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A GALERKIN PROCEDURE FOR APPROXIMATING THE FLUX ON THE BOUNDARY FOR ELLIPTIC AND PARABOLIC BOUNDARY VALUE PROBLEMS (*)

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Abstract. — *Elliptic and parabolic boundary value problems are first solved approximately on a rectangle and then simple auxiliary calculations are performed on the Galerkin solution to produce higher order correct approximations to the boundary flux than arise from the straightforward differentiation of the Galerkin solution.* $L^2(\partial R)$ *and* $L^\infty(\partial R)$ *estimates are derived for the error.*

1. INTRODUCTION

In many physical problems the major interest is in obtaining flux values on the boundary, rather than the values of the solution to the particular partial differential equation being considered. In this paper a method proposed by J. Wheeler [5] for approximating the boundary flux will be analyzed for second order elliptic and parabolic boundary value problems when the domain is a rectangle.

Let $R = I \times I$, where $I = (0, 1)$. Suppose that $a(x)$, $c(x)$, and $\rho(x)$ are $L^\infty(R)$ functions. Further assume that there exist constants ρ_0, ρ_1, a_0, a_1, c_0 and c_1 such that $0 < \rho_0 \leqslant \rho(x) \leqslant \rho_1$, $0 < a_0 \leqslant a(x) \leqslant a_1$, $0 \leqslant c_0 \leqslant c(x) \leqslant c_1$, $x \in \bar{R}$.

We shall consider the following two boundary value problems :

$$(1.1) \qquad \begin{aligned} Lu &= f, \qquad x \in R, \\ u(x) &= 0, \qquad x \in \partial R, \end{aligned}$$

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and

(1.2)
$$\rho(x)\frac{\partial w}{\partial t} + L(w) = f(x, t), \quad (x, t) \in R \times (0, T),$$
$$w(x, t) = 0, \qquad (x, t) \in \partial R \times (0, T),$$
$$w(x, 0) = w_0(x) \quad , \quad x \in R,$$

where

$$Ly = -\nabla \cdot (a(x)\nabla y) + c(x)y.$$

For nonnegative integers s, the Sobolev space $H^s(R)$ is the set of all functions in $L^2(R)$ whose distributional derivatives of order not greater than s are also in $L^2(R)$; $H^s(R)$ is normed by

$$\|w\|_s^2 = \sum_{|\alpha| \leqslant s} \|D^\alpha w\|^2,$$

where $\alpha = \{\alpha_1, \alpha_2\}$, α_i a nonnegative integer, $|\alpha| = \alpha_1 + \alpha_2$, and

$$D^\alpha w = \frac{\partial^{|\alpha|} w}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}.$$

Here $\|\cdot\|$ denotes the $L^2(R)$ norm, and $(\cdot, \cdot)$ denotes the (real) $L^2(R)$ inner product,

$$(\varphi, \psi) = \int_R \varphi(x)\psi(x)\, dx.$$

We define

$$\langle \varphi, \psi \rangle = \int_{\partial R} \varphi(x)\psi(x)\, d\gamma.$$

Also $H_0^1(I) = \{ v \in H^1(I) : v(0) = v(1) = 0 \}$. For X a normed space with norm $\|\cdot\|_X$ and $\varphi : [0, T] \to X$, we adopt the notations

$$\|\varphi\|_{L^2(X)}^2 = \int_0^T \|\varphi(t)\|_X^2\, dt$$

and

$$\|\varphi\|_{L^\infty(X)} = \operatorname*{ess\,sup}_{0 \leqslant t \leqslant T} \|\varphi(t)\|_X.$$

The fractional order $H^s(R)$-spaces are to be interpreted in the sense of interpolation of Hilbert spaces [4] for $s > 0$. The space $H^{-s}(R)$, $s > 0$, is the dual of $H^s(R)$ and is normed as follows :

$$\|\varphi\|_{-s} = \sup_{0 \neq \psi \in H^s(R)} \frac{(\varphi, \psi)}{\|\psi\|_s}.$$

Let $\Delta : 0 = x_0 < x_1 < \dots < x_N = 1$ be a partition of I and set $I_j = [x_{j-1}, x_j]$. Let

$$M_k(r, \Delta) = \{ v \mid v \in C^k(I),\ v \in P_r(I_j), \quad j = 1, 2, \dots, N \},$$

where $P_r(E)$ is the class of functions defined on I with restrictions to the set E agreeing with a polynomial of degree not greater than r. Let

$$h = \max_{1 \leqslant j \leqslant N} h_j \quad , \quad h_j = x_j - x_{j-1},$$

and assume that there exists a constant $c_2 > 0$ such that $c_2 h \leqslant h_j$; such a partition is called quasi-uniform. Let

$$\mathcal{M}_{k,r} = M_k(r, \Delta) \otimes M_k(r, \Delta),$$

and

$$\mathcal{M}^0_{k,r} = \{ v \in \mathcal{M}_{k,r} \mid v(x) = 0,\ x \in \partial R \}.$$

