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Congruences modulo ℓ between ε factors for cuspidal representations of $GL(2)$

par MARIE-FRANCE VIGNÉRAS

Pour Jacques Martinet

RÉSUMÉ. Titre français : Congruences modulo ℓ entre facteurs ε des représentations cuspidales de $GL(2)$

Soient $\ell \neq p$ deux nombres premiers distincts, F un corps local non archimédien de caractéristique résiduelle p , $\overline{\mathbf{Q}}_\ell$ une clôture algébrique du corps des nombres ℓ -adiques, et $\overline{\mathbf{F}}_\ell$ le corps résiduel de $\overline{\mathbf{Q}}_\ell$. On conjecture que la correspondance locale de Langlands pour $GL(n, F)$ sur $\overline{\mathbf{Q}}_\ell$ respecte les congruences modulo ℓ entre les facteurs L et ε de paires, et que la correspondance locale de Langlands sur $\overline{\mathbf{F}}_\ell$ est caractérisée par des identités entre de nouveaux facteurs L et ε . Nous allons le démontrer lorsque $n = 2$.

ABSTRACT. Let $\ell \neq p$ be two different prime numbers, let F be a local non archimedean field of residual characteristic p , and let $\overline{\mathbf{Q}}_\ell, \overline{\mathbf{Z}}_\ell, \overline{\mathbf{F}}_\ell$ be an algebraic closure of the field of ℓ -adic numbers \mathbf{Q}_ℓ , the ring of integers of $\overline{\mathbf{Q}}_\ell$, the residual field of $\overline{\mathbf{Z}}_\ell$. We proved the existence and the unicity of a Langlands local correspondence over $\overline{\mathbf{F}}_\ell$ for all $n \geq 2$, compatible with the reduction modulo ℓ in [V5], without using L and ε factors of pairs.

We conjecture that the Langlands local correspondence over $\overline{\mathbf{Q}}_\ell$ respects congruences modulo ℓ between L and ε factors of pairs, and that the Langlands local correspondence over $\overline{\mathbf{F}}_\ell$ is characterized by identities between new L and ε factors. The aim of this short paper is prove this when $n = 2$.

Introduction

The Langlands local correspondence is the unique bijection between all irreducible $\overline{\mathbf{Q}}_\ell$ -representations of $GL(n, F)$ and certain ℓ -adic representations of an absolute Weil group W_F of dimension n , for all integers $n \geq 1$,

which is induced by the reciprocity law of local class field theory

$$W_F^{ab} \simeq F^*$$

when $n = 1$ (W_F^{ab} is the biggest abelian Hausdorff quotient of W_F), and which respects L and ε factors of pairs [LRS], [HT], [H2].

Let $\psi : F \rightarrow \overline{\mathbf{Z}}_\ell^*$ be a non trivial character. We denote by $\text{Cusp}_R GL(n, F)$ the set of isomorphism classes of irreducible cuspidal R -representations of $GL(n, F)$. When $\pi \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} GL(n, F)$, Henniart [H1] showed that π is characterized by the epsilon factors of pairs $\varepsilon(\pi, \sigma)$ for all $\sigma \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} GL(m, F)$ and for all $m \leq n - 1$ (note that $L(\pi, \sigma) = 1$), using the theory of Jacquet, Piatetski-Shapiro, and Shalika [JPS1].

Does this remain true for cuspidal irreducible $\overline{\mathbf{F}}_\ell$ -representations of $GL(n, F)$? We need first to define the epsilon factors of pairs.

Let $\pi \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} GL(n, F)$. It is known that the constants of the epsilon factors of pairs $\varepsilon(\pi, \sigma)$ belong to $\overline{\mathbf{Z}}_\ell$ for all $\sigma \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} GL(m, F)$ and for all $m \leq n - 1$, and that the conductor does not change by reduction modulo ℓ (this is proved by Deligne [D] for the irreducible representations of the Weil group, and by the local Langlands correspondence over $\overline{\mathbf{Q}}_\ell$ is true for cuspidal representations).

