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## Continued Fractions, Multidimensional Diophantine Approximations and Applications

par NIKOLAI G. MOSHCHEVITIN

**RÉSUMÉ.** Cet article rassemble des résultats généraux d'approximation diophantienne, sur les meilleures approximations et leurs applications à la théorie de répartition uniforme.

**ABSTRACT.** This paper is a brief review of some general Diophantine results, best approximations and their applications to the theory of uniform distribution.

### 1. DIOPHANTINE APPROXIMATIONS.

#### 1.1. One-dimensional approximations.

1.1.1. *Lagrange spectrum.* Let  $\alpha$  be an irrational number. Dirichlet's theorem states that there are infinitely many positive integers  $q$  such that

$$\|q\alpha\| < \frac{1}{q}$$

holds, where  $\|\cdot\|$  denotes the distance to the nearest integer. Hurwitz obtained a more precise result: for any irrational number  $\alpha$ , the inequality

$$\|q\alpha\| < \frac{1}{\sqrt{5}q}$$

has infinitely many solutions in  $q$ . Moreover, there is a countable set of numbers  $\alpha$  for which this inequality is an exact one, that is, for any positive  $\varepsilon$  there are only finitely positive integers  $q$  such that the inequality

$$\|q\alpha\| < \left(\frac{1}{\sqrt{5}} - \varepsilon\right) \frac{1}{q}$$

holds.

We define the *Lagrange spectrum* to be the set of the real numbers  $\lambda$  for which there exists  $\alpha = \alpha(\lambda)$  such that the inequality

$$\|q\alpha\| < \lambda \frac{1}{q}$$

has infinitely many solutions, and for any positive  $\varepsilon$  the inequality

$$\|q\alpha\| < (\lambda - \varepsilon) \frac{1}{q}$$

has only a finite number of solutions. It is well-known that Lagrange spectrum has a discrete part

$$\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{8}}, \dots,$$

and the minimal  $\lambda$  for which there are uncountably many  $\alpha = \alpha(\lambda)$  is  $\lambda = 1/3$ . Also it is well-known that Lagrange spectrum contains an interval  $[0, \lambda^*]$ .

Moreover, for any decreasing function  $\psi$  satisfying  $\psi(y) = o(y^{-1})$ , as  $y$  tends to infinity, there is an uncountable set of real numbers  $\alpha$  such that the inequality

$$\|q\alpha\| < \psi(q)$$

has infinitely many solutions, but for any  $\varepsilon > 0$ , the stronger inequality

$$\|q\alpha\| < (1 - \varepsilon)\psi(q)$$

has only a finite number of solutions.

One can find the above results in [5]. All of them can be obtained from the continued fraction expansion [14].

1.1.2. *Best approximations and continued fractions.* Any real number  $\alpha$  may be written as

$$\alpha = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \dots}}}$$

where  $b_0 \in \mathbb{Z}$  and, for  $j > 0$ ,  $b_j$  are a nonnegative integers. For convenience, we use the notation

$$\alpha = [b_0; b_1, b_2, b_3, \dots].$$

This representation is infinite and unique when  $\alpha$  is irrational. If  $\alpha$  is rational, we have  $\alpha = [b_0; b_1, b_2, b_3, \dots, b_i]$ , and this representation is unique if we impose the condition  $b_i \neq 0, 1$ .

Convergents to  $\alpha$  of the order  $\nu$  are defined as

$$\frac{p_\nu}{q_\nu} = [b_0; b_1, b_2, b_3, \dots, b_\nu]$$

A simple theorem states that these fraction and only these form *the best approximations*, that is the relation

$$\|q_\nu \alpha\| = \min_{q < q_\nu} \|q\alpha\|$$

holds for the denominators  $q_\nu$  and only for them (see [14]). We now give two other easy facts.