Let q_0 and q_1 be linearly independent elements of $M_k(r, \Delta)$ such that $q_0(0) = q_1(1) = 1$, $\|q_j\|_{L^2(I)} \leqslant Ch^{1/2}$, and $\|q_j\|_{H^1(I)} \leqslant Ch^{-1/2}$; such elements were constructed in the proof of Lemma 2.1 of Douglas, Dupont and Wahlbin [1] and can be chosen to satisfy the additional constraint that the support of q_j is contained in an interval of length $O(h)$. Let $N_k(r, \Delta)$ be the span of q_0 and q_1. Let

$$\partial\mathcal{M}_{k,r} = (N_k(r, \Delta) \otimes M_k(r, \Delta)) + (M_k(r, \Delta) \otimes N_k(r, \Delta))$$

denote a particular algebraic complement of $\mathcal{M}^0_{k,r}$ in $\mathcal{M}_{k,r}$.

The space $\mathcal{M}_{k,r}$ has the following property.

Lemma 1.1. There exists a constant C such that, for any $\gamma \in \mathcal{M}_{k,r}$, there exists an element $Q \in \mathcal{M}_{k,r}$ such that $Q = \gamma$ on ∂R and

$$\|Q\|_1 \leqslant Ch^{-1/2} \|\gamma\|_{L_2(\partial R)},$$

$$\|Q\| \leqslant Ch^{1/2} \|\gamma\|_{L_2(\partial R)}.$$

Proof. Take Q to be the unique element in $\partial\mathcal{M}_{k,r}$ agreeing with γ on ∂R. q.e.d.

For convenience we denote by $\beta(\cdot, \cdot)$ the bilinear form

$$\beta(\varphi, \psi) = (a\nabla\varphi, \nabla\psi) + (c\varphi, \psi).$$

The Galerkin solution $U \in \mathcal{M}^0_{k,r}$ to (1.1) is defined by

$$\beta(U, V) = (f, V) \quad , \quad V \in \mathcal{M}^0_{k,r}. \tag{1.3}$$

The approximation to the flux $a(x)\frac{\partial u}{\partial \nu}$ is defined by the boundary values of the element $\Gamma \in \partial\mathcal{M}_{k,r}$ satisfying the relations

$$\langle \Gamma, V \rangle = \beta(U, V) - (f, V), \qquad V \in \mathcal{M}_{k,r}. \tag{1.4}$$

For the parabolic problem we define an analogous procedure. Let

$$Z : [0, T] \to \mathcal{M}^0_{k,r}$$

be given by

$$\begin{aligned} \left(\rho\frac{\partial Z}{\partial t}, V\right) + \beta(Z, V) &= (f, V), \quad V \in \mathcal{M}^0_{k,r}, \quad t > 0, \\ \beta(Z - w_0, V) &= 0, \quad V \in \mathcal{M}^0_{k,r}, \quad t = 0. \end{aligned} \tag{1.5}$$

We define $\Gamma : [0, T] \to \partial\mathcal{M}_{k,r}$ by the equations

$$\langle \Gamma(t), V \rangle = \left(\rho\frac{\partial Z}{\partial t}, V\right) + \beta(Z, V) - (f, V), t > 0, V \in \mathcal{M}_{k,r}. \tag{1.6}$$

Again the flux is approximated by the boundary values of Γ. Note that the evaluation of $\Gamma(t)$ involves the solution at time t only; thus, it can be evaluated at any convenient selection of time arguments.

In Section 2 we derive error estimates of $\Gamma - a\frac{\partial u}{\partial \nu}$ for the elliptic problem with homogeneous data. For the case $a \equiv 1$ we obtain optimal L^2 and L^∞ estimates. Here optimal means optimal rate of convergence $(O(h^{r+1}))$ with minimal norm on the solution.

For smooth functions $a(x)$ we show that $\left\|\Gamma - a\frac{\partial u}{\partial \nu}\right\|_{L^2(\partial R)} \leqslant C(u)h^{r+1/2}$ for the special case $k = 0$. This result does not give an optimal rate of convergence since the exponent on h is $r + 1/2$ and not $r + 1$; however, the norm on the solution is minimal.

