c+9}{66}, \frac{5c+3}{12})} L^8.$$

Proof. Let $N \geq 1$, $M = M_j = \frac{x^{\frac{c}{2}}}{N^{\frac{1}{2}2^j}}$ for $j = 1, 2, \dots$, and $F = H(M^2N)^\gamma$, take $J = (M^2N)^{\frac{1-\gamma}{2}}$. By lemma 2 we have

$$(4.14) \quad \begin{aligned} T_1(N, M) &= \sum_{n \sim N} \sum_{\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}2^j}} < d \leq \frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}2^{j-1}}} r(d) (\psi(-(d^2n+1)^\gamma) - \psi(-(d^2n)^\gamma)) \\ &\ll MNJ^{-1} + \sum_H \frac{1}{H} |T_1(H, N, M)| + \sum_H \frac{1}{H} |T_2(H, N, M)|, \end{aligned}$$

where H runs through $J, \frac{J}{2}, \frac{J}{2^2}, \dots$, and

$$(4.15) \quad T_1(H, N, M) = \sum_{h \sim H} \sum_{n \sim N} \sum_{\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}2^j}} < d \leq \frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}2^{j-1}}} c(h)r(d)e(-h(d^2n)^\gamma),$$

$$T_2(H, N, M) = \sum_{h \sim H} \sum_{n \sim N} \sum_{\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}2^j}} < d \leq \frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}2^{j-1}}} c_1(h)r_1(d)e(-h(d^2n+1)^\gamma),$$

for some $c(h) \ll 1, c_1(h) \ll 1, r(d) \ll 1, r_1(d) \ll 1$.

Since $e(-h(d^2n+1)^\gamma) - e(-h(d^2n)^\gamma) \ll |h|(d^2n)^{\gamma-1}$, we have

$$\begin{aligned} T_2(H, N, M) &= \sum_{h \sim H} \sum_{n \sim N} \sum_{\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}2^j}} < d \leq \frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}2^{j-1}}} c_1(h)r_1(d)e(-h(d^2n)^\gamma) \\ &+ \sum_{h \sim H} \sum_{n \sim N} \sum_{\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}2^j}} < d \leq \frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}2^{j-1}}} c_1(h)r_1(d)\Phi_h(d^2n)e(-h(d^2n)^\gamma) \\ &\ll \sum_{h \sim H} \sum_{n \sim N} \sum_{\frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}2^j}} < d \leq \frac{x^{\frac{c}{2}}}{n^{\frac{1}{2}2^{j-1}}} c_1(h)r_1(d)e(-h(d^2n)^\gamma) \\ &\quad + MN(M^2N)^{\gamma-1}LH. \end{aligned}$$

So it suffices to bound $T_1(H, M, N)$.

Now first applying Lemma 6 to the variable d and then using Lemma 8 to estimate the sum directly with $(\alpha, \beta_1, \beta_2) = (2\gamma, \gamma, 1)$, we get

$$L^{-4}T_1(N, M) \ll \sqrt{(MN)^2(M^2N)^{\gamma-1}} + \sum_H \frac{1}{H} \left\{ {}^{22}\sqrt{(HN)^{19}M^{13}F^3} \right. \\ \left. + HNM^{\frac{5}{8}} \sqrt[16]{1 + M^7F^{-4}} + \sqrt[4]{(HN)^3M^4(1 + FM^{-2})} \right\}.$$

Using the bound $M^2N \ll x^c$, one has

$$(4.16) \quad L^{-6}T_1(N, M) \ll \sqrt{(x^c)^\gamma N} + \sqrt[44]{(x^c)^{13+6\gamma}N^{25}} + \sqrt[16]{x^{5c}N^{11}} \\ + \sqrt[32]{x^{17c-8}N^{15}} + \sqrt[4]{x^{2c}N} + \sqrt[4]{x^{1+c}N^2}.$$

Now we note that $N \leq x^{\frac{5}{3}}$, by (4.16) and simple splitting argument, the bound (4.13) could be obtained at once. \square

Finally, Combining Lemma 13, Lemma 14, (4.1) and the basic Proposition completes the proof of Theorem .

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