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## Symmetry and folding of continued fractions

par ALFRED J. VAN DER POORTEN

*To Michel Mendès France on his 65th birthday*

RÉSUMÉ. Le ‘lemme de pliage’ de Michel Mendès France fournit une nouvelle justification de la symétrie du développement en fraction continue d’un irrationnel quadratique.

ABSTRACT. Michel Mendès France’s ‘Folding Lemma’ for continued fraction expansions provides an unusual explanation for the well known symmetry in the expansion of a quadratic irrational integer.

### 1. Continued fractions

A *continued fraction* expansion

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5 + \dots}}}}}$$

is conveniently represented by  $[a_0, a_1, a_2, \dots]$ . Set  $[a_0, a_1, \dots, a_h] = x_h/y_h$  for  $h = 0, 1, 2, \dots$ . Then the definition  $[a_0, a_1, \dots, a_h] = a_0 + 1/[a_1, \dots, a_h]$  immediately implies by induction on  $h$  that there is a correspondence

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_h & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x_h & x_{h-1} \\ y_h & y_{h-1} \end{pmatrix} \\ \longleftrightarrow [a_0, a_1, \dots, a_h] = x_h/y_h$$

between certain products of two by two matrices and continued fractions. It follows by transposition of matrices that

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**Proposition 1.**  $[a_h, a_{h-1}, \dots, a_1] = y_h/y_{h-1}$ , for  $h = 1, 2, \dots$ ,

and by taking determinants that

**Proposition 2.**  $x_h y_{h-1} - x_{h-1} y_h = (-1)^{h+1}$ , for  $h = 0, 1, 2 \dots$ .

One then sees that  $x_h/y_h = a_0 + 1/y_0 y_1 - 1/y_1 y_2 + \dots + (-1)^{h-1}/y_{h-1} y_h$ . It follows that if  $\alpha = [a_0, a_1, a_2, \dots] = \lim_{h \rightarrow \infty} x_h/y_h$  then

$$\alpha - x_h/y_h = (-1)^h (1/y_h y_{h+1} - 1/y_{h+1} y_{h+2} + 1/y_{h+2} y_{h+3} - \dots).$$

Specifically, if the *partial quotients*  $a_1, a_2, \dots$  are positive integers, as is the case for the simple continued fraction expansions we discuss below, and  $\alpha$  is irrational, then

$$|y_h \alpha - x_h| < 1/y_{h+1} < 1/a_{h+1} y_h \quad \text{for } h = 0, 1, 2, \dots,$$

illustrating the quality of approximation of the *convergents*  $x_h/y_h$  of  $\alpha$ .

Continued fractions are well studied objects, so new elementary results concerning their structure are a surprise.

**Proposition 3** (Folding Lemma [2]). *We have*

$$x_h/y_h + (-1)^h/cy_h^2 = [a_0, w_h, c - y_{h-1}/y_h] = [a_0, w_h, c, \overleftarrow{w}_h].$$

Here we have set  $w_h = a_1, a_2 \dots, a_h$ ; accordingly the notation  $\overleftarrow{w}_h$  denotes  $-a_h, -a_{h-1}, \dots, -a_1$ . It will save space to write  $\bar{a}$  for  $-a$ .

*Proof.* Here  $\longleftrightarrow$  denotes the ‘correspondence’ between two by two matrices and continued fractions. We have

$$[a_0, w_h, c - y_{h-1}/y_h] \longleftrightarrow \begin{pmatrix} x_h & x_{h-1} \\ y_h & y_{h-1} \end{pmatrix} \begin{pmatrix} c - y_{h-1}/y_h & 1 \\ 1 & 0 \end{pmatrix} = \\ \begin{pmatrix} cx_h - (x_h y_{h-1} - x_{h-1} y_h)/y_h & x_h \\ cy_h & y_h \end{pmatrix} \longleftrightarrow x_h/y_h - (-1)^{h+1}/cy_h^2,$$

as alleged. Moreover,  $[c - y_{h-1}/y_h] = [c, \overleftarrow{w}_h]$  by Proposition 1. □

The adjective ‘folding’ is appropriate because iteration of the perturbed symmetry  $w \rightarrow w, c, \overleftarrow{w}$  yields a pattern of signs on the word  $w$  corresponding to the pattern of creases in a sheet of paper repeatedly folded in half; see [1].

