

MARTIN EPKENHANS

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An Analogue of Pfister's Local-Global Principle in the Burnside Ring

par MARTIN EPKENHANS

RÉSUMÉ. Soit N/K une extension galoisienne de groupe de Galois \mathcal{G} . On étudie l'ensemble $\mathcal{T}(\mathcal{G})$ des combinaisons linéaires sur \mathbb{Z} de caractères de l'anneau de Burnside $\mathcal{B}(\mathcal{G})$, qui induisent des combinaisons \mathbb{Z} -linéaires des formes trace de sous-extensions de N/K qui sont triviales dans l'anneau de Witt $W(K)$ de K . On montre que le sous-groupe de torsion de $\mathcal{B}(\mathcal{G})/\mathcal{T}(\mathcal{G})$ est le noyau de l'homomorphisme signature.

ABSTRACT. Let N/K be a Galois extension with Galois group \mathcal{G} . We study the set $\mathcal{T}(\mathcal{G})$ of \mathbb{Z} -linear combinations of characters in the Burnside ring $\mathcal{B}(\mathcal{G})$ which give rise to \mathbb{Z} -linear combinations of trace forms of subextensions of N/K which are trivial in the Witt ring $W(K)$ of K . In particular, we prove that the torsion subgroup of $\mathcal{B}(\mathcal{G})/\mathcal{T}(\mathcal{G})$ coincides with the kernel of the total signature homomorphism.

1. INTRODUCTION

Let L/K be a finite, separable extension of fields of characteristic $\neq 2$. With it we associate the 'trace form' which is defined by $\mathrm{tr}_{L/K} : L \rightarrow K : x \mapsto \mathrm{tr}_{L/K} x^2$. P.E. Conner started to investigate the connection of the trace form of L/K and the trace form of a normal closure N/K of L/K . His work yields some polynomial vanishing theorems for trace forms (see [1]). These identities come from identities in the Burnside ring of the Galois group $\mathcal{G} = G(N/K)$ of N/K . We study the trace ideal $\mathcal{T}(\mathcal{G})$ in $\mathcal{B}(\mathcal{G})$, which is roughly speaking the set of \mathbb{Z} -linear combinations of trace forms of subextensions of N/K which are trivial in the Witt ring $W(K)$ of K .

We first recall the definition of the Burnside ring $\mathcal{B}(\mathcal{G})$ of a finite group \mathcal{G} . A theorem of Springer [6] gives rise to a homomorphism $h_{N/K} : \mathcal{B}(\mathcal{G}) \rightarrow W(K)$. The trace ideal $\mathcal{T}(\mathcal{G})$ is a finitely generated subgroup of the free abelian group $\mathcal{B}(\mathcal{G})$. We introduce a signature homomorphism $\mathrm{sign}_\sigma : \mathcal{B}(\mathcal{G}) \rightarrow \mathbb{Z}$ for each element $\sigma \in \mathcal{G}$ of order ≤ 2 . These signature homomorphisms correspond to signatures of the Witt ring. We conclude that $\mathcal{T}(\mathcal{G})$ is contained in the intersection $L(\mathcal{G})$ of all kernels of signatures. The

main theorem states that $\mathcal{T}(\mathcal{G})$ and $L(\mathcal{G})$ are of equal rank. Hence the torsion subgroup of $\mathcal{B}(\mathcal{G})/\mathcal{T}(\mathcal{G})$ is given by the kernel of the total signature homomorphism. In section 7 we reduce our approach to 2-groups. The general case follows by induction via the Frattini subgroup of \mathcal{G} .

2. NOTATION

We first fix our notations. Let K be a field. Then K^* denotes the multiplicative group of K , K^{*2} is the group of squares in K^* . We write K_s for a separable closure of K .

Let N/K be a Galois extension, then $G(N/K)$ denotes the Galois group of N/K . If $\mathcal{H} < G(N/K)$ then $N^{\mathcal{H}}$ is the fixed field of \mathcal{H} in N . Let $\text{Aut}(K)$ be the group of field automorphisms of K .

Now let K be a field of characteristic $\neq 2$. Let ψ, φ be non-degenerate quadratic forms over K . Then $\det_K \psi$ is the determinant of ψ . If \mathfrak{p} is a real place of K then $\text{sign}_{\mathfrak{p}} \psi$ is the signature of ψ with respect to \mathfrak{p} . $\psi \otimes \varphi$ is the product of ψ and φ . For $m \in \mathbb{Z}$, $m \times \psi$ is the m -fold sum of ψ . $\psi \simeq \varphi$ indicates the isometry of ψ and φ over K . Let L/K be a field extension. Then ψ_L is the lifting of ψ to a form over L by scalar extension. $W(K)$ is the Witt ring of K . Let $a_1, \dots, a_n \in K^*$. Then $\langle a_1, \dots, a_n \rangle$ is the diagonal form $a_1 X_1^2 + \dots + a_n X_n^2$ over K . $\langle\langle a_1, \dots, a_n \rangle\rangle = \otimes_{i=1, \dots, n} \langle 1, -a_i \rangle$ is the n -fold Pfister form defined by a_1, \dots, a_n .

Let L/K be a finite and separable field extension. The trace form of L/K is the non-degenerate quadratic form $\text{tr}_{L/K} \langle 1 \rangle : L \rightarrow K : x \mapsto \text{tr}_{L/K}(x^2)$. We denote the trace form also by $\langle L/K \rangle$, resp $\langle L \rangle$ if no confusion can arise.

Let M be a set. Then $\#M$ is the cardinality of M . $\text{ord}(\mathcal{G})$, $\text{ord}(\sigma)$ is the order of the finite group \mathcal{G} , resp. of the element $\sigma \in \mathcal{G}$.

3. THE BURNSIDE RING $\mathcal{B}(\mathcal{G})$

Let \mathcal{G} be a finite group and let $\mathcal{H} < \mathcal{G}$ be a subgroup of \mathcal{G} . We denote the transitive action of \mathcal{G} on the set of left cosets $\mathcal{G}/\mathcal{H} = \{a\mathcal{H}, a \in \mathcal{G}\}$ by $(\mathcal{G}, \mathcal{G}/\mathcal{H})$. The transitive and faithful actions of \mathcal{G} on finite sets are in one-to-one correspondence with the set of conjugacy classes of subgroups of \mathcal{G} . A subgroup \mathcal{H} of \mathcal{G} induces a transitive action of degree $[\mathcal{G} : \mathcal{H}]$, hence a representation of dimension $[\mathcal{G} : \mathcal{H}]$. Let $\chi_{\mathcal{H}}$ denote the corresponding character. We sometimes write $\chi_{\mathcal{H}}^{\mathcal{G}}$ to indicate that the character is defined on \mathcal{G} .