Now let $\pi \in \text{Cusp}_{\overline{\mathbf{F}}_\ell} GL(n, F)$. Then π lifts to $\text{Cusp}_{\overline{\mathbf{Q}}_\ell} GL(n, F)$ [V1, III.5.10]. By reduction modulo ℓ , one can define epsilon factors of pairs $\varepsilon(\pi, \sigma)$ for all $\sigma \in \text{Cusp}_{\overline{\mathbf{F}}_\ell} GL(m, F)$ and for all $m \leq n - 1$. Let q be the order of the residual field of F . We expect that π is characterized by the epsilon factors $\varepsilon(\pi, \sigma)$ for all σ , when the multiplicative order of q modulo ℓ is $> n - 1$; otherwise, π should be characterized by less naive but natural epsilon factors. The same should be true when π is replaced by an $\overline{\mathbf{F}}_\ell$ -irreducible representation of the Weil group W_F .

The existence [V4] of an integral Kirillov model for $\pi \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} GL(n, F)$ seems to be an adequate tool to solve the problem. The description of the representation π on the Kirillov model is given by the central character ω_π and by the action of the symmetric group S_n (the Weyl group of $GL(n, F)$). The action of S_n is related with the $\varepsilon(\pi, \sigma)$ for all σ as above [GK, see the end of paragraph 7]. When $n = 2$ Jacquet and Langlands [JL] described the action of S_2 on the Kirillov model in terms of $\varepsilon(\pi, \chi) = \varepsilon(\pi \otimes \chi)$ for all $\overline{\mathbf{Q}}_\ell$ -characters χ of F^* , using the Fourier transform on F^* .

In the case $n = 2$ and only in this case, we will prove that two integral $\pi, \pi' \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} GL(2, F)$ have the same reduction modulo ℓ if and only if their central characters have the same reduction modulo ℓ and the factors $\varepsilon(\pi \otimes \chi), \varepsilon(\pi' \otimes \chi)$ have the same reduction modulo ℓ for integral $\overline{\mathbf{Q}}_\ell$ -characters χ of F^* when ℓ does not divide $q - 1$. When ℓ divides $q - 1$ this remains true with new epsilon factors taking into account the natural

congruences modulo ℓ satisfied by the $\varepsilon(\pi \otimes \chi)$ for all χ . By reduction modulo ℓ , we get that the local Langlands $\overline{\mathbf{F}}_\ell$ -correspondence for $n = 2$ is characterized by the equality on L and new ε factors of pairs. The field $\overline{\mathbf{F}}_\ell$ can be replaced by any algebraically closed field R of characteristic ℓ .

The case $n = 3$ could be treated probably, but the general case $n \geq 4$ remains an open and interesting question.

1. Integral Kirillov model

The definition of the L and ε factors of pairs [JPS1] uses the Whittaker model, or what is equivalent the Kirillov model. We showed [V4] that these models are compatible with the reduction modulo ℓ .

We denote by O_F the ring of integers of F . Let R be an algebraically closed field of characteristic $\neq p$, and let $\psi : F \rightarrow R^*$ be a character such that O_F is the biggest ideal on which ψ is trivial. We extend ψ to a R -character of the group N of strictly upper triangular matrices of $G = GL(n, F)$ by $\psi(n) = \psi(\sum n_{i,i+1})$ for $n = (n_{i,j}) \in N$. The mirabolic subgroup P of G is the semi-direct product of the group $GL(n - 1, F)$ embedded in $GL(n, F)$ by

$$g \rightarrow \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

and of the group F^{n-1} embedded in $GL(n, F)$ by

$$x \rightarrow \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

The representation $\tau_R := \text{ind}_{P,N} \psi$ of the mirabolic subgroup P (compact induction) is called mirabolic. It is irreducible (this is a corollary of [V4 prop.1]), but it is not admissible when $n \geq 2$.

Lemma. $\text{End}_{RP} \tau_R \simeq R$.