In Section 3 we extend the elliptic results to the parabolic problem. For $a \equiv 1$, optimal convergence rates in L^∞ in space and time are established. For variable functions a, we again treat the case $k = 0$ and lose a factor of $h^{1/2}$.

2. FLUX ESTIMATES FOR ELLIPTIC PROBLEMS WITH HOMOGENEOUS DIRICHLET BOUNDARY VALUES

We first consider the case $a \equiv 1$ in (1.3.) For $w \in H^1_0(I)$ let Pw denote the H^1_0 projection of w into $M^0_{k,r} = M_k(r, \Delta) \cap H^1_0(I)$ defined by

$$\int_I (w - Pw)'v' \, dx = 0 \quad , \quad v \in M^0_{k,r}.$$

Let $W = P \otimes Pu$. We have

$$(2.1)\qquad (\nabla(u - W), \nabla V) = \left\langle (I - P)\frac{\partial u}{\partial \nu}, V \right\rangle - (\eta, V) \quad , \quad V \in \mathcal{M}_{k,r},$$

where $I - P$ has the obvious one-dimensional interpretation on each piece of ∂R and

$$\eta = I \otimes (I - P)u_{xx} + (I - P) \otimes Iu_{yy}.$$

Subtracting (2.1) from (1.4), we obtain

$$(2.2)\qquad \left\langle \Gamma - P\frac{\partial u}{\partial \nu}, V \right\rangle = \beta(U - W, V) + (\eta + c(u - W), V), \quad V \in \mathcal{M}_{k,r}.$$

If we set $V = U - W$ in (2.2), we see that

$$(2.3)\qquad \|W - U\|_1 \leqslant C(\|\eta\|_{-1} + \|u - W\|_{-1}),$$

since $U - W = 0$ on ∂R. For $w \in H^{r+1}(I)$ it is well known (see, for instance, Douglas, Dupont, and Wheeler [2]) that, for $0 \leqslant s \leqslant r - 1$ and $1 \leqslant p \leqslant r + 1$,

$$\|(I - P)w\|_{H^{-s}(I)} \leqslant Ch^{p+s} \|w\|_{H^p(I)}.$$

We remark that, if one uses fractional H^s spaces, then in particular

$$\|(I - P)w\|_{H^{-1}(I)} \leqslant Ch^{r+3/2} \|w\|_{H^{r+1/2}(I)}.$$

Thus,

$$(2.4)\qquad \|\eta\|_{-1} \leqslant Ch^{r+3/2} \|u\|_{r+5/2} \quad , \quad r \geqslant 2,$$

and

$$(2.5)\qquad \|\eta\|_{-1} \leqslant Ch^{r+2} \|u\|_{r+3} \quad , \quad r \geqslant 2.$$

We now derive an estimate of $\|u - W\|_{-1}$. Let $\psi \in H^1(R)$ and define $\varphi \in H^3(R)$ by

$$\varphi_{xxyy} = \psi, \qquad x \in R,$$
$$\varphi = 0, \qquad x \in \partial R.$$

Then, $\|\varphi_{xy}\|_2 \leqslant C\|\psi\|_1$. For properly chosen $\chi \in \mathcal{M}^0_{k,r}$,

$$\begin{aligned}(u - W, \psi) &= ((u - W)_{xy}, \varphi_{xy} - \chi_{xy}) \\ &= O(\|u_{xy}\|_r h^r \|\varphi_{xy}\|_2 h^2) \\ &= O(\|u\|_{r+2} \|\psi\|_1 h^{r+2}),\end{aligned}$$

since it is well known that the approximability property

$$(2.6)\quad \inf_{\chi\in\mathcal{M}^0_{k,r}} \|(v-\chi)_{xy}\| \leqslant C\,\|v_{xy}\|_{p-1}h^{p-1},\ v\in H^p(R)\cap H^1_0(R), v_{xy}\in H^{p-1}(R),$$

holds on $\mathcal{M}^0_{k,r}$ for $1 \leqslant p \leqslant r+1$. Hence, for $r \geqslant 2$, (2.3), (2.4), and (2.5) show that

$$(2.7)\qquad \|u-W\|_{-1} \leqslant Ch^{r+2}\,\|u\|_{r+2}.$$

Then,

$$(2.8)\qquad \|W-U\|_1 \leqslant Ch^{r+\alpha}\,\|u\|_{r+\alpha+1}\quad ,\quad \alpha = 3/2 \text{ or } 2.$$