Definition 1. Let \mathcal{G} be a finite group. The *Burnside ring* $\mathcal{B}(\mathcal{G})$ of \mathcal{G} is the free abelian group freely generated by the set $\{\chi_{\mathcal{H}} \mid \mathcal{H} \text{ runs over representatives of conjugacy classes of subgroups of } \mathcal{G}\}$

and with multiplication given by

$$\chi_{\mathcal{U}_1} \cdot \chi_{\mathcal{U}_2} = \bigoplus_{\sigma \in \mathcal{U}_1 \backslash \mathcal{G} / \mathcal{U}_2} \chi_{\mathcal{U}_1 \cap \sigma \mathcal{U}_2 \sigma^{-1}},$$

where the sum runs over a set of representatives of the double cosets in $\mathcal{U}_1 \backslash \mathcal{G} / \mathcal{U}_2$.

Remark 2. $\chi_{\mathcal{G}}$ is the multiplicative identity, $\chi_{\{e\}} =: \chi_1$ is the regular character.

Another way of defining the multiplication is as follows. Let $\rho_i : \mathcal{G} \rightarrow \mathrm{GL}(V_i), i = 1, 2$ be representations of \mathcal{G} . Then $\rho_1 \otimes \rho_2 : \mathcal{G} \times \mathcal{G} \rightarrow \mathrm{GL}(V_1 \otimes V_2)$ is a representation of $\mathcal{G} \times \mathcal{G}$ on $V_1 \otimes V_2$. According to the diagonal embedding $\mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$ the representation $\rho_1 \otimes \rho_2$ restricts to a representation of \mathcal{G} on $V_1 \otimes V_2$. For $\rho_i = (\mathcal{G}, \mathcal{G}/\mathcal{U}_i)$ we get $\rho_1 \otimes \rho_2|_{\mathcal{G}} = \bigoplus_{\sigma \in \mathcal{U}_1 \backslash \mathcal{G} / \mathcal{U}_2} (\mathcal{G}, \mathcal{G}/(\mathcal{U}_1 \cap \sigma \mathcal{U}_2 \sigma^{-1}))$.

4. THE HOMOMORPHISM $h_{N/K} : \mathcal{B}(G(N/K)) \rightarrow \mathrm{W}(K)$

Proposition 3 (T.A. Springer). *Let N/K be a finite Galois extension with Galois group $G(N/K) = \mathcal{G}$. Then there is a well-defined ring homomorphism*

$$h_{N/K} : \mathcal{B}(\mathcal{G}) \rightarrow \mathrm{W}(K)$$

with

$$h_{N/K}(\chi_{\mathcal{H}}) = \langle N^{\mathcal{H}} \rangle$$

for all subgroups \mathcal{H} of \mathcal{G} .

Proof. Let $\mathcal{H} < \mathcal{G}$ be a subgroup of \mathcal{G} . Then $h_{N/K}$ is well-defined as a group homomorphism since $\langle N^{\sigma \mathcal{H} \sigma^{-1}} \rangle = \langle \sigma(N^{\mathcal{H}}) \rangle = \langle N^{\mathcal{H}} \rangle$. Now the assertion follows from the next lemma. \square

Lemma 4. *Let N/K be a finite Galois extension with Galois group $\mathcal{G} = G(N/K)$. Let $\mathcal{U}_1, \mathcal{U}_2$ be subgroups of $G(N/K)$. Then*

$$\langle N^{\mathcal{U}_1} \rangle \otimes \langle N^{\mathcal{U}_2} \rangle = \bigoplus_{\sigma \in \mathcal{U}_1 \backslash \mathcal{G} / \mathcal{U}_2} \langle N^{\mathcal{U}_1 \cap \sigma \mathcal{U}_2 \sigma^{-1}} \rangle,$$

where the sum runs over a set of representatives of the double cosets $\mathcal{U}_1 \backslash \mathcal{G} / \mathcal{U}_2$.

Proof. (see [2], I.6.2) Let $\alpha \in N$ with $N^{\mathcal{U}_1} = K(\alpha)$ and let $f \in K[X]$ be the minimal polynomial of α over K . Set $L := N^{\mathcal{U}_2}$. From Frobenius reciprocity [5], 2.5.6 we get

$$\begin{aligned} \langle N^{\mathcal{U}_1} \rangle \otimes \langle N^{\mathcal{U}_2} \rangle &= \langle K(\alpha) \rangle \otimes \langle L \rangle = \mathrm{tr}_{L/K}((\mathrm{tr}_{K(\alpha)/K} \langle 1 \rangle)_L) \\ &= \mathrm{tr}_{L/K} \langle (L[X]/(f)) / L \rangle \\ &= \bigoplus_{i=1, \dots, r} \mathrm{tr}_{L/K} \langle (L[X]/(f_i)) / L \rangle, \end{aligned}$$

where $f = f_1 \cdots f_r$ is the decomposition of f into monic irreducible polynomials in $L[X]$. Now consider $\mathrm{tr}_{L/K} \langle (L[X]/(g)) / L \rangle$ for some monic prime

divisor $g \in L[X]$ of f . Then g is the minimal polynomial of some conjugate $\sigma(\alpha)$ of α over L . Hence

$$\mathrm{tr}_{L/K}\langle(L[X]/(g))/L\rangle = \mathrm{tr}_{L/K}(\mathrm{tr}_{L(\sigma(\alpha))/L}\langle 1 \rangle) = \langle L(\sigma(\alpha)) \rangle .$$

Now

$$L(\sigma(\alpha)) = L \cdot K(\sigma(\alpha)) = L \cdot \sigma(K(\alpha)) = N^{\mathcal{U}_2} \cdot \sigma(N^{\mathcal{U}_1}) = N^{\mathcal{U}_1 \cap \sigma \mathcal{U}_2 \sigma^{-1}} .$$

The action of \mathcal{G} on the roots of f induces an action of \mathcal{U}_2 on the roots of f , which is equivalent to the action of \mathcal{U}_2 on $\mathcal{G}/\mathcal{U}_1$. Each orbit of this action corresponds to a monic irreducible factor $g \in L[X]$ of f . \square

5. THE TRACE IDEAL IN $\mathcal{B}(\mathcal{G})$

Definition 5. Let \mathcal{G} be a finite group. Set

$$\mathcal{T}(\mathcal{G}) := \bigcap \ker(h_{N/K}) ,$$

where the intersection is taken over all Galois extensions N/K over all fields K of characteristic $\neq 2$ with Galois group $G(N/K) \simeq \mathcal{G}$. We call $\mathcal{T}(\mathcal{G})$ the *trace ideal of $\mathcal{B}(\mathcal{G})$* .