**Theorem 1.** *We have*

$$\|q_\nu \alpha\| \asymp (q_{\nu+1})^{-1}, \quad (\text{in order of approximation}).$$

**Proposition 2.** *We have*

$$\Delta_\nu = \begin{vmatrix} p_\nu & q_\nu \\ p_{\nu+1} & q_{\nu+1} \end{vmatrix} = \pm 1.$$

1.1.3. *Klein polygons.* We now consider the integer lattice  $\mathbb{Z}^2 \subset \mathbb{R}^2$ . Let  $(q, a) \in \mathbb{Z}^2$  be a primitive point ( $\gcd(q, a) = 1$ ) and  $q, a > 0$ . We define the two angles  $\varphi_+$  and  $\varphi_-$  by

$$\varphi_+ = \left\{ Z = (x, y) \in \mathbb{R}^2 : x \geq 0, y \geq \frac{a}{q}x \right\},$$

$$\varphi_- = \left\{ Z = (x, y) \in \mathbb{R}^2 : y \geq 0, y \leq \frac{a}{q}x \right\},$$

*Klein polygons*  $\mathcal{K}_+(a, q)$  and  $\mathcal{K}_-(a, q)$  are defined to be respectively the following borders

$$\partial(\text{conv}(\varphi_+ \cap (\mathbb{Z}^2 \setminus \{0\})))$$

and

$$\partial(\text{conv}(\varphi_- \cap (\mathbb{Z}^2 \setminus \{0\})))$$

which consist of finite (nontrivial) intervals.

We now define  $\Delta(a, q)$  to be the domain:

$$\Delta(a, q) = \left\{ Z = (x, y) \in \mathbb{R}^2 : x > 0, y > 0, \right. \\ \left. Z \notin \text{conv}(\varphi_+ \cap (\mathbb{Z}^2 \setminus \{0\})), \quad Z \notin \text{conv}(\varphi_- \cap (\mathbb{Z}^2 \setminus \{0\})) \right\}.$$

We have

**Theorem 3** ([7, 9]). 1. *The vertices of  $\mathcal{K}_-(a, q)$ , (different from  $(q, a)$ ) are integer points of the form  $(q_{2\mu}, p_{2\mu})$ , where  $(p_{2\mu}/q_{2\mu})$  is the  $2\mu$ -th convergent to  $a/q$ .*

2. *The vertices of  $\mathcal{K}_+(a, q)$  (different from  $(q, a)$ ) are integer points of the form  $(q_{2\nu+1}, p_{2\nu+1})$ , where  $p_{2\nu+1}/q_{2\nu+1}$  is the  $(2\nu + 1)$ -th convergent to  $a/q$ .*

3. *If  $(u, v) \in (\mathcal{K}_+(a, q) \cup \mathcal{K}_-(a, q)) \cap \mathbb{Z}^2$  is an integer point then  $v/u$  is a convergent to  $a/q$  or one of the intermediate fractions  $(wp_\nu + p_{\nu-1})/(wq_\nu + q_{\nu-1})$ ,  $1 \leq w \leq b_{\nu+1}$ .*

4.  $\Delta(a, q) \cap \mathbb{Z}^2 = \emptyset$ .

One can easily verify the same results for infinite continued fractions (i.e. for irrational numbers).

Recently, several papers [1, 21, 45, 46, 27] devoted to multidimensional generalization of Klein polygons have appeared. Unfortunately one must notice that there is something incorrect in papers [45, 46].

1.1.4. *Representation of rationals.* The rationals  $a/q$  with bounded partial quotients  $b_j$  are of great interest (see [22, 23, 24, 11]).

Let  $N(k, q)$  be the number of integers  $A$ ,  $1 \leq A < q$ ,  $\gcd(A, q) = 1$  such that any component  $b_i$  of the continued fraction expansion

$$\frac{A}{q} = [0; b_1, \dots, b_{n(A)}]$$

is bounded by  $k$ :  $b_i \leq k$ ,  $i = 1, \dots, n(A)$ . It is known ([22, 4, 52]) that if  $k > \gamma \log q$  with  $\gamma$  sufficiently large, then  $N(k, q) \geq 1$ . Moreover we can show that for *almost all* positive integers  $q$  and  $A$  with  $1 \leq A < q$ , all partial quotients are bounded by  $O(\log q)$ .

By the way we may recall a famous and still open conjecture which asserts that for any  $q \geq 1$ , we have  $N(6, q) \geq 1$ . However it is known that the conjecture holds when  $q = 2^\alpha$  or  $q = 3^\alpha$  ([39]).