Proof. This is a general fact: the representation τ_R is absolutely irreducible [V1, I.6.10], hence $\text{End}_{RP} \tau_R \simeq R$. From the Schur's lemma [V1, I.6.9] $\text{End}_{RP} \tau_R \simeq R$ when the cardinal of R is strictly bigger than $\dim_R \tau_R$ (countable dimension). There exists an algebraically closed field R' which contains R and of uncountable cardinal. Two RP -endomorphisms of τ_R which are proportional over R' are proportional over R . \square

Theorem. *An irreducible R -representation π of G is cuspidal if and only if extends the mirabolic representation τ_R .*

Proof. This results from [BZ] and [V1]. Suppose that π is cuspidal. Then $\pi|_P$ is the mirabolic representation: when $R = \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$ see [BZ, 5.13 & 5.20], when $R = \overline{\mathbf{F}}_\ell$, π lifts to $\overline{\mathbf{Q}}_\ell$ [V1, III.5.10] where it is true then reduce. Conversely, suppose $\pi|_P = \tau_R$ and $R = \overline{\mathbf{Q}}_\ell$ or $\overline{\mathbf{F}}_\ell$. Then π is cuspidal [V1,

III.1.8]. The case of a general R is deduced from this two cases by the next lemma. □

Let G be the group of rational points of a reductive connected group over F . We denote by $\text{Irr}_R G$ the set of isomorphism classes of irreducible R -representations of G .

Lemma. (1) *A non zero homomorphism of algebraically closed fields $f : R \rightarrow R'$ gives a natural injective map $\pi \rightarrow f_*(\pi) : \text{Irr}_R G \rightarrow \text{Irr}_{R'} G$ which respects cuspidality.*

(2) *Let $\pi' \in \text{Cusp}_R G$. Then there exists an unramified character χ of G such that $\pi' \otimes \chi = f_*(\pi)$ with $\pi \in \text{Cusp}_R G$.*

Proof. This results from [V1].

(1) f_* respects irreducibility [V1, II.4.5], and commutes with the parabolic restriction. Hence it respects cuspidality. The linear independence of characters [V1, I.6.13] shows that if $\pi, \pi' \in \text{Irr}_R G$ are not isomorphic then $f_*\pi, f_*\pi'$ are not isomorphic.

(2) Let Z be the center of G . The group of rational characters $X(Z)$ is a subgroup of finite index in the group $X(G)$. This implies that there exists an unramified character χ of G such that the quotient Z/Z_o of Z by the kernel Z_o of the central character ω of $\pi' \otimes \chi$ is profinite. Hence the values of ω are roots of unity. We deduce that $\pi' \otimes \chi$ has a model on R [V1, II.4.9]. □

Let $\pi \in \text{Cusp}_R GL(n, F)$ of central character ω . The realisation of π on the mirabolic representation τ_R is called the Kirillov model $K(\pi)$ of π . It is sometimes useful to use the Whittaker model instead of the Kirillov model. By adjonction and the theorem $\text{Hom}_{RG}(\pi, \text{Ind}_{G,N} \psi) \simeq R$ (the unicity of the Whittaker model); the Whittaker model $W(\pi)$ is the unique realisation of π in $\text{Ind}_{G,N} \psi$. By definition

$$W(g) = (\pi(g)W)(1)$$

for all $g \in G$ and for all Whittaker functions $W \in W(\pi)$. We denote by $\Gamma(j)$ the subgroup of matrices $k \in GL(n, O_F)$ of the form

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a \in GL(n-1, O_F), d \in O_F^*, c \in \mathfrak{p}_F^j O_F$$

for any integer $j > 0$. The smallest $j > 0$ such that π contains a non-zero vector transforming under $\Gamma(j)$ according to the one dimensional character

$$\omega_j(k) = \omega(d)$$

for $k \in \Gamma(j)$ as above, is called the conductor of π and denoted f .

Theorem. Let $\pi \in \text{Cusp}_R GL(n, F)$ of central character $\omega = \omega$ and conductor f .

(1) The restriction from G to P induces a G -equivariant isomorphism

$$W \rightarrow W|_P : W(\pi) \simeq K(\pi)$$

from the Whittaker model to the Kirillov model.