6. THE MAIN RESULTS

Theorem 6. *Let \mathcal{G} be a finite group. Then the trace ideal $\mathcal{T}(\mathcal{G})$ of $\mathcal{B}(\mathcal{G})$ is a free abelian group of rank*

$$\mathrm{rank}(\mathcal{T}(\mathcal{G})) = \mathrm{rank}(\mathcal{B}(\mathcal{G})) - \#\{\text{conjugacy classes of elements } \sigma \in \mathcal{G} \text{ of order } \leq 2\} .$$

The proof of theorem 6 will be organized as follows. We start by defining in a rather canonical way signatures for elements in the Burnside ring. By lemma 8, the trace ideal is contained in the kernel $L(\mathcal{G})$ of the total signature homomorphism. We compute the rank $L(\mathcal{G})$ in lemma 14. Now the assertion follows from the equality of the ranks of $\mathcal{T}(\mathcal{G})$ and $L(\mathcal{G})$, whose proof will be the subject of sections 7 and 8. In section 7 we reduce the proof of theorem 6 to 2-groups. Section 8 contains the proof of theorem 6 for 2-groups. It runs via induction over the Frattini subgroup of \mathcal{G} .

If \mathcal{G} is a finite group then $\mathrm{RC}(\mathcal{G})$ denotes a set of representatives of the conjugacy classes of subgroups of \mathcal{G} . Further, $\mathrm{RC}_2(\mathcal{G})$ denotes a set of representatives of the conjugacy classes of elements of order 1 or 2 in \mathcal{G} . Let \mathcal{G}_2 be a 2-Sylow subgroup of \mathcal{G} . Then we can choose $\mathrm{RC}_2(\mathcal{G}) \subset \mathcal{G}_2$.

In the sequel we will use the following proposition of Sylvester.

Proposition 7. *Let K be field, \mathfrak{p} be an ordering of K . Then for any separable polynomial $f(X) \in K[X]$ the signature of the trace form of $K[X]/(f(X))$ over K equals the number of real roots of $f(X)$ with respect to the ordering \mathfrak{p} .*

For a proof see [7].

Lemma 8. *Let \mathcal{G} be a finite group and let $\sigma \in \mathcal{G}$ be an element of order ≤ 2 . Then there is a Galois extension N/K of algebraic number fields and an isomorphism $\iota : \mathcal{G} \xrightarrow{\sim} G(N/K)$ such that*

1. $K \subset \mathbb{R}$ and $N \subset \mathbb{C}$.
2. $\iota(\sigma)$ is induced by the complex conjugation.

Proof. Set $n := \text{ord}(\mathcal{G})$.

1. $\text{ord}(\sigma) = 2$. If $n = 2$, set $K = \mathbb{Q}, N = \mathbb{Q}(\sqrt{-1})$.

Now let $n = 2m \geq 4$. Consider the quadratic form $\psi = (m-1) \times \langle 1, -1 \rangle \perp \langle 1, -2 \rangle$ as a form over \mathbb{Q} . Then $\det_{\mathbb{Q}} \psi \notin \mathbb{Q}^{*2}$ and $\text{sign}_{\mathbb{Q}} \psi = 0$. By theorems 1 and 3 of [4] there is a field extension L/\mathbb{Q} with normal closure N/\mathbb{Q} such that $N \subset \mathbb{C}, G(N/\mathbb{Q}) \simeq \mathfrak{S}_n$ and L/\mathbb{Q} has trace form ψ . Here \mathfrak{S}_n denotes the symmetric group on n elements.

Let $\alpha \in L$ be a primitive element of L/\mathbb{Q} . Since $\text{sign}_{\mathbb{Q}} \langle L \rangle = 0$ no conjugate of α is real (see proposition 7). Let $M := \{\alpha_1, \bar{\alpha}_1, \dots, \alpha_m, \bar{\alpha}_m\}$ be the set of conjugates of α . $\bar{\alpha}$ is the complex conjugate of $\alpha \in \mathbb{C}$. Let $\varphi : \mathcal{G} \rightarrow M$ be a bijection such that for each $a \in \mathcal{G}$ the set $\varphi(\{a, \sigma(a)\})$ consists of a pair of complex conjugate elements of M . Now according to the identification given by φ we get a monomorphism $\iota : \mathcal{G} \hookrightarrow S(\mathcal{G}) \xrightarrow{\sim} S(M) \xrightarrow{\sim} G(N/\mathbb{Q})$. Then $\iota(\sigma)$ is given by the complex conjugation on N . Set $K := N^{\iota(\mathcal{G})}$. Since $\iota(\sigma) \in \iota(\mathcal{G})$ the field K is real.

2. $\sigma = \text{id}$. Set $\psi = (n-1) \times \langle 1 \rangle \perp \langle 2 \rangle$.

Then $\det_{\mathbb{Q}} \psi \notin \mathbb{Q}^{*2}$ and $\text{sign}_{\mathbb{Q}} \psi = n$. Now choose L, N and $\alpha \in L$ as above. Since $\text{sign}_{\mathbb{Q}} \psi = \text{sign}_{\mathbb{Q}} \langle L \rangle = n$ all conjugates of α are real. Hence $L \subset N \subset \mathbb{R}$. Choose any injection $\iota : \mathcal{G} \hookrightarrow G(N/\mathbb{Q})$ and set $K := N^{\iota(\mathcal{G})} \subset \mathbb{R}$. \square

Set

$$X = \sum_{\mathcal{H} \in \text{RC}(\mathcal{G})} m_{\mathcal{H}} \cdot \chi_{\mathcal{H}}, \quad m_{\mathcal{H}} \in \mathbb{Z}.$$

Let N/K be a Galois extension with Galois group $G(N/K) = \mathcal{G}$. Let \mathfrak{p} be a real place of K . Then

$$h_{N/K}(X) = \sum_{\mathcal{H} \in \text{RC}(\mathcal{G})} m_{\mathcal{H}} \times \langle N^{\mathcal{H}} \rangle = 0$$

gives

$$(I) \quad \text{sign}_{\mathfrak{p}} h_{N/K}(X) = 0 = \sum_{\mathcal{H} \in \text{RC}(\mathcal{G})} m_{\mathcal{H}} \cdot \text{sign}_{\mathfrak{p}} \langle N^{\mathcal{H}} \rangle$$

Let $\mathcal{H} < \mathcal{G}$ and $N^{\mathcal{H}} = K(\alpha)$. By proposition 7, $\text{sign}_{\mathfrak{p}} \langle N^{\mathcal{H}} \rangle$ equals the number of real conjugates of α with respect to the ordering \mathfrak{p} . Let $\sigma \in G(N/K)$ be the automorphism which is induced by the complex conjugation. Then $\text{sign}_{\mathfrak{p}} \langle N^{\mathcal{H}} \rangle$ is the number of fixed points of the action of $\langle \sigma \rangle$ on the set of conjugates of α , which equals the number of fixed points of the action of $\langle \sigma \rangle$ on \mathcal{G}/\mathcal{H} . Therefore the equation (I) is already determined by \mathcal{G} and the conjugacy class of the complex conjugation in \mathcal{G} . This leads to the following definition.