Sergei Konyagin (see [17]), by means of Farey fractions, proved the following upper bound for  $N(k, q)$ :

**Theorem 4.** *For any  $\gamma < 1$  and for any  $k \geq k(\gamma)$  we have*

$$N(k, q) \ll \varphi(q)q^{-\frac{\gamma}{k \log k}},$$

where  $\varphi$  denotes the Euler function.

We define the sequence  $A_1 < A_2 < \dots < A_d$  to be an *almost arithmetic progression* if

$$\exists w > 1 : w \leq A_{j+1} - A_j \leq 3w, \quad j = 1, \dots, d-1.$$

In [32], the author shows that numbers with bounded partial quotients cannot appear very regularly: they cannot form long almost arithmetic progressions. The following theorem improves the result from [32].

**Theorem 5.** *For  $d \geq 3$ , let  $A_0, \dots, A_d$  be positive integers. Suppose*

- (i)  $0 < A_0 < \dots < A_d$  form an almost arithmetic progression;
- (ii)  $\gcd(A_i, q) = 1$ ,  $i = 0, \dots, d$ .

Let  $A_\nu/q = [b_{\nu,1}, \dots, b_{\nu,s(\nu)}]$ . Then there exist  $\nu_0$  and  $\mu_0$  such that

$$0 \leq \nu_0 \leq d, \quad 1 \leq \mu_0 \leq s(\nu_0)$$

and

$$b_{\nu_0, \mu_0} \gg d^{1/2}.$$

Theorem 5 is proved by means of Klein polygons. The same result is true for real-valued (not integer) almost arithmetic progressions and in the last case S. Konyagin showed that the result for real-valued progressions is exact in order.

**1.2. Simultaneous approximations.** Let  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a positive and real-valued function. For given  $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ , a positive integer  $p$  is said to be a  $\psi$ -approximation of  $\alpha$ , if

$$\max_{j=1, \dots, s} \|p\alpha_j\| = \max_{j=1, \dots, s} \min_{a \in \mathbb{Z}} |p\alpha_j - a| \leq \psi(p).$$

**1.2.1. Dirichlet and Liouville's theorems.** Dirichlet's theorem states that for any  $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ , where  $1, \alpha_1, \dots, \alpha_s$  linearly independent over  $\mathbb{Z}$ , there are infinitely many  $\psi$ -approximations of  $\alpha$  with  $\psi(y) = y^{-1/s}$ .

On the other hand, Liouville's theorem ([2], ch.5) shows that for any  $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$  such that  $1, \alpha_1, \dots, \alpha_s$  form a basis of a real algebraic field of degree  $s + 1$ , there exists  $C(\alpha)$  such that

$$\max_{j=1, \dots, s} \|p\alpha_j\| \geq C(\alpha)p^{-1/s}, \quad \forall p \in \mathbb{N}.$$

One can see that there are only countably many algebraic  $\alpha = (\alpha_1, \dots, \alpha_s)$ .

**1.2.2. Theorem by Cassels and Davenport and the result by Jarnik.** In [3, 6] the following result is obtained.

**Theorem 6.** *There exists a constant  $C_s$  for which there exists an uncountable set of elements  $\alpha \in \mathbb{R}^s$  which do not have any  $\psi$ -approximation where  $\psi(y) = C_s y^{-1/s}$ .*

V. Jarnik [12, 13] proved another result:

**Theorem 7.** *Let  $\psi$  and  $\lambda$  be positive real-valued functions such that  $\psi(y)y^{1/s}$  decreases as  $y \rightarrow \infty$  and  $\lambda(y) \rightarrow 0$  as  $y \rightarrow \infty$ . Then there exists an uncountable set of elements  $\alpha \in \mathbb{R}^s$  for which there are infinitely many  $\psi$ -approximations, but only finitely many  $\psi\lambda$ -approximations.*

A review of other results can be found in [43, 10, 2].

**1.2.3. Exact results in terms of the order of approximation.** Generalizing the work [3] by means of chains of parallelepipeds [28, 50, 7, 8] we improve Jarnik's result.