(2) Let $\pi' \in \text{Cusp}_R GL(n, F)$. There is a natural isomorphism $W \rightarrow W' : W(\pi) \rightarrow W(\pi')$ of R -vector spaces defined by the condition $W|_P = W'|_P$.

(3) There is unique function $W_\pi \in W(\pi)$ such that

$$W_\pi|_{GL(n-1, F)} = 1_{GL(n-1, O_F)}.$$

The function W_π is called the new vector of π and generates the space of vectors of π transforming under $\Gamma(f)$ according to ω_f .

(4) $W(\pi)$ is contained in the compactly induced representation $\text{ind}_{G, NZ} \psi \otimes \omega_\pi$.

Proof. (1) There exists $W \in W(\pi)$ with $W(1) \neq 0$, and $f : W \rightarrow W_P$ is a non zero P -equivariant map from π to $\text{Ind}_N^P \psi$. The map f is injective of image $\text{ind}_N^P \psi$, because $\text{End}_R \tau_R \simeq R$. We get also (2).

(3) The space of τ_R is isomorphic by restriction to $G' = GL(n - 1, F)$, to the space of $\text{ind}_{N', G'} \psi$ where $N' = N \cap G'$. As ψ is trivial on O_F , the characteristic function of $GL(n - 1, O_F)$ belongs to $\text{ind}_{N'}^{G'} \psi$. For the conductor [JPS2].

(4) Let $W \in W(\pi)$. The function $x \rightarrow W(xg)$ on the parabolic standard subgroup PZ is locally constant of compact support modulo NZ for all $g \in G$. As $G = PZGL(n, O_F)$, the function W is of compact support modulo NZ . □

Let $\pi \in \text{Irr}_{\overline{\mathbf{Q}}_\ell} G$. Let E/\mathbf{Q}_ℓ be an extension contained in a finite extension of the maximal unramified extension of \mathbf{Q}_ℓ . Example: the extension E/\mathbf{Q}_ℓ generated by the values of ψ . The ring of integers O_E is principal. An O_E -free module L with an action of G such that L is a finite type $O_E G$ -module and such that $\overline{\mathbf{Q}}_\ell \otimes_{O_E} L \simeq \pi$ is called an O_E -integral structure of π . If such an L exists, π is called integral, the representation $r_\ell L = L \otimes_{O_E} \overline{\mathbf{F}}_\ell$ is of finite length. One calls $\overline{\mathbf{Z}}_\ell \otimes_{O_E} L$ an integral structure of π . When L, L' are two integral structures of π , then the semi-simplifications of $r_\ell L, r_\ell L'$ are isomorphic (see [V1, II.5.11.b] when E/\mathbf{Q}_ℓ is finite, and [Vig4, proof of theorem 2, page 416] in general). When $\pi \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} G$ is integral, $r_\ell L = L \otimes_{O_E} \overline{\mathbf{F}}_\ell$ is irreducible; the isomorphism class $r_\ell \pi$ of $r_\ell L$ is called the reduction of π ; any irreducible cuspidal $\overline{\mathbf{F}}_\ell$ -representation of G is the reduction of an integral irreducible cuspidal $\overline{\mathbf{Q}}_\ell$ -representation of G . For all these facts see [V1, III.5.10].

A function with values in $\overline{\mathbf{Q}}_\ell$ is called integral, when its values belong to $\overline{\mathbf{Z}}_\ell$. We denote by $K(\pi, \overline{\mathbf{Z}}_\ell)$, resp. $W(\pi, \overline{\mathbf{Z}}_\ell)$, the set of integral functions in the Kirillov model, resp. Whittaker model, of $\pi \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} G$. Let Λ be the maximal ideal of $\overline{\mathbf{Z}}_\ell$. The reduction modulo ℓ of an integral function f is the fonction $r_\ell f$ with values in $\overline{\mathbf{Z}}_\ell/\Lambda \simeq \overline{\mathbf{F}}_\ell$ deduced from f .