Definition 9. Let $\sigma \in \mathcal{G}$ be an element of order ≤ 2 . Let \mathcal{H} be a subgroup of \mathcal{G} and let $\chi_{\mathcal{H}} \in \mathcal{B}(\mathcal{G})$ be the corresponding character. Set

$$\text{sign}_{\sigma} \chi_{\mathcal{H}} = \#\{\text{fixed points of } (\langle \sigma \rangle, \mathcal{G}/\mathcal{H})\}.$$

Of course, $\text{sign}_{\sigma} \chi_{\mathcal{H}} = \chi_{\mathcal{H}}(\sigma)$. Since our approach is motivated by quadratic form considerations we feel it is more convenient to talk about signatures.

As usual $C_{\mathcal{G}}(\sigma)$ denotes the centralizer of σ in \mathcal{G} . Let $\mathcal{G}\sigma = \{\rho^{-1}\sigma\rho \mid \rho \in \mathcal{G}\}$ be the set of conjugates of σ in \mathcal{G} .

Proposition 10. Let \mathcal{G} be a finite group, $\mathcal{H} < \mathcal{G}$ a subgroup of \mathcal{G} . Let $\sigma \in \mathcal{G}$ be an element of order ≤ 2 . Then

$$\text{sign}_{\sigma} \chi_{\mathcal{H}} = \frac{\text{ord}(C_{\mathcal{G}}(\sigma)) \#(\mathcal{G}\sigma \cap \mathcal{H})}{\text{ord}(\mathcal{H})} = \frac{[\mathcal{G} : \mathcal{H}] \#(\mathcal{G}\sigma \cap \mathcal{H})}{\#\mathcal{G}\sigma}$$

Proof. Consider the action of $\langle \sigma \rangle$ on \mathcal{G}/\mathcal{H} . Let $\rho \in \mathcal{G}$. Then $\rho\mathcal{H}$ is a fixed point if and only if $\rho^{-1}\sigma\rho \in \mathcal{H}$. Hence we can assume that

$$\mathcal{G}\sigma \cap \mathcal{H} = \{\sigma_1, \dots, \sigma_r\}$$

is a set of $r > 0$ elements. Let

$$M = \{(\rho, \sigma_i) \mid \rho^{-1}\sigma\rho = \sigma_i\} \subset \mathcal{G} \times \{\sigma_1, \dots, \sigma_r\}.$$

Obviously the cardinality of M is the product of $\text{ord}(\mathcal{H})$ and the number of fixed points. Further, for $i = 1, \dots, r$ we get

$$\#\{\rho \in \mathcal{G} \mid (\rho, \sigma_i) \in M\} = \text{ord}(C_{\mathcal{G}}(\sigma)).$$

Hence $\#M = \text{ord}(C_{\mathcal{G}}(\sigma)) \cdot \#\mathcal{G}\sigma \cap \mathcal{H}$. □

We abbreviate $\chi_{\langle \tau \rangle}$ to χ_{τ} .

Corollary 11. In the situation of proposition 10 we get

1. $\text{sign}_{\sigma} \chi_{\mathcal{H}} \equiv [\mathcal{G} : \mathcal{H}] \pmod{2}$.
2. $\text{sign}_{i_d} \chi_{\mathcal{H}} = [\mathcal{G} : \mathcal{H}]$.

3. $\text{sign}_\sigma \chi_{\mathcal{H}} \neq 0$ if and only if \mathcal{H} contains some conjugate of σ .
4. Let $\tau \in \mathcal{G}$ be an element of order ≤ 2 . Then $\text{sign}_\sigma \chi_\tau \neq 0$ if and only if σ and τ are conjugate or $\sigma = \text{id}$.
5. Let τ and σ be two conjugate involutions. Then

$$2 \cdot \#\mathcal{G}\sigma \cdot \text{sign}_\sigma \chi_\tau = \text{ord}(\mathcal{G}).$$

6. If \mathcal{H} is a normal subgroup of \mathcal{G} , then $\text{sign}_\sigma \chi_{\mathcal{H}} = 0$ or $= [\mathcal{G} : \mathcal{H}]$.

sign_σ extends to a homomorphism on $\mathcal{B}(\mathcal{G})$.

Proposition 12. *Let \mathcal{G} be a finite group and let $\sigma \in \mathcal{G}$ be an element of order ≤ 2 . Then there is a unique homomorphism*

$$\text{sign}_\sigma : \mathcal{B}(\mathcal{G}) \rightarrow \mathbb{Z}$$

with $\text{sign}_\sigma \chi_{\mathcal{U}} = \#\{\text{fixed points of } \langle \sigma \rangle, \mathcal{G}/\mathcal{U}\}$ for all subgroups \mathcal{U} of \mathcal{G} .

Proof. We consider the representations and characters over fields of characteristic 0. Let $\rho : \mathcal{G} \rightarrow GL(V)$ be the underlying representation of $\chi_{\mathcal{U}}$. Hence we get $\text{sign}_\sigma \chi_{\mathcal{U}} = \text{trace}(\rho(\sigma)) = \chi_{\mathcal{U}}(\sigma)$. Since $\text{trace}(A \otimes B) = \text{trace}(A) \cdot \text{trace}(B)$, sign_σ is a ring homomorphism. \square

We conclude that $\mathcal{T}(\mathcal{G})$ is contained in the intersection of all kernels of signature homomorphisms.

Definition 13. Let \mathcal{G} be a finite group. Set

$$L(\mathcal{G}) := \left\{ \sum_{\mathcal{H} \in \text{RC}(\mathcal{G})} m_{\mathcal{H}} \chi_{\mathcal{H}} \mid \sum_{\mathcal{H} \in \text{RC}(\mathcal{G})} m_{\mathcal{H}} \cdot \text{sign}_\sigma \chi_{\mathcal{H}} = 0 \right. \\ \left. \text{for all } \sigma \in \text{RC}_2(\mathcal{G}) \right\} \subset \mathcal{B}(\mathcal{G}).$$

Lemma 14. *Let \mathcal{G} be a finite group of order n . The system of linear equations given by*

$$\sum_{\mathcal{H} \in \text{RC}(\mathcal{G})} \text{sign}_\sigma \chi_{\mathcal{H}} \cdot x_{\mathcal{H}} = 0, \quad \sigma \in \text{RC}_2(\mathcal{G})$$

has rank $\#\text{RC}_2(\mathcal{G})$.