**Theorem 8.** *For  $y \geq 1$ , let  $\psi$  and  $\omega$  such that Let  $\psi(y) = y^{-1/s}\omega(y)$ , where  $\omega(y)$  decreases as  $y \rightarrow \infty$  and*

$$\omega(1) \leq 2^{-(s+1)(s+2)}(s!)^{-1/s}.$$

*Then there exists a vector  $\alpha = (\alpha_1, \dots, \alpha_s)$  which has infinitely many  $\psi$ -approximations but not any  $2^{-(s+3)}\psi$ -approximation.*

**Theorem 9.** *Let  $\omega$  and  $\psi$  be as in Theorem 8 and suppose that*

$$\omega(1) \leq 2^{-(s+1)(s+3)}(s!)^{-1/s}.$$

*Then there exists an uncountable set of vectors  $\alpha = (\alpha_1, \dots, \alpha_s)$ , each of them having infinitely many  $\psi$ -approximations but not any  $2^{-(s+3)\psi}$ -approximation.*

It follows that in Cassels Theorem 6 we may put

$$C_s = 2^{-(s+2)(s+3)}(s!)^{-1/s}.$$

We say that  $\alpha = (\alpha_1, \dots, \alpha_s)$  satisfies the  $\psi$ -condition if  $\alpha$  has infinitely many  $\psi$ -approximations but not any  $c\psi$ -approximation for some  $c = c(\alpha)$

**Theorem 10.** *Let  $\psi$  be defined by  $\psi(y) = y^{-1/s}\omega(y)$  where  $\omega$  is decreasing positive function. Then in any Jordan  $s$ -dimensional domain  $\Omega$  with  $\text{Vol } \Omega > 0$ , there exists an uncountable set of  $\alpha \in \mathbb{R}^s$  satisfying the  $\psi$ -condition.*

Theorems 8, 9, 10 are discussed in [33].

1.2.4. *Successive best approximations.* Let  $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ . We define a *best simultaneous approximation* (b.a) of  $\alpha$  to be any integer point  $\zeta = (p, a_1, \dots, a_s) \in \mathbb{Z}^{s+1}$  such that  $\forall q, \forall (b_1, \dots, b_s) \in \mathbb{Z}^s, 1 \leq q \leq p, (q, b_1, \dots, b_s) \neq (p, a_1, \dots, a_s)$ , we have

$$D(\zeta) = \max_{j=1, \dots, s} |p\alpha_j - a_j| < \max_{j=1, \dots, s} |q\alpha_j - b_j|.$$

Let  $\alpha_j \notin \mathbb{Q}, j = 1, \dots, s$ . Then all b.a. of  $\alpha$  form infinite sequences

$$\zeta^\nu = (p^\nu, a_1^\nu, \dots, a_s^\nu), \nu = 1, 2, \dots$$

where  $p^1 < p^2 < \dots < p^\nu < p^{\nu+1} < \dots$  and

$$D(\zeta^1) > D(\zeta^2) > \dots > D(\zeta^\nu) > D(\zeta^{\nu+1}) > \dots .$$

Let

$$M_\nu[\alpha] = \begin{pmatrix} p^\nu & a_1^\nu & \dots & a_s^\nu \\ \dots & \dots & \dots & \dots \\ p^{\nu+s} & a_1^{\nu+s} & \dots & a_s^{\nu+s} \end{pmatrix}.$$

For  $\alpha = (\alpha_1, \dots, \alpha_s)$  satisfying  $\alpha_j \notin \mathbb{Q}, j = 1, \dots, s$ , we define  $R(\alpha) \in [2, s + 1]$  to be the integer

$$R(\alpha) = \min \{n : \text{there exist a lattice } \Lambda \subseteq \mathbb{Z}^{s+1} \text{ with } \dim \Lambda = n \text{ and a natural } \nu_0 \text{ such that } \zeta^\nu \in \Lambda, \forall \nu > \nu_0\}.$$

**Proposition 11.** *Let  $s = 1$ . Then for any  $\nu \geq 1$  we have  $\det M_\nu[\alpha] = \pm 1$  ( $\text{rank } M_\nu[\alpha] = 2, \forall \nu$ ).*

**Proposition 12.** *For any  $s \geq 1$  we have  $R(\alpha) = \dim_{\mathbb{Z}} (\alpha_1, \dots, \alpha_s, 1)$ .*

**Proposition 13.** *Let  $s = 2$  and  $\alpha_1, \alpha_2$  such that  $1, \alpha_1, \alpha_2$  are linearly independent over  $\mathbb{Z}$ . Then for infinitely many  $\nu$  we have*

$$\text{rank } M_\nu[\alpha] = 3 = \dim_{\mathbb{Z}}(\alpha_1, \alpha_2, 1).$$

Proposition 11 – 13 can be easily verified. The following result is proved in [36].