Theorem. (A) *Let $\pi \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} G$ with central character ω_π . Then the following properties are equivalent:*

- (A.1) ω_π is integral.
- (A.2) π is integral.
- (A.3) $K(\pi, \overline{\mathbf{Z}}_\ell)$ is a $\overline{\mathbf{Z}}_\ell$ -structure of π , called the integral Kirillov model.
- (A.4) $W(\pi, \overline{\mathbf{Z}}_\ell)$ is a $\overline{\mathbf{Z}}_\ell$ -structure of π , called the integral Whittaker model.

(B) *When π is integral, we have*

- (B.1) *The restriction to P from $W(\pi, \overline{\mathbf{Z}}_\ell)$ to $K(\pi, \overline{\mathbf{Z}}_\ell)$ is an isomorphism.*
- (B.2) *The integral Kirillov model is $\overline{\mathbf{Z}}_\ell P$ - generated by any function f with $f(1) = 1$. The integral Whittaker model $W(\pi, \overline{\mathbf{Z}}_\ell)$ is $\overline{\mathbf{Z}}_\ell G$ generated by the new vector.*
- (B.3) $\overline{\mathbf{F}}_\ell \otimes_{\overline{\mathbf{Z}}_\ell} K(\pi, \overline{\mathbf{Z}}_\ell) = K(r_\ell \pi, \overline{\mathbf{F}}_\ell)$ is the Kirillov model, and $\overline{\mathbf{F}}_\ell \otimes_{\overline{\mathbf{Z}}_\ell} W(\pi, \overline{\mathbf{Z}}_\ell) = W(r_\ell \pi, \overline{\mathbf{F}}_\ell)$ is the Whittaker model of $r_\ell \pi$.

Proof. The equivalence of (A1) (A2) [V1, II.4.12]; for the rest [V4 th.2] and the last theorem. □

Corollary. *Let $\pi, \pi' \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} G$ integral, with central character $\omega_\pi, \omega_{\pi'}$. Then $r_\ell \pi = r_\ell \pi'$ if and only if*

$$(*) \quad r_\ell \omega_\pi = r_\ell \omega_{\pi'}, \quad r_\ell \pi(w)(f) = r_\ell \pi'(w)(f)$$

for all $w \in S_n$, and for all f in the integral Kirillov model.

Proof. Use (B.3) and $\text{End}_{\overline{\mathbf{F}}_\ell} \tau_{\overline{\mathbf{F}}_\ell} \simeq \overline{\mathbf{F}}_\ell$. □

Questions. Can one define an integral Kirillov or Whittaker model for $\pi \in \text{Irr}_{\overline{\mathbf{Q}}_\ell} G$ integral and not cuspidal ? What is the action of S_n in the Kirillov model ?

2. The case $n = 2$

We can go further in the case $n = 2$. Let $\pi \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} G$ where $G = GL(2, F)$. The restriction of $GL(2, F)$ to $GL(1, F) = F^*$ gives an isomorphism from $K(\pi)$ to the space $C_c^\infty(F^*, \overline{\mathbf{Q}}_\ell)$ of locally constant functions $F^* \rightarrow \overline{\mathbf{Q}}_\ell$ with compact support, which respects the natural $\overline{\mathbf{Z}}_\ell$ -structures $K(\pi, \overline{\mathbf{Z}}_\ell) \simeq C_c^\infty(F^*, \overline{\mathbf{Z}}_\ell)$. The unique non trivial element of S_2 is represented by

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The action of $\pi(w)$ on the Kirillov model was described by Jacquet and Langlands [JL, Prop. 2.10 p.46], using Fourier transform for complex representations.

We choose a \mathbf{Q}_ℓ -Haar measure dx on F^* . The Fourier transform of $f \in C_c^\infty(F^*, \overline{\mathbf{Q}}_\ell)$ with respect to dx is

$$\hat{f}(\chi) := \int_{F^*} f(x)\chi(x)dx$$

for any character $\chi : F^* \rightarrow \overline{\mathbf{Q}}_\ell^*$.