Proof. Let $\sigma_1 = \text{id}, \sigma_2, \dots, \sigma_r$ be the r distinct elements of $\text{RC}_2(\mathcal{G})$. Consider the coefficients $\text{sign}_{\sigma_j} \chi_{\langle \sigma_i \rangle}$ for $i, j = 1, \dots, r$. We get $\text{sign}_{\text{id}} \chi_{\langle \sigma_i \rangle} = \text{ord}(\mathcal{G})/\text{ord}(\sigma_i) \in \{n, n/2\}$ for $i = 1, \dots, r$. For $j = 2, \dots, r$ we have $\text{sign}_{\sigma_j} \chi_{\langle \sigma_i \rangle} \neq 0$ if and only if $i = j$. \square

Remark 15. By lemma 14, $L(\mathcal{G})$ is a free abelian group of rank

$$\text{rank}(\mathcal{B}(\mathcal{G})) - \#\text{RC}_2(\mathcal{G}).$$

Further, $\mathcal{T}(\mathcal{G}) \subset L(\mathcal{G})$ by lemma 8 and the remarks following it. We get $\text{rank}(\mathcal{T}(\mathcal{G})) = \text{rank}(L(\mathcal{G}))$ if and only if there exists a positive integer $a \in \mathbb{Z}$ with $a \cdot L(\mathcal{G}) \subset \mathcal{T}(\mathcal{G})$.

By Pfisters local-global principle, $L(\mathcal{G})$ is the set of all $X \in \mathcal{B}(\mathcal{G})$ such that $h_{N/K}(X)$ is a torsion form for any Galois extension N/K with $G(N/K) \simeq \mathcal{G}$. Hence the rank formula of theorem 6 is equivalent to the existence of an integer $l \in \mathbb{Z}, l \geq 0$ depending only on \mathcal{G} such that 2^l annihilates $h_{N/K}(L(\mathcal{G}))$ for any Galois extension N/K with Galois group \mathcal{G} .

Since $\mathcal{T}(\mathcal{G}) \subset L(\mathcal{G})$ each signature homomorphism sign_σ induces a unique signature homomorphism $\text{sign} : \mathcal{B}(\mathcal{G})/\mathcal{T}(\mathcal{G}) \rightarrow \mathbb{Z}$. Hence we easily get from Theorem 6:

Theorem 16 (Local-Global Principle). *An element $X \in \mathcal{B}(\mathcal{G})$ is a torsion element in $\mathcal{B}(\mathcal{G})/\mathcal{T}(\mathcal{G})$ if and only if $\text{sign}_\sigma(X) = 0$ for every $\sigma \in \mathcal{G}$ of order ≤ 2 . Every torsion element of $\mathcal{B}(\mathcal{G})/\mathcal{T}(\mathcal{G})$ has 2-power order.*

7. REDUCTION TO 2-GROUPS

Proposition 17. *Let \mathcal{G} be a group of odd order. Then*

$$\mathcal{T}(\mathcal{G}) = L(\mathcal{G}).$$

Hence $\text{rank}(\mathcal{T}(\mathcal{G})) = \text{rank}(\mathcal{B}(\mathcal{G})) - 1$.

Proof. Let N/K be a Galois extension with Galois group $G(N/K) \simeq \mathcal{G}$. Let L be an intermediate field of N/K . Then $\langle L \rangle = [L : K] \times \langle 1 \rangle$ (see [2], cor. I.6.5). Let $X = \sum_{\mathcal{H} \in \text{RC}(\mathcal{G})} m_{\mathcal{H}} \cdot \chi_{\mathcal{H}}$. Then $h_{N/K}(X) = \sum_{\mathcal{H} \in \text{RC}(\mathcal{G})} m_{\mathcal{H}} \cdot [\mathcal{G} : \mathcal{H}] \times \langle 1 \rangle$. Since $\text{ord}(\mathcal{G})$ is odd, $L(\mathcal{G})$ is defined by the equation $\sum_{\mathcal{H} \in \text{RC}(\mathcal{G})} m_{\mathcal{H}} \cdot [\mathcal{G} : \mathcal{H}] = 0$ (see corollary 11). Now the statement about the ranks follows from remark 15. \square

Let \mathcal{H}, \mathcal{U} be subgroups of \mathcal{G} . Then the representation defined by the action of \mathcal{G} on \mathcal{G}/\mathcal{U} restricts to a representation of \mathcal{H} on \mathcal{G}/\mathcal{U} . This defines a ring homomorphism

$$\text{res}_{\mathcal{H}}^{\mathcal{G}} : \mathcal{B}(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{H}),$$

the ‘restriction map’. We get

$$\text{res}_{\mathcal{H}}^{\mathcal{G}} \chi_{\mathcal{U}}^{\mathcal{G}} = \bigoplus_{\sigma \in \mathcal{H} \backslash \mathcal{G}/\mathcal{U}} \chi_{\mathcal{H} \cap \sigma \mathcal{U} \sigma^{-1}}^{\mathcal{H}} \in \mathcal{B}(\mathcal{H}),$$

where $\chi_{\mathcal{H} \cap \sigma \mathcal{U} \sigma^{-1}}^{\mathcal{H}}$ is a character of \mathcal{H} .

Proposition 18. *Let \mathcal{G} be a finite group and let $\mathcal{H} < \mathcal{G}$. Let $\sigma \in \mathcal{H}$ be an element of order ≤ 2 . Then*

$$\begin{array}{ccc}
 \mathcal{B}(\mathcal{G}) & \xrightarrow{\text{sign}_\sigma} & \mathbb{Z} \\
 \text{res}_\mathcal{H}^\mathcal{G} \downarrow & & \nearrow \\
 \mathcal{B}(\mathcal{H}) & & \text{sign}_\sigma
 \end{array}$$

commutes.

Proof. Let $\mathcal{U} < \mathcal{G}$. We compute the signature of $\text{res}_\mathcal{H}^\mathcal{G} \chi_\mathcal{U}^\mathcal{G}$ as follows: Restrict the action of \mathcal{G} on \mathcal{G}/\mathcal{U} to \mathcal{H} . Then count the number of fixed points of $\langle \sigma \rangle$ according to this action. Of course, this number equals $\text{sign}_\sigma \chi_\mathcal{U}^\mathcal{G}$. \square

There is an additive but not multiplicative *corestriction* map $\text{cor}_\mathcal{H}^\mathcal{G} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{G})$ defined by $\text{cor}_\mathcal{H}^\mathcal{G} \chi_\mathcal{U}^\mathcal{H} = \chi_\mathcal{U}^\mathcal{G}$.