**Theorem 14.** *Let  $s \geq 3$ . There exists an uncountable set of elements  $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$  such that*

(i)  $1, \alpha_1, \dots, \alpha_s$  are linearly independent over  $\mathbb{Z}$ ,

(thus  $\dim_{\mathbb{Z}}(\alpha_1, \dots, \alpha_s, 1) = s + 1$ ),

and

(ii)  $\text{rank } M_\nu[\alpha] \leq 3, \forall \nu \geq 1$ ,

(Hence for all  $\nu \geq 1$  we have  $\det M_\nu[\alpha] = 0$ ).

Theorem 14 represents a counterexample to the conjecture of J.S. Lagarias [26]. It shows that the successive b.a. have no such an asymptotic property as a reader can see in Proposition 12. The idea of the proof was suggested to the author by N.P. Dolbilin.

**1.3. Linear forms.** Again, let  $\alpha_1, \dots, \alpha_s$  be real numbers such that  $1, \alpha_1, \dots, \alpha_s$  are linearly independent over  $\mathbb{Z}$ , and put  $\alpha = (\alpha_1, \dots, \alpha_s)$ .

For  $m = (m_0, m_1, \dots, m_s) \in \mathbb{Z}^{s+1} \setminus \{0\}$  we define

$$\zeta(m) = m_0 + m_1\alpha_1 + \dots + m_s\alpha_s, \quad M = \max_{j=0,1,\dots,s} |m_j|.$$

A vector  $m \in \mathbb{Z}^{s+1} \setminus \{0\}$  is a best approximation of  $\alpha$  in sense of linear form if

$$\zeta(m) = \min_{\substack{n \in \mathbb{Z}^{s+1} \setminus \{0\} \\ \max_j |n_j| \leq M}} |\zeta(n)|.$$

All best approximations form sequences

$$\zeta_1 > \zeta_2 > \dots > \zeta_\nu > \zeta_{\nu+1} > \dots,$$

$$M_1 < M_2 < \dots < M_\nu < M_{\nu+1} < \dots$$

where  $m_\nu = (m_{0,\nu}, \dots, m_{s,\nu})$  is the vector of the  $\nu$ -th b.a.,  $\zeta_\nu = \zeta(m_\nu)$  and  $M_\nu = \max_j |m_{j,\nu}|$ .

By Minkowski's Theorem we have  $\zeta_\nu M_{\nu+1}^s \leq 1$ .

**1.3.1. Singular systems.** The theorem on the order of approximations from §1.1.2 does not admit multidimensional generalization in the sense of linear form.

**Theorem 15** (see [29, 35]). *Let  $s$  be an integer  $\geq 1$  and  $\psi$  a function such that  $\psi(y)$  decreases to zero when  $y$  tends to infinity. Then there exists an uncountable set of elements  $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$  such that*

(i)  $1, \alpha_1, \dots, \alpha_s$  are linearly independent over  $\mathbb{Z}$ ,



and

(ii) the sequence of the best approximations of  $\alpha$  satisfies

$$\zeta_\nu \leq \psi(M_{\nu+s-1}).$$

In the case  $s = 1$  this theorem means that there are real numbers with any given order of the best approximations. In higher dimensions it gives something more.

Khinchin [15] defined a vector  $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$  to be a  $\psi$ -singular system if for any  $T > 0$  the system

$$\|m_1\alpha_1 + \dots + m_s\alpha_s\| < \psi(T), \quad M = \max_{1 \leq j \leq s} |m_j| < T$$

has a nontrivial solution  $(m_1, m_2, \dots, m_s) \in \mathbb{Z}^s$ .