We choose a uniformizing parameter p_F of F . A function $f \in C_c^\infty(F^*, \overline{\mathbf{Q}}_\ell)$ is determined by the set of functions $f_n \in C_c^\infty(O_F^*, \overline{\mathbf{Q}}_\ell)$ defined by $f_n(x) := f(p_F^{-n}x)$ for all $n \in \mathbf{Z}$. The functions f_n depend on the choice of p_F . Extension by zero allows to consider $C_c^\infty(O_F^*, \overline{\mathbf{Q}}_\ell)$ as a subspace of $C_c^\infty(F^*, \overline{\mathbf{Q}}_\ell)$, because O_F^* is open in F^* . We have

$$\hat{f}(\chi) = \sum_n \hat{f}_n(\chi)\chi(p_F^{-n}).$$

For a given character χ , the sum is finite. The functions $\hat{f}_n(\chi)$ depend only on the restriction of χ to O_F^* . Set $\hat{O}_F^* := \text{Hom}(O_F^*, \overline{\mathbf{Q}}_\ell)$. One introduces the formal series

$$f(x, X) := \sum_{n \in \mathbf{Z}} f_n(x)X^n, \quad \hat{f}(\chi, X) := \sum_{n \in \mathbf{Z}} \hat{f}_n(\chi)X^n$$

for all $x \in O_F^*$ and for all $\chi \in \hat{O}_F^*$.

Jacquet and Langlands [JL Prop. 2.10 page 46] proved that the action of $\pi(w)$ on the Kirillov model is given by:

$$(\pi(w)f)_n^\wedge(\chi) = c(\pi \otimes \chi^{-1}) \hat{f}_m(\chi^{-1}\omega_\pi^{-1})$$

for all $\chi \in \hat{O}_F^*$, all integers $n \in \mathbf{Z}$, where $m = -n - f(\pi \otimes \chi^{-1})$, for some constant $c(?) \in \overline{\mathbf{Q}}_\ell^*$ and some integer $f(?) \in \mathbf{Z}$. The formula and $c(\pi \otimes \chi^{-1})$ are independent of the choice of dx . The formula is equivalent to

$$(\pi(w)f)^\wedge(\chi, X) = \varepsilon(\pi \otimes \chi^{-1}) \hat{f}(\chi^{-1}\omega_\pi^{-1}, X^{-1})$$

for all $\overline{\mathbf{Q}}_\ell$ -characters χ of O_F^* , where the epsilon factor is

$$\varepsilon(\pi \otimes \chi^{-1}) = c(\pi \otimes \chi^{-1})X^{f(\pi \otimes \chi^{-1})}.$$

One calls $c(\pi)$ the constant and $f(\pi)$ the conductor of the epsilon factor $\varepsilon(\pi)$. They both depend on the choice of the non trivial character $\psi : F \rightarrow \overline{\mathbf{Z}}_\ell^*$ which was fixed, but not on the choice of dx or on p_F . Jacquet and Langlands used complex representations but their method is valid when the field of complex numbers is replaced by $\overline{\mathbf{Q}}_\ell$, because one uses only integrals of locally constant functions on compact sets. There is no problem of vanishing because we work on $\overline{\mathbf{Q}}_\ell$.

We suppose that dx is a \mathbf{Z}_ℓ -Haar measure on F^* which is not divisible by ℓ . Let

$$\mathcal{L} = \text{the Fourier transform of } C_c^\infty(O_F^*, \overline{\mathbf{Z}}_\ell).$$