Proposition 19. *Let \mathcal{G} be a finite group, $\mathcal{H} < \mathcal{G}$. Let N/K be a Galois extension with $G(N/K) = \mathcal{G}$. Let $s^* : W(K) \rightarrow W(N^\mathcal{H})$ be the lifting homomorphism. Then*

$$\begin{array}{ccc}
 \mathcal{B}(\mathcal{G}) & \xrightarrow{h_{N/K}} & W(K) \\
 \text{res}_\mathcal{H}^\mathcal{G} \downarrow & & \downarrow s^* \\
 \mathcal{B}(\mathcal{H}) & \xrightarrow{h_{N/N^\mathcal{H}}} & W(N^\mathcal{H})
 \end{array}$$

and

$$\begin{array}{ccc}
 \mathcal{B}(\mathcal{H}) & \xrightarrow{h_{N/N^\mathcal{H}}} & W(N^\mathcal{H}) \\
 \text{cor}_\mathcal{H}^\mathcal{G} \downarrow & & \downarrow \text{tr}_{N^\mathcal{H}/K} \\
 \mathcal{B}(\mathcal{G}) & \xrightarrow{h_{N/K}} & W(K)
 \end{array}$$

commute.

Proof. We use the notation of lemma 4 and its proof. Set $L := N^\mathcal{H}$. Then

$$\begin{aligned}
 h_{N/L}(\text{res}_\mathcal{H}^\mathcal{G}(\chi_\mathcal{U}^\mathcal{G})) &= \perp_{\sigma \in \mathcal{U} \setminus \mathcal{G}/\mathcal{H}} h_{N/L}(\chi_{\mathcal{H} \cap \sigma \mathcal{U} \sigma^{-1}}^\mathcal{H}) \\
 &= \perp_{\sigma \in \mathcal{U} \setminus \mathcal{G}/\mathcal{H}} \langle N^{\mathcal{H} \cap \sigma \mathcal{U} \sigma^{-1}} / L \rangle = \perp_{i=1, \dots, r} \langle (L[X]/(f_i)) / L \rangle \\
 &= \langle (L[X]/(f_1 \cdots f_r)) / L \rangle = \langle (K[X]/(f)) \otimes L \rangle \\
 &= s^*(\langle N^\mathcal{U} / K \rangle) = s^* \circ h_{N/K}(\chi_\mathcal{U}^\mathcal{G}).
 \end{aligned}$$

Let $\mathcal{U} < \mathcal{H} < \mathcal{G}$. Then

$$\begin{aligned} h_{N/K} \circ \text{cor}_{\mathcal{H}}^{\mathcal{G}}(\chi_{\mathcal{U}}^{\mathcal{H}}) &= h_{N/K}(\chi_{\mathcal{U}}^{\mathcal{G}}) = \langle N^{\mathcal{U}} \rangle = \\ &= \text{tr}_{N^{\mathcal{H}}/K} \langle N^{\mathcal{U}}/N^{\mathcal{H}} \rangle = \text{tr}_{N^{\mathcal{H}}/K}(h_{N/N^{\mathcal{H}}}(\chi_{\mathcal{U}}^{\mathcal{H}})). \end{aligned}$$

□

Lemma 20. *Let $\mathcal{H} < \mathcal{G}$ be finite groups.*

1. *Then $\text{res}_{\mathcal{H}}^{\mathcal{G}}(L(\mathcal{G})) \subset L(\mathcal{H})$.*
2. *Let $[\mathcal{G} : \mathcal{H}]$ be odd.*
 - (a) *Then $\text{res}_{\mathcal{H}}^{\mathcal{G}}(X) \in L(\mathcal{H})$ if and only if $X \in L(\mathcal{G})$.*
 - (b) *$\text{res}_{\mathcal{H}}^{\mathcal{G}}(X) \in \mathcal{T}(\mathcal{H})$ implies $X \in \mathcal{T}(\mathcal{G})$.*

Proof. 1. follows from proposition 18.

2. Choose $\text{RC}_2(\mathcal{G}) \subset \text{RC}_2(\mathcal{H})$ and apply proposition 18.

(b) Let N/K be a Galois extension with $G(N/K) = \mathcal{G}$ and let $X \in \mathcal{B}(\mathcal{G})$ with $\text{res}_{\mathcal{H}}^{\mathcal{G}}(X) \in \mathcal{T}(\mathcal{H})$. Now $h_{N/N^{\mathcal{H}}} \circ \text{res}_{\mathcal{H}}^{\mathcal{G}}(X) = 0 = s^* \circ h_{N/K}(X)$ by proposition 19. By a theorem of Springer s^* is injective (see [5], 2.5.3). Thus $h_{N/K}(X) = 0$. □

From $X \in \mathcal{T}(\mathcal{G})$ we get $X \in \ker(h_{N/K})$, hence $\text{res}_{\mathcal{H}}^{\mathcal{G}}(X) \in \ker(h_{N/N^{\mathcal{H}}})$. But we do not get $\text{res}_{\mathcal{H}}^{\mathcal{G}}(X) \in \mathcal{T}(\mathcal{H})$. We only get $\text{res}_{\mathcal{H}}^{\mathcal{G}}(X) \in \bigcap \ker(h_{N/K})$, where the intersection runs over all Galois extensions N/K with Galois group \mathcal{G} and such that $\mathcal{G} < \text{Aut}(N)$.

Let $\text{exp}(\mathcal{G})$ denote the exponent of \mathcal{G} .

Proposition 21. *Let \mathcal{G} be a finite group and let \mathcal{G}_2 be a 2-Sylow subgroup of \mathcal{G} .*

1. *Then the rank formula of theorem 6 holds for \mathcal{G} if it holds for any 2-Sylow subgroup of \mathcal{G} , in which case $\text{exp}(L(\mathcal{G})/\mathcal{T}(\mathcal{G}))$ divides the exponent of $L(\mathcal{G}_2)/\mathcal{T}(\mathcal{G}_2)$.*
2. *Suppose there is a set \mathcal{X} of fields such that $\mathcal{G} < \text{Aut}(N)$ for any $N \in \mathcal{X}$ and such that*

$$\mathcal{T}(\mathcal{G}_2) = \bigcap_{N \in \mathcal{X}} \bigcap_{\mathcal{U} < \text{Aut}(N), \mathcal{U} \simeq \mathcal{G}_2} \ker(h_{N/N^{\mathcal{U}}}).$$

Then $X \in \mathcal{T}(\mathcal{G})$ if and only if $\text{res}_{\mathcal{G}_2}^{\mathcal{G}}(X) \in \mathcal{T}(\mathcal{G}_2)$. Hence $L(\mathcal{G})/\mathcal{T}(\mathcal{G})$ is isomorphic to a subgroup of $L(\mathcal{G}_2)/\mathcal{T}(\mathcal{G}_2)$.