**Proposition 16.** *System is  $\psi$ -singular  $\iff \zeta_\nu < \psi(M_{\nu+1}), \forall \nu$ .*

1.3.2. *Successive best approximations for linear form.* Here we define  $\Delta_\nu^s$  to be the determinant of the successive best approximations

$$\Delta_\nu^s = \begin{vmatrix} m_{0,\nu} & m_{1,\nu} & \dots & m_{s,\nu} \\ \dots & \dots & \dots & \dots \\ m_{0,\nu+s} & m_{1,\nu+s} & \dots & m_{s,\nu+s} \end{vmatrix}.$$

The proposition below follows from Minkowski theorem on convex body. It seems to me that it is a well-known fact, but I could not find the corresponding reference.

**Proposition 17.** *Let  $s = 2$ . Then for infinitely many  $\nu$  we have  $\Delta_\nu^2 \neq 0$ .*

The theorem below was proved by the author in [35] by means of singular systems.

**Theorem 18.** *Lets  $\geq 3$ . Then there exists a uncountable set of vectors  $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$  such that*

(i)  $1, \alpha_1, \dots, \alpha_s$  are linearly independent over  $\mathbb{Z}$ ,

and

(ii) there exists a linear subspace  $\mathcal{L}_\alpha \subset \mathbb{R}^{s+1}$ ,  $\dim \mathcal{L}_\alpha = 3$  satisfying the condition

$$m_\nu \in \mathcal{L}_\alpha, \quad \forall \nu > \nu_0.$$

We see that for  $s \geq 3$  almost all best approximations may asymptotically lie in a three-dimensional plane but they cannot lie in two-dimensional plane. Of course these examples are degenerated in sense of measure. For almost all vectors  $\alpha \in \mathbb{R}^s$  (in the sense of Lebesgue) best approximations are asymptotically  $(s + 1)$ -dimensional.

2. INTEGRALS FROM QUASIPERIODIC FUNCTIONS

In the text below we discuss applications of diophantine results to certain problems of uniform distribution of irrational rotations on torus. A full review of the methods and results of the theory of uniform distribution is given in [25].

2.1. **Uniform distribution.** Let  $\mathbf{T}$  be the one-dimensional torus and  $f : \mathbf{T}^s \rightarrow \mathbb{R}$  defined by the series

$$f(x_1, \dots, x_s) = \sum_{\substack{m \in \mathbb{Z}^s \\ m \neq 0}} f_m \exp(2\pi i(m_1 x_1 + \dots + m_s x_s)).$$

We also define the integral

$$I(T, \varphi) = I_{f, \omega}(T, \varphi) = \int_0^T f(\omega_1 t + \varphi_1, \dots, \omega_s t + \varphi_s) dt.$$

where  $\omega_1, \dots, \omega_s \in \mathbb{R}$  are linearly independent over  $\mathbb{Z}$ , and  $\varphi = (\varphi_1, \dots, \varphi_s) \in \mathbb{R}^s$ .

H. Weyl [51] proved that if  $f$  is a continuous function, then for any  $\varphi$  we have  $I(T, \varphi) = o(T)$ ,  $T \rightarrow \infty$ . This equality holds uniformly in  $\varphi$  if we suppose moreover that  $f$  is smooth.

V.V. Kozlov conjectured that the integral  $I(T, \varphi)$  is *recurrent* that is the following condition holds:

$$(*) \quad \forall \varepsilon > 0, \quad \forall T, \quad \exists T^* > T : |I(T^*, \varphi)| < \varepsilon.$$

This conjecture is true when  $f$  is any trigonometric polynomial, and in this case (\*) holds uniformly in  $\varphi$ . This implies that for any trigonometric polynomial  $f$  of finite degree,  $J^\infty$  defined by

$$J^\infty(T) = J_{f, \omega}^\infty(T) = \sup_{\varphi \in \mathbb{R}^s} |I(T, \varphi)|,$$

is itself recurrent, that is

$$(\$) \quad \forall \varepsilon > 0, \quad \forall T, \quad \exists T^* > T : |J^\infty(T^*)| < \varepsilon.$$

2.2. **Case  $s = 2$ .** In the two-dimensional case, the conjecture above was proved by V.V. Kozlov himself for functions  $f \in C^2(\mathbf{T}^2)$  in [18] (see also [19]). It is also easy to see that when  $f$  is a smooth function, then (\*) holds uniformly in  $\varphi$ , that is (\$) is true. E.A. Sidorov [44] obtained a similar result for “absolutely” continuous  $f$ .