Incidentally, it is often convenient to drop subscripts and to use the convention whereby if, say,  $y$  denotes  $y_h$ , then  $y'$  denotes the previous  $y$ ; that is,  $y' = y_{h-1}$ .

The folding lemma makes it easy, given the expansion of a partial sum of a series, to adjust a continued fraction expansion for an appended term if the intervening ‘gap’ is wide enough. Our example is from the function field case, where ‘degree’ in  $X$  replaces the logarithm of the absolute value. Specifically if the partial sum has degree  $-n$  one can readily append a term

of degree  $-2n$  or less. Of course if the degree of the appended term is much less, that introduces a partial quotient of correspondingly high degree.

As an application of the lemma: the continued fraction expansion of the sum

$$F = X \sum_{h \geq 0} X^{-2^h} = 1 + X^{-1} + X^{-3} + X^{-7} + X^{-15} + X^{-31} + X^{-63} + X^{-127} + \dots$$

is given sequentially by  $1 + X^{-1} = [1, X]$ ,  $1 + X^{-1} + X^{-3} = [1, X, \overline{X}, \overline{X}]$ ,  $1 + X^{-1} + X^{-3} + X^{-7} = [1, X, \overline{X}, \overline{X}, \overline{X}, X, X, \overline{X}]$ , ... , where the addition of each term is done by a 'fold' with  $c = -X$ ; see [4]\*.

### 2. Symmetry

The computation

$$\begin{aligned} -y &= 0 + \overline{y} \\ -1/y &= \overline{1} + (y - 1)/y \\ y/(y - 1) &= 1 + 1/(y - 1) \\ y - 1 &= \overline{1} + y \\ 1/y &= 0 + 1/y \\ y &= \end{aligned}$$

shows how to make negative partial quotients positive. Specifically

$$\begin{aligned} [a_0, w', a_h, c, -\overleftarrow{w}] &= [a_0, w', a_h, c, 0, \overline{1}, 1, \overline{1}, 0, a_h, \overleftarrow{w}'] \\ &= [a_0, w', a_h, c - 1, 1, a_h - 1, \overleftarrow{w}']. \end{aligned}$$

Here we use  $[\dots, a, 0, b, \dots] = [\dots, a + b, \dots]$ , having recalled that

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a + b & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus folding yields a symmetric continued fraction expansion, perturbed (cf [3]) in the given example by the word  $a_h, c - 1, 1, a_h - 1$ .

This raises the question as to whether every such 'symmetric' expansion arises from folding. Mind you, it should not raise any such thing. If the given symmetric expansion yields the convergent  $X/Y$  and its 'first half' is  $x/y$  then necessarily  $X/Y - x/y$  is of the shape  $\pm 1/cy^2$ , for some appropriate  $c$ .

So, a better question is to ask what one might learn from viewing a symmetric expansion as a folded expansion. The best known case of a symmetric expansion is the period of a real quadratic irrational integer. Below

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\* Jeffrey Shallit recently noticed that the evaluation just given is foreshadowed in remarks of Walter Leighton and W. T. Scott, 'A general continued fraction expansion', *Bull. Amer. Math. Soc.* **45** (1939) 596-605.

we study just that case and rediscover the conditions for being halfway in such an expansion.

### 3. Periodic continued fractions

Suppose the real quadratic irrational integer  $\delta$  has norm  $n$  and trace  $t$ . That is,  $\delta^2 - t\delta + n = 0$  with integers  $n$  and  $t$ , where  $t^2 - 4n > 0$  is not a square in  $\mathbb{Z}$ . To fix  $\delta$ , we suppose that  $\delta > \bar{\delta}$ , where  $\bar{\delta}$  denotes the conjugate of  $\delta$ .

Let  $x_h/y_h = [a_0, a_1, \dots, a_h] = [a_0, w]$  be an initial segment of the continued fraction expansion of  $\delta$ . We will play with the matrices

$$N_h = \begin{pmatrix} x_h & -ny_h \\ y_h & x_h - ty_h \end{pmatrix} = \begin{pmatrix} x & -ny \\ y & x - ty \end{pmatrix},$$

noticing first that the decomposition

$$(1) \quad N_h = \begin{pmatrix} x_h & -ny_h \\ y_h & x_h - ty_h \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_h & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & P_{h+1} \\ 0 & Q_{h+1} \end{pmatrix}$$

defines sequences  $(P_h)_{h \geq 0}$  and  $(Q_h)_{h \geq 0}$  of integers. Here,  $(-1)^{h+1}Q_{h+1}$  is the determinant  $x^2 - txy + ny^2$  of  $N_h$ . Also

$$(2) \quad xP + x'Q = -ny, \quad yP + y'Q = x - ty.$$

Because  $x/y$  is a convergent of  $\delta$  we have  $|x - \delta y| < 1/y$ , and so  $Q$  is at most  $\delta - \bar{\delta}$ . It follows from several applications of the box principle that there are positive integers  $X$  and  $Y$  providing a solution to ‘Pell’s equation’  $X^2 - tXY + nY^2 = \pm 1$ .