We have $\mathcal{L} \subset C_c^\infty(\hat{O}_F^*, \overline{\mathbf{Z}}_\ell)$ and $\mathcal{L} = C_c^\infty(\hat{O}_F^*, \overline{\mathbf{Z}}_\ell)$ if and only if $q \not\equiv 1 \pmod{\ell}$ [V2]. In general, we separate the ℓ -regular part X of O_F^* from the ℓ -part Y of O_F^* , which is a cyclic group of order $m = \ell^a$. The volume of X for dx should be a unit in \mathbf{Z}_ℓ^* ; we can suppose it is equal to 1. The group of $\overline{\mathbf{Q}}_\ell$ -characters satisfy $\hat{O}_F^* \simeq \hat{X} \times \hat{Y}$. A general character in \hat{O}_F^* is now written as $\chi\mu$ where $\chi \in \hat{X}$ and $\mu \in \hat{Y}$, and a function $v : \hat{O}_F^* \rightarrow \overline{\mathbf{Q}}_\ell$ is thought as a function $v : \hat{X} \rightarrow C(\hat{Y}, \overline{\mathbf{Q}}_\ell)$ with $v(\chi)(\mu) := v(\chi\mu)$.

The $\overline{\mathbf{Z}}_\ell$ -module \mathcal{L} consists of all functions $v : \hat{X} \rightarrow L$ with compact support, where

$$L \subset C_c^\infty(\hat{Y}, \overline{\mathbf{Z}}_\ell)$$

is the free $\overline{\mathbf{Z}}_\ell$ -module with basis the characters $\underline{y} : \mu \rightarrow \mu(y^{-1})$ of \hat{Y} for all $y \in Y$.

We need some elementary linear algebra. The $\overline{\mathbf{Z}}_\ell$ -module L is the set of functions $v \in C_c^\infty(\hat{Y}, \overline{\mathbf{Q}}_\ell)$ such that

$$y \mapsto \langle v, y \rangle := |Y|^{-1} \sum_{\mu \in \hat{Y}} v(\mu)\mu(y)$$

belongs to $C(Y, \overline{\mathbf{Z}}_\ell)$. The orthogonality formula of characters gives

$$v = \sum_{y \in Y} \langle v, y \rangle \underline{y}$$

for all $v \in C(\hat{Y}, \overline{\mathbf{Q}}_\ell)$. For the usual product, $C_c^\infty(\hat{Y}, \overline{\mathbf{Q}}_\ell)$ is an algebra.

Lemma. *Let $v \in C_c^\infty(\hat{Y}, \overline{\mathbf{Q}}_\ell)$.*

- (i) *The inclusion $vL \subset L$ is equivalent to $v \in L$.*
- (ii) *The equality $vL = L$ is equivalent to $v \in L$ and $v(\mu) \in \overline{\mathbf{Z}}_\ell^*$ for all $\mu \in \hat{Y}$.*
- (iii) *The inclusion $vL \subset \Lambda L$ is equivalent to $\langle v, y \rangle \in \Lambda$ for all $y \in Y$ (Λ is the maximal ideal of $\overline{\mathbf{Z}}_\ell$).*

Proof. (i) The inclusion $vL \subset L$ is equivalent to $\langle v\underline{z}, z' \rangle = \langle v, z^{-1}z' \rangle \in \overline{\mathbf{Z}}_\ell$ for all $z, z' \in Y$, which is equivalent to $v \in L$.

(ii) $vL = L$ means that $v\underline{z}$ for $z \in Y$ is a basis of L . We have $v\underline{z} = \sum_{z' \in Y} \langle v, z^{-1}z' \rangle \underline{z}'$, hence $vL = L$ means that

$$\langle v, z^{-1}z' \rangle_{z, z'} \in SL(m, \overline{\mathbf{Z}}_\ell).$$

The Dedekind determinant $\det(\langle v, z^{-1}z' \rangle_{z, z'})$ is equal to $\prod_{\mu \in \hat{Y}} v(\mu)$ (see [L] exercise 28 page 495).