Proof. 1. If the rank formula holds for \mathcal{G}_2 , then by remark 15 there is a positive integer a with $a \cdot L(\mathcal{G}_2) \subset \mathcal{T}(\mathcal{G}_2)$. Let $X \in L(\mathcal{G})$. Then $\text{res}_{\mathcal{G}_2}^{\mathcal{G}}(X) \in L(\mathcal{G}_2)$ and $\text{res}_{\mathcal{G}_2}^{\mathcal{G}}(aX) = a \cdot \text{res}_{\mathcal{G}_2}^{\mathcal{G}}(X) \in a \cdot L(\mathcal{G}_2) \subset \mathcal{T}(\mathcal{G}_2)$. Hence $aX \in \mathcal{T}(\mathcal{G})$ by

lemma 20(2)(b). The proof of (2) is left to the reader. \square

8. PROOF OF THEOREM 6

Let $\mathcal{J}_2(\mathcal{G})$ be the set of involutions of the 2-group \mathcal{G} . For a subgroup \mathcal{H} of \mathcal{G} define

$$X_{\mathcal{H}}^{\mathcal{G}} := X_{\mathcal{H}} := \text{ord}(\mathcal{H}) \cdot \chi_{\mathcal{H}}^{\mathcal{G}} - \chi_1^{\mathcal{G}} + \sum_{\tau \in \text{RC}_2(\mathcal{G}), \tau \neq 1} \#(\mathcal{G}\tau \cap \mathcal{H}) \cdot (\chi_1^{\mathcal{G}} - 2 \cdot \chi_{\tau}^{\mathcal{G}})$$

and let

$$M_{\mathcal{G}} := \{X_{\mathcal{H}} \mid \mathcal{H} \in \text{RC}(\mathcal{G}) - \text{RC}_2(\mathcal{G})\}.$$

By proposition 10 and corollary 11, M is a free subset of $L(\mathcal{G})$ which consists of $\text{rank}(L(\mathcal{G}))$ elements. We will prove by induction that $M_{\mathcal{G}}$ is contained in $\mathcal{T}(\mathcal{G})$.

Lemma 22. *Let \mathcal{G} be a 2-group. Then $M_{\mathcal{G}}$ is a free subset of $\mathcal{T}(\mathcal{G})$ consisting of $\text{rank}(L(\mathcal{G}))$ elements.*

Proof. Observe that $\mathcal{T}(\mathbb{Z}_2) = 0$. Let \mathcal{G} be a group of order $2^l \geq 4$ and let N/K be a Galois extension with Galois group \mathcal{G} . Now we proceed by induction.

1. Let \mathcal{H} be a subgroup with $\mathcal{H} \neq \mathcal{G}$. Let $\tau, \tau' \in \mathcal{G}$ be involutions. Then $\chi_{\tau} = \chi_{\tau'}$ if and only if $\tau' \in \mathcal{G}\tau$. Since $\mathcal{J}_2(\mathcal{G})$ is the disjoint union of the conjugacy classes of the involutions of \mathcal{G} we get

$$\mathcal{J}_2(\mathcal{H}) = \mathcal{J}_2(\mathcal{G}) \cap \mathcal{H} = \bigcup_{\tau \in \text{RC}_2(\mathcal{G}), \tau \neq 1} \mathcal{G}\tau \cap \mathcal{H}.$$

Let $\mathcal{U} < \mathcal{G}$ be a maximal subgroup of \mathcal{G} which contains \mathcal{H} . Then

$$\begin{aligned} X_{\mathcal{H}}^{\mathcal{U}} &= \text{ord}(\mathcal{H}) \cdot \chi_{\mathcal{H}}^{\mathcal{U}} - \chi_1^{\mathcal{U}} + \sum_{\tau \in \text{RC}_2(\mathcal{U}), \tau \neq 1} \#(\mathcal{U}\tau \cap \mathcal{H})(\chi_1^{\mathcal{U}} - 2 \cdot \chi_{\tau}^{\mathcal{U}}) \\ &= \text{ord}(\mathcal{H}) \cdot \chi_{\mathcal{H}}^{\mathcal{U}} - \chi_1^{\mathcal{U}} + \sum_{\tau \in \mathcal{J}_2(\mathcal{H})} (\chi_1^{\mathcal{U}} - 2 \cdot \chi_{\tau}^{\mathcal{U}}). \end{aligned}$$

Now $X_{\mathcal{H}}^{\mathcal{U}} \in \mathcal{T}(\mathcal{U})$ by induction hypothesis. Hence $h_{N/N\mathcal{U}}(X_{\mathcal{H}}^{\mathcal{U}}) = 0$, which gives $h_{N/K}(X_{\mathcal{H}}^{\mathcal{G}}) = \text{tr}_{N\mathcal{U}/K}(h_{N/N\mathcal{U}}(X_{\mathcal{H}}^{\mathcal{U}})) = 0$ (see proposition 19). Hence $X_{\mathcal{H}}^{\mathcal{G}} \in \mathcal{T}(\mathcal{G})$ if $\mathcal{H} \neq \mathcal{G}$.

2. Next we have to prove $X_{\mathcal{G}}^{\mathcal{G}} \in \mathcal{T}(\mathcal{G})$. First we consider an elementary abelian group. Then

$$X_{\mathcal{G}}^{\mathcal{G}} = 2^l \cdot \chi_{\mathcal{G}}^{\mathcal{G}} + (2^l - 2) \cdot \chi_1^{\mathcal{G}} - 2 \cdot \sum_{\tau \in \mathcal{G}, \tau \neq 1} \chi_{\tau}^{\mathcal{G}}.$$

Let $N = K(\sqrt{a_1}, \dots, \sqrt{a_l})$. We know that $\langle N \rangle = \langle 2^l \rangle \otimes \langle \langle -a_1, \dots, -a_l \rangle \rangle$ (see [3], prop. 1).

Now expand the Pfister form $\langle \langle -a_1, \dots, -a_l \rangle \rangle = \langle 1, b_2, \dots, b_{2^l} \rangle$. Then the entries b_2, \dots, b_{2^l} are in one-to-one correspondence with the quadratic subextensions of N/K . There are exactly $2^{l-1} - 1$ elements $\tau \in \mathcal{G}, \tau \neq \text{id}$ such that $K(\sqrt{b_i}) \subset N^\tau$. Hence

$$\begin{aligned} h_{N/K}(X_{\mathcal{G}}^{\mathcal{G}}) &= 2^l \times \langle 1 \rangle \perp (2^l - 2) \times \langle N \rangle - 2 \sum_{\tau \in \mathcal{G}, \tau \neq \text{id}} \langle N^\tau \rangle \\ &= 0. \end{aligned}$$