**Proposition 4.** *If  $X^2 - tXY + nY^2 = \pm 1$  and  $X/Y = [a_0, a_1, \dots, a_r]$ , then*

$$\delta = [\overline{a_0, a_1, \dots, a_r, 0}] = [a_0, \overline{a_1, \dots, a_{r-1}, a_r + a_0}].$$

*Proof.* Set  $[\overline{a_0, a_1, \dots, a_r, 0}] = \gamma$ ; that is,  $\gamma = [a_0, a_1, \dots, a_r, 0, \gamma]$ . By the matrix correspondence<sup>†</sup> this is

$$\gamma \longleftrightarrow \begin{pmatrix} X & -nY \\ Y & X - tY \end{pmatrix} \begin{pmatrix} \gamma & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \gamma X - nY & X \\ X - tY + \gamma Y & Y \end{pmatrix} \longleftrightarrow \frac{\gamma X - nY}{X - tY + \gamma Y},$$

and so  $(\gamma^2 - t\gamma + n)Y = 0$ . But  $Y \neq 0$  and  $\gamma > \bar{\gamma}$ , so  $\gamma = \delta$ . □

Notice moreover that

$$[a_0, a_1, \dots, a_r, 0, t] \longleftrightarrow \begin{pmatrix} X & -nY \\ Y & X - tY \end{pmatrix} \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} tX - nY & X \\ X & Y \end{pmatrix}.$$

Thus the word  $a_0, a_1, \dots, a_{r-1}, a_r + t$  must be a palindrome<sup>‡</sup>.

<sup>†</sup>There is a subtle point here at which I tacitly suppose that  $X^2 - tXY + nY^2 = (-1)^r$ . However, that loses no generality, because if  $X/Y = [\dots, a_r]$  also  $X/Y = [\dots, a_r - 1, 1]$ .

<sup>‡</sup>Anubis: Dog as a devil deified lived as a god.

#### 4. Halfway in the period

Suppose, first, that the period of  $\delta$  is of even length  $r = 2h + 2$ . Then  $N = N_h$  corresponds to half the period in the sense that  $N^2 = \begin{pmatrix} X & -nY \\ Y & X-tY \end{pmatrix}$  provides  $X/Y$  yielding the period as in Proposition 4.

Thus we consider

$$\begin{aligned} \frac{X}{Y} - \frac{x}{y} &= \frac{(x^2 - ny^2)y - (2xy - ty^2)x}{Yy} \\ &= \frac{-(x^2 - txy + ny^2)y}{(2x - ty)y^2} = (-1)^h \frac{Qy}{(2x - ty)y^2}; \end{aligned}$$

whence

$$\frac{x}{y} + \frac{(-1)^h}{\frac{2x-ty}{Qy}y^2} = [a_0, w, \frac{2x-ty}{Qy} - \frac{y'}{y}].$$

By (2) we have

$$\frac{2x - ty}{Qy} = \frac{(2P + t)y + 2Qy'}{Qy} = \frac{2P + t}{Q} + 2\frac{y'}{y},$$

and so, indeed, the Folding Lemma entails that

$$\frac{X}{Y} = \frac{x}{y} + \frac{(-1)^h}{\frac{2x-ty}{Qy}y^2} = [a_0, w, \frac{2P+t}{Q}, \frac{y}{y'}] = [a_0, w, (2P+t)/Q, \overline{w}].$$

**Proposition 5.** *Suppose that the quadratic irrational integer  $\delta$  satisfies  $\delta > \bar{\delta}$  and has a continued fraction expansion period of even length  $2h + 2$ . Then*

$$\delta = [a_0, \overline{w, (2P+t)/Q, \overline{w}, 2a_0 - t}]$$

*if and only if  $Q$  divides  $2P + t$ .*

*Proof.* The necessity of the condition is clear since otherwise the given expansion is not admissible. For its sufficiency we need to confirm that  $X/Y$  does yield a solution to ‘Pell’s equation’. That is not obvious: plainly  $X^2 - tXY + nY^2 = Q^2$ .