(iii) see the proof of (i). □

Let $\pi \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} G$ integral. As $\pi(w)$ is an isomorphism of the integral Kirillov model, the function

$$c(\pi \otimes \chi) : \mu \in \hat{Y} \rightarrow c(\pi \otimes \chi\mu) \in \overline{\mathbf{Q}}_\ell$$

satisfies $c(\pi \otimes \chi)L = L$ for all character $\chi \in \hat{X}$. We apply the lemma to $c(\pi \otimes \chi)$. We define **new epsilon factors**

$$\varepsilon(\pi, y) := \langle c(\pi), y \rangle X^{f(\pi)}, \quad \langle c(\pi), y \rangle = |Y|^{-1} \sum_{\mu \in \hat{Y}} c(\pi \otimes \mu)\mu(y),$$

for all $y \in Y$. As have $f(\pi) \geq 2$ for $\pi \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} G$, we have $f(\pi) = f(\pi \otimes \mu) \geq 2$ for all $\mu \in \hat{Y}$. When Y is trivial (i.e. $q \not\equiv 1 \pmod{\ell}$), they are simply the usual ones.

Theorem. (1) Let $\pi \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} G$ integral. Then the constant of the epsilon factor is a unit $c(\pi) \in \overline{\mathbf{Z}}_\ell^*$ and the new constants $\langle c(\pi), y \rangle \in \overline{\mathbf{Z}}_\ell$ are integral, for all $y \in Y$.

(2) Let $\pi, \pi' \in \text{Cusp}_{\overline{\mathbf{Q}}_\ell} G$ integral with central characters $\omega_\pi, \omega_{\pi'}$. Then $r_\ell \pi = r_\ell \pi'$ if and only if $r_\ell \omega_\pi = r_\ell \omega_{\pi'}$ and their new epsilon factors have the same reduction modulo ℓ : the conductors $f(\pi \otimes \chi) = f(\pi' \otimes \chi)$ are equal, and the new constants have the same reduction modulo ℓ :

$$r_\ell \langle c(\pi \otimes \chi), y \rangle = r_\ell \langle c(\pi' \otimes \chi), y \rangle$$

for all $y \in Y$, and all $\overline{\mathbf{Q}}_\ell$ -characters $\chi \in \hat{X}$.

Proof. With the last corollary of the paragraph (1), $r_\ell \pi = r_\ell \pi'$ if and only if $r_\ell \omega_\pi = r_\ell \omega_{\pi'}$ and

$$(*) \quad c(\pi \otimes \chi)\hat{f}_m(\chi^{-1}\omega_\pi^{-1}) = c(\pi' \otimes \chi)\hat{f}_{m'}(\chi^{-1}\omega_{\pi'}^{-1}) \pmod{\Lambda\mathcal{L}}$$

for all $f_n \in C_c^\infty(O_F^*, \overline{\mathbf{Z}}_\ell)$ and all $n \in \mathbf{Z}$. With the lemma, we deduce the theorem. \square

We apply now the theorem to representations over $\overline{\mathbf{F}}_\ell$. Any $\pi \in \text{Cusp}_{\overline{\mathbf{F}}_\ell} G$ lifts to $\overline{\mathbf{Q}}_\ell$ and we can define epsilon factors

$$\varepsilon(\pi \otimes \chi, y) := \langle c(\pi \otimes \chi), y \rangle X^{f(\pi \otimes \chi)}$$

for all $y \in Y$ and all $\chi \in \text{Hom}(O_F^*, \overline{\mathbf{F}}_\ell^*) = \text{Hom}(X, \overline{\mathbf{F}}_\ell^*)$, by reduction modulo ℓ . They are not zero for any (y, χ) .

Corollary. $\pi, \pi' \in \text{Cusp}_{\overline{\mathbf{F}}_\ell} G$ are isomorphic if and only if they have the same central character and the same epsilon factors

$$\varepsilon(\pi \otimes \chi, y) = \varepsilon(\pi' \otimes \chi, y)$$

for all $y \in Y$, and for all character $\chi \in \text{Hom}(O_F^*, \overline{\mathbf{F}}_\ell^*)$.

Final remarks. a) When $n > 2$, the groups $GL(m, O_F)^*$ for $m \leq n - 1$ replace O_F^* .

b) Using the explicit description for the irreducible representations of dimension n of W_F [V3], one could try to prove a similar theorem for the irreducible integral $\overline{\mathbf{Q}}_\ell$ -representations of W_F of dimension n . To my knowledge this is a known and harder problem, which is not solved in the complex case.

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