Now we can assume that \mathcal{G} is not an elementary abelian group. Let $\mathcal{U}_1, \dots, \mathcal{U}_m$ be the maximal subgroups of \mathcal{G} . Since \mathcal{G} is not a group of order 2, we get $\mathcal{J}_2(\mathcal{G}) \subset \cup_{i=1}^m \mathcal{U}_i$. This gives

$$\sum_{\tau \in \mathcal{J}_2(\mathcal{G})} (\chi_1 - 2 \cdot \chi_\tau) = \sum_{\mathcal{U} = \mathcal{U}_{i_1} \cap \dots \cap \mathcal{U}_{i_r}} (-1)^{r+1} \sum_{\tau \in \mathcal{J}_2(\mathcal{U})} (\chi_1 - 2 \cdot \chi_\tau),$$

where the sum runs over the set of all non-empty subsets of $\{1, \dots, m\}$. Let $\Phi(\mathcal{G})$ denote the Frattini subgroup of \mathcal{G} . Let 2^k be its order and set $\mathcal{V} = \mathcal{G}/\Phi(\mathcal{G})$. Let F be the fixed field of $\Phi(\mathcal{G})$. Then F/K is an elementary abelian extension. Let $\{i_1, \dots, i_r\} \subset \{1, \dots, m\}$ be a set of r different indices. Set $\mathcal{H} = \mathcal{U}_{i_1} \cap \dots \cap \mathcal{U}_{i_r}$. Then $X_{\mathcal{H}}^{\mathcal{H}} \in \mathcal{T}(\mathcal{H})$ by induction hypothesis. We get $h_{N/N^{\mathcal{H}}}(X_{\mathcal{H}}^{\mathcal{H}}) = 0$, which implies

$$\sum_{\tau \in \mathcal{J}_2(\mathcal{H})} (\langle N/N^{\mathcal{H}} \rangle - 2 \times \langle N^\tau/N^{\mathcal{H}} \rangle) = \langle N/N^{\mathcal{H}} \rangle - \text{ord}(\mathcal{H}) \times \langle 1 \rangle.$$

Set $\mathcal{V}' = \mathcal{H}/\Phi(\mathcal{G})$ and suppose $\mathcal{H} \neq \Phi(\mathcal{G})$. By (1) we know that $X_{\mathcal{V}'}^{\mathcal{V}'} \in \mathcal{T}(\mathcal{V}')$ for all subgroups \mathcal{V}' of \mathcal{V} with $\mathcal{V}' \neq 1$. This gives

$$\text{ord}(\mathcal{H}/\Phi(\mathcal{G})) \times \langle 1 \rangle = \langle F/N^{\mathcal{H}} \rangle - \sum_{\tau \in \mathcal{J}_2(\mathcal{V}')} (\langle F/N^{\mathcal{H}} \rangle - 2 \cdot \langle F^\tau/N^{\mathcal{H}} \rangle).$$

We further get

$$\begin{aligned}
 h_{N/K} \left(\sum_{\tau \in \mathcal{J}_2(\mathcal{H})} (\chi_1^{\mathcal{G}} - 2 \cdot \chi_{\tau}^{\mathcal{G}}) \right) &= \sum_{\tau \in \mathcal{J}_2(\mathcal{H})} (\langle N \rangle - 2 \times \langle N^{\tau} \rangle) \\
 &= \text{tr}_{N^{\mathcal{H}}/K} \left[\sum_{\tau \in \mathcal{J}_2(\mathcal{H})} (\langle N/N^{\mathcal{H}} \rangle - 2 \times \langle N^{\tau}/N^{\mathcal{H}} \rangle) \right] \\
 &= \text{tr}_{N^{\mathcal{H}}/K} (\langle N/N^{\mathcal{H}} \rangle - \text{ord}(\mathcal{H}) \times \langle 1 \rangle) \\
 &= (\langle N \rangle - \text{ord}(\mathcal{H}) \times \langle N^{\mathcal{H}} \rangle) \\
 &= \langle N \rangle - 2^k \times \text{tr}_{N^{\mathcal{H}}/K} (\text{ord}(\mathcal{H}/\Phi(\mathcal{G})) \times \langle 1 \rangle) \\
 &= \langle N \rangle - 2^k \times \text{tr}_{N^{\mathcal{H}}/K} (\langle F/N^{\mathcal{H}} \rangle \\
 &\quad - \sum_{\tau \in \mathcal{J}_2(\mathcal{H}/\Phi(\mathcal{G}))} (\langle F/N^{\mathcal{H}} \rangle - 2 \times \langle F^{\tau}/N^{\mathcal{H}} \rangle)) \\
 &= \langle N \rangle - 2^k \times \langle F \rangle \\
 &\quad + 2^k \times \sum_{\tau \in \mathcal{J}_2(\mathcal{H}/\Phi(\mathcal{G}))} (\langle F \rangle - 2 \times \langle F^{\tau} \rangle)
 \end{aligned}$$

If $\mathcal{H} = \Phi(\mathcal{G})$, then $\mathcal{J}_2(\mathcal{H}/\Phi(\mathcal{G}))$ is empty and $N^{\mathcal{H}} = F$. Hence the formula also holds in this situation.

Now $\sum_{r=0}^n (-1)^r \binom{n}{r} = 0$ implies

$$\begin{aligned}
 h_{N/K}(X_{\mathcal{G}}^{\mathcal{G}}) &= 2^l \times \langle 1 \rangle - \langle N \rangle + \sum_{\mathcal{H}} (-1)^{r+1} \sum_{\tau \in \mathcal{J}_2(\mathcal{H})} (\langle N \rangle - 2 \times \langle N^{\tau} \rangle) \\
 &= 2^l \times \langle 1 \rangle - 2^k \times \langle F \rangle \\
 &\quad + 2^k \times \sum_{\mathcal{H}} (-1)^{r+1} \sum_{\tau \in \mathcal{J}_2(\mathcal{H}/\Phi(\mathcal{G}))} (\langle F \rangle - 2 \times \langle F^{\tau} \rangle) \\
 &= 2^k \times [\text{ord}(\mathcal{V}) \times \langle 1 \rangle - \langle F \rangle] + h_{F/K} \left(\sum_{\tau \in \mathcal{J}_2(\mathcal{V})} (\chi_1^{\mathcal{V}} - 2 \cdot \chi_{\tau}^{\mathcal{V}}) \right) \\
 &= 2^k h_{F/K}(X_{\mathcal{V}}^{\mathcal{V}}) = 0
 \end{aligned}$$

by the above. □

9. OPEN QUESTIONS

We conclude with some open questions. How does the exponent of $B(\mathcal{G})/\mathcal{T}(\mathcal{G})$ depend on \mathcal{G} ?

Proposition 23. *Let \mathcal{G} be a finite group. If a 2-Sylow subgroup \mathcal{G}_2 of \mathcal{G} is a normal subgroup of \mathcal{G} , then the restriction homomorphism induces an epimorphism*

$$\text{res} : L(\mathcal{G}) \rightarrow L(\mathcal{G}_2)/\mathcal{T}(\mathcal{G}_2)$$