**2.3. The general result.** The author [34] proved the conjecture in the general case:

**Theorem 19.** *Suppose that*

$$f(x_1, \dots, x_s) = \sum_{\substack{m \in \mathbb{Z}^s \\ m \neq 0}} f_m \exp(2\pi i(m_1 x_1 + \dots + m_s x_s))$$

*belongs to the class  $C^d(\mathbf{T}^s)$ , where  $d > Cs^{s^3}$  and  $\omega_1, \dots, \omega_s$  are linearly independent over  $\mathbb{Z}$ . Then for any  $\varphi$ , the integral  $I(T, \varphi)$  satisfies (\*).*

The proof is based on consideration of best approximations in the sense of linear form (see §1.3.2.).

**2.4. Metric results.** It is known [49, 48] that for almost all (in the sense of Lebesgue) vectors  $\omega = (\omega_1, \dots, \omega_s) \in \mathbb{R}^s$ , if  $f$  is smooth enough, then the integral  $I(T, \varphi)$  is bounded when  $T \rightarrow \infty$  uniformly in  $\varphi$ . Hence the integral  $I(T, \varphi)$  satisfies (\*), uniformly in  $\varphi$ . But even in the case  $s \geq 3$ , this result is not universal.

Let  $\Phi$  be a decreasing function and assume that the series  $\sum_{m \in \mathbb{Z}^s} \Phi(m)$  converges. We define a periodic function  $\Theta : \mathbf{T}^s \rightarrow \mathbb{R}$  to be of the type  $\Phi$  if, the coefficients  $\Theta_{m_1, \dots, m_s}$  in the expansion

$$\Theta(x_1, \dots, x_s) = \sum \Theta_{m_1, \dots, m_s} e^{2\pi i m_1 x_1 + \dots + m_s x_s},$$

satisfy

$$|\Theta_{m_1, \dots, m_s}| \leq \Phi(M), \quad \text{where } M = \max_j |m_j|.$$

We consider

$$J^\infty(T) = J_{f, \omega}^\infty(T) = \max_{\varphi \in \mathbf{T}^s} |I_{f, \omega}(T, \varphi)|;$$

$$J^2(T) = J_{f, \omega}^2(T) = \left( \int_{\mathbf{T}^s} |I_{f, \omega}(T, \varphi)|^2 d\varphi \right)^{1/2}.$$

The result below is proved in [29].

**Theorem 20.** *Let  $s \geq 3$ . Then for any function  $\Phi$  which decreases to zero as  $y \rightarrow \infty$  and for any function  $\psi$  with  $\psi(y) = o(1)$  as  $y \rightarrow \infty$ , there exist  $\omega_1, \omega_2, \dots, \omega_s$  which are linearly independent over  $\mathbb{Z}$ , and a function  $f$  of type  $\Phi$  such that  $\int_{\mathbf{T}^s} f(x) dx = 0$  and*

$$J^l(T) \gg T\psi(T) \quad \forall T, \quad l = 2, \infty.$$

We will reformulate Theorem 20 in the following way.

**Theorem 21.** *Let  $s \geq 3$ , and  $f : \mathbf{T}^s \rightarrow \mathbb{R}$  be smooth with zero mean value. Assume that in the expansion*

$$f(x_1, \dots, x_s) = \sum_{(m_1, \dots, m_s) \neq 0} f_{m_1, \dots, m_s} e^{2\pi i(m_1 x_1 + \dots + m_s x_s)}$$

*the coefficients  $f_{m_1, \dots, m_s}$ , where  $(m_1, \dots, m_s) \neq 0$ , are all different from zero.*

*Then there exist  $\omega_1, \omega_2, \dots, \omega_s$  which are linearly independent over  $\mathbb{Z}$  such that*

$$J^l(t) \gg t\psi(t) \quad \forall t; \quad l = 2, \infty.$$

An improvement of the latter result was obtained recently by E.V. Kolo-meikina [20].

One can see that the behaviour of integrals  $J_l$  in two-dimensional case radically differs from the case  $s \geq 3$ .