However, it turns out that both  $Q|X$  and  $Q|Y$ . To see that, consider the conditions (2) modulo  $Q$ , that is,  $Px \equiv -ny$  and  $Py \equiv x - ty$ .

Recall we suppose  $2P + t \equiv 0$ . Then  $X = x^2 - ny^2 \equiv (P+t)xy + Pxy \equiv 0$  and  $Y = 2xy - ty^2 = (x - ty + x)y \equiv (Py + (P+t)y)y \equiv 0$ , as suggested.  $\square$

Suppose, second, that the period of  $\delta$  is of odd length  $r = 2h + 1$ . Then  $N$  corresponds to half the period in the sense that  $NN' = \begin{pmatrix} X & -nY \\ Y & X-tY \end{pmatrix}$  provides  $X/Y$  yielding the period; here  $N'$  denotes  $N_{h-1}$ .

We now consider

$$\begin{aligned} \frac{X}{Y} - \frac{x}{y} &= \frac{(xx' - nyy')y - (xy' + x'y - tyy')x}{Yy} \\ &= \frac{-(x^2 - txy + ny^2)y'}{(xy' + x'y - tyy')y} = (-1)^h \frac{Qy'}{(xy' + x'y - tyy')y}; \end{aligned}$$

whence

$$\frac{x}{y} + \frac{(-1)^h}{\left(\frac{x-ty}{Qy} + \frac{x'}{Qy'}\right)y^2} = [a_0, w, \frac{x-ty}{Qy} + \frac{x'}{Qy'} - \frac{y'}{y}].$$

By (2)

$$\frac{x-ty}{Qy} = \frac{P}{Q} + \frac{y'}{y} \quad \frac{x'}{Qy'} = \frac{P'+t}{Q} + \frac{Q'y''}{Qy'},$$

and so

$$\begin{aligned} X/Y &= x/y + \frac{(-1)^h}{\left(\frac{x-ty}{Qy} + \frac{x'}{Qy'}\right)y^2} \\ &= [a_0, w, \frac{P'+P+t}{Q}, \frac{Qy'}{Q'y''}] = [a_0, w, (P'+P+t)/Q, \overleftarrow{w'}], \end{aligned}$$

provided that  $Q = Q'$ ; here  $w'$  denotes the word  $a_1, a_2, \dots, a_{h-1}$ .

**Proposition 6.** *Suppose that the quadratic irrational integer  $\delta$  satisfies  $\delta > \bar{\delta}$  and has a continued fraction expansion period of odd length  $2h + 1$ . Then*

$$\delta = [a_0, w, \overline{(P'+P+t)/Q}, \overleftarrow{w'}, 2a_0 - t]$$

*if and only if  $Q = Q'$ .*

*Remark.* We will see below that  $P_k + P_{k+1} + t = a_k Q_k$  for  $k = 0, 1, \dots$ , so when  $Q' = Q$  the requirement  $Q | (P' + P + t)$  is automatically satisfied.

*Proof.* The necessity of the condition is clear since otherwise the given expansion is not symmetric; in particular we would not have  $a_h = (P' + P + t)/Q$ . For its sufficiency we need to confirm that  $X/Y$  does yield a solution to 'Pell's equation'. That's not obvious: plainly  $X^2 - tXY + nY^2 = -Q'Q = -Q^2$ .

However, both  $Q | X$  and  $Q | Y$ . Recall we have  $Q' = Q$  and so  $P' + P + t \equiv 0$  modulo  $Q$ . Then  $xx' - nyy' \equiv (P + t)x'y + P'x'y \equiv 0$  and  $Y = xy' + x'y - tyy' = (P + t)yy' + (P' + t)y'y - tyy' \equiv 0$ .  $\square$

## 5. Remarks and explanations

**5.1. Some identities.** The equations  $Px + Qx' = -ny$  and  $P'y + Q'y' = x - ty$  together entail that

$$(xy' - x'y)P = -(xx' + nyy' - tx'y), \quad \text{or} \quad P = (-1)^h (xx' + nyy' - tx'y).$$