**2.5. Odd functions. Sergei Konyagin's result.** Recently, S. Konyagin [16] obtained the following result.

**Theorem 22.** *The Kozlov's conjecture is true (that is (\*) holds) for arbitrary  $s \geq 1$  and any function  $f$  satisfying the condition*

$$f(-x_1, \dots, -x_s) = f(x_1, \dots, x_s), \quad f \in C^\tau(\mathbf{T}^s), \quad \tau \asymp s^{2s}.$$

**2.6. The smoothness.** In [42],[41] it is shown that we need some kind of smoothness conditions on  $f$  to insure that (\*) is true : indeed in the two-dimensional case ( $s = 2$ ), there exists a function  $f : \mathbf{T}^2 \rightarrow \mathbb{R}$  (with zero mean value) of the class  $C \setminus C^1(\mathbf{T}^2)$  such that  $I(T, 0)$  tends to infinity when  $T \rightarrow \infty$  (with the choice  $\omega_1 = 1$  and  $\omega_2 = \sqrt{2}$ ). On the other hand, in [44] it is shown that when  $s = 2$ , a sufficient condition on  $f$  for having (\*), is  $f$  to be absolutely continuous.

Developping an idea of D.V. Treshchev, the author, in [31], generalized Poincaré's example. He proved that for any real  $\omega_1, \dots, \omega_s$  which form a basis of a real algebraic field, there exists a function  $f \in C^{s-2}(T^s) \setminus C^{s-1}(T^s)$  such that  $I(\cdot, 0)$  does not satisfy the property (\*) with  $\varphi = 0$ .

One may find some results on algebraic numbers in [40] and [38]. Recently, S.V. Konyagin [16] proved that for some Liouville transcendental numbers, there exists  $f \in C^d(\mathbf{T}^s)$  with  $d \asymp 2^s/s$  such that (\*) is not satisfied.

Some early results are reviewed in [37].

**2.7. Vector-functions: counterexample in dimension  $s = 3$ .** Let  $f^j : \mathbf{T}^s \rightarrow \mathbb{R}$ ,  $j = 1, 2$  be defined by

$$f^j(x_1, \dots, x_s) = \sum_{\substack{k \in \mathbb{Z}^s \\ k \neq 0}} f_k^j \exp(2\pi i(k_1 x_1 + \dots + k_s x_s)).$$

For  $\omega_1, \dots, \omega_s \in \mathbb{R}$  be linearly independent over  $\mathbb{Z}$ , we put

$$I^j(T) = \int_0^T f^j(\omega_1 t, \dots, \omega_s t) dt, \quad j = 1, 2.$$

The analogue of property (\*) for the vector-integral  $I = (I^1, I^2) : \mathbb{R} \rightarrow \mathbb{R}^2$  becomes

$$(\%) \quad \forall \varepsilon > 0 \quad \forall T \quad \exists T^* > T : |I^1(T^*)| + |I^2(T^*)| < \varepsilon.$$

**Proposition 23.** *In the case when  $s = 2$  and  $f$  is a smooth vector-function, then (%) holds.*

**Proposition 24.** *The analogue of Theorem 22 holds for vector-function, that is (%) is satisfied for any odd smooth vector-function  $f$ .*

Let  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a positive function such that  $\sum_{k \in \mathbb{Z}^s} \Phi(\max_{1 \leq j \leq s} |k_j|)$  converges. A vector-function  $f = (f^1, f^2) : \mathbb{T}^s \rightarrow \mathbb{R}^2$  is defined to be a function of type  $\Phi$  if we have

$$|f_k^j| \leq \Phi(\max_{1 \leq j \leq s} |k_j|) \quad \forall k, \quad j = 1, 2.$$

Recently, the author [34] constructed the following example.

**Theorem 25.** *For any given positive function  $\Phi$ , there exist a vector-function  $f = (f^1, f^2) : \mathbb{T}^3 \rightarrow \mathbb{R}^2$  of the type  $\Phi$  with zero mean value ( $\int_{\mathbb{T}^3} f^j(x) dx = 0, j = 1, 2$ ) and numbers  $\omega_1, \omega_2, \omega_3$ , which are linearly independent over  $\mathbb{Z}$  such that*

$$|I^1(T)| + |I^2(T)| \rightarrow \infty, \quad \text{as } T \rightarrow +\infty.$$

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