Brute computation now shows that

$$\det NN' = -QQ' = n + tP + P^2.$$

We will avoid the need for such brutality by showing that our also taking the equations (2) in the form  $P'x' + Q'x'' = -ny'$  and  $P'y' + Q'y'' = x' - ty'$  provides a base from which one can prove the facts we want by induction. Here we note that  $x = ax' + x''$  and  $y = ay' + y''$ , where  $a = a_h$ . Then

$$\begin{aligned} 0 &= P'y' + Q'y'' - x' + ty' \\ &= P'y' + Q'y - x' + ty' - aQ'y', \end{aligned}$$

and

$$\begin{aligned} 0 &= P'x' + Q'x'' + ny' \\ &= P'x' + Q'x + ny' - aQ'x'. \end{aligned}$$

Multiplying these equations respectively by  $x$  and by  $y$ , and subtracting, yields

$$(xy' - x'y)aQ' = (xy' - x'y)P' - (xx' + nyy' - tx'y) + (xy' - x'y)t.$$

That is, as promised,

$$(3) \quad aQ' = P' + P + t.$$

As just noted, we therefore have

$$\begin{aligned} 0 &= (P' + t - aQ')y' + Q'y - x' \\ &= Q'y - Py' - x', \end{aligned}$$

and

$$\begin{aligned} 0 &= (P' - aQ')x' + Q'x + ny' \\ &= Q'x - (P + t)x' + ny'. \end{aligned}$$

Multiplying these equations respectively by  $n$  and by  $P$ , and adding, yields

$$\begin{aligned} 0 &= Q'(Px + ny) - (n + tP + P^2)x' \\ &= -Q'Qx' - (n + tP + P^2)x', \end{aligned}$$

and so, indeed

$$(4) \quad -QQ' = n + tP + P^2.$$



**5.2. The continued fraction expansion.** The significance of (3) is its announcement that

$$(5) \quad (\delta + P_h)/Q_h = a_h - (\bar{\delta} + P_{h+1})/Q_h$$

is step  $h$  in the continued fraction expansion of  $\delta$ . Just so, (4) fixes the reciprocal of the *remainder* in (5) above and thus  $Q_{h+1}$  and the next *complete quotient*  $(\delta + P_{h+1})/Q_{h+1}$ . Moreover, because the  $Q$ 's are positive it follows from (4) alone that  $-\delta < P_{h+1} < -\bar{\delta}$  for  $h = 0, 1, \dots$ . Mind you, we also know that the complete quotients satisfy  $(\delta + P)/Q > 1$  and the remainders  $-1 < (\bar{\delta} + P)/Q' < 0$ .

Given the rules (3) and (4) defining the continued fraction expansion it is an interesting exercise to retrieve the conditions (2) and so the decompositions (1) of the matrices  $N_h$ . The trick is to use induction, showing that  $P'x' + Q'x'' = -ny'$  and  $P'y' + Q'y'' = x' - ty'$  entails  $Px + Qx' = -ny$  and  $Py + Qy' = x - ty$ . The point of this all is to emphasise that the two approaches *are* equivalent, confirming that the matrices  $N_h$  detail the continued fraction expansion of  $\delta$ .

**5.3. Binary quadratic forms.** The matrix  $N_h$  determines a quadratic form  $(-1)^{h+1}Qu^2 - (2P+t)uv + (-1)^hQ'v^2$  of discriminant  $4P^2 + 4Pt + t^2 + 4QQ' = t^2 - 4n = (\delta - \bar{\delta})^2$ . That is, it provides  $P$  and  $Q$ , and the discriminant, and therefore  $Q'$ .

It might take us a little too far afield to provide details, so I leave confirmation of the following to the interested reader. It turns out that multiplication of a pair  $N_{i-1}$  and  $N_{j-1}$  is composition of the corresponding quadratic forms. Indeed, this is a natural way to define composition. There are provisos. Whereas the forms determined by the  $N_h$  are reduced, the composite, which a priori has leading coefficient  $Q_iQ_j$ , is in general not reduced. Moreover, it may not be primitive. Its coefficients will have the common factor  $G^2$ , where  $G = \gcd(Q_i, Q_j, P_i + P_j + t)$ . One must therefore always consider the 'primitive' matrix product  $G^{-1}N_{i-1}N_{j-1}$ .

**5.4.** The 'results', specifically Propositions 5 and 6, given here are of course well known. I doubt however whether they have heretofore been proved by the Folding Lemma. My remarks on the matrices  $N_h$  contain some previously unpublished observations. One motivation for viewing continued fractions of quadratic irrationals as a study of the  $N_h$  is that doing it suggests an immediate generalisation to approximation algorithms for higher degree algebraic irrationals.

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