Let $Q\in\mathcal{M}_{k,r}$ be the extension of $\Gamma - P\dfrac{\partial u}{\partial \nu}$ given by Lemma 1.1, and take the test function v in (2.2) to be Q. Then,

$$\left\|\Gamma - P\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\partial R)} \leqslant C[\|U-W\|_1\,\|Q\|_1 + (\|\eta\| + \|u-W\|)\,\|Q\|].$$

By Lemma 1.1 and the fact that $\|u-W\| \leqslant C\,\|u\|_{r+1}h^{r+1}$,

$$(2.9)\qquad \left\|\Gamma - P\frac{\partial u}{\partial \nu}\right\|_{L^2(\partial R)} \leqslant Ch^{-1/2}\|U-W\|_1 + Ch^{r+\beta}\,\|u\|_{r+\beta+3/2},$$

$$\beta = 1 \text{ or } 3/2.$$

Again, use the property of P that $\|Pw-w\|_{L^2(I)} \leqslant C\,\|w\|_{H^{r+1}(I)}h^{r+1}$ to see that (with $\beta = 1$ and $\alpha = 3/2$)

$$(2.10)\qquad \left\|\Gamma - \frac{\partial u}{\partial \nu}\right\|_{L^2(\partial R)} \leqslant C\,\|u\|_{r+5/2}h^{r+1}.$$

Next, recall [1] that

$$\|Pw-w\|_{L^\infty(I)} \leqslant C\,\|w^{(r+1)}\|_{L^\infty(I)}h^{r+1}.$$

Thus, with $\alpha = 2$ and $\beta = 3/2$,

$$(2.11)\qquad \left\|\Gamma - \frac{\partial u}{\partial \nu}\right\|_{L^\infty(\partial R)} \leqslant C\,(\|u\|_{r+3} + \|u\|_{W^\infty_{r+2}})h^{r+1},$$

where

$$\|u\|_{W^\infty_{r+2}} = \sum_{j+k\leqslant r+2}\left\|\frac{\partial^{j+k}u}{\partial x_1^j\partial x_2^k}\right\|_{L^\infty(R)}.$$

We can summarize the above results as below.

Theorem 2.1. Let

$$(2.12)\qquad \begin{aligned} -\Delta u + c(x)u &= f, & x &\in R,\\ u &= 0, & x &\in \partial R, \end{aligned}$$

and let $U \in \mathcal{M}^0_{k,r}$ be the solution of (1.3) corresponding to (2.12). Let $\Gamma \in \partial\, \mathcal{M}_{k,r}$, be defined by the equations

$$\langle \Gamma, v \rangle = (\nabla U, \nabla v) + (cU - f, v), \qquad v \in \mathcal{M}_{k,r}.$$

Then, the error estimates (2.10) and (2.11) hold.

We now consider the case of variable $a(x)$ and restrict the choice of k to $k = 0$. Let us show that

$$(2.13)\qquad |(a\nabla(W - u), \nabla v)| \leqslant C\, \|a\|_{W_1^\infty} \|u\|_{r+2}\, \|v\|_1 h^{r+1},\ v \in \mathcal{M}_{k,r}.$$

It is clear that (using $x = (x, y)$)

$$(a\nabla(W - u), \nabla v) = \sum_{i,j=1}^{N} a(x_{i-1/2}, y_{j-1/2}) \int_{I_i \times I_j} \nabla(W - u) \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y$$

$$+ O(\|a\|_{W_1^\infty} \| W - u\|_1\, \|v\|_1 h).$$

Since Pw interpolates w at the knots when $k = 0$ (and this is the source of our restriction to $k = 0$), $P \otimes Pu = W$ is locally determined and

$$\int_{I_i \times I_j} (u_x - (P \otimes Pu)_x) v_x \,\mathrm{d}x\,\mathrm{d}y = \int_{I_i \times I_j} (u_x - (I \otimes Pu)_x) v_x \,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{I_i \times I_j} v_x I \otimes (I - P) u_x \,\mathrm{d}x\,\mathrm{d}y.$$

A similar reduction can be made on the y-terms. Thus,

$$|(a\nabla(W - u), \nabla v)| \leqslant C\|a\|_{L^\infty(R)} \{ \|I \otimes (I - P)u_x\| + \|(I - P) \otimes I u_y\| \} \|v\|_1$$

$$+ C\, \|a\|_{W_1^\infty} \|u\|_{r+1}\, \|v\|_1 h^{r+1}$$

$$\leqslant C\, \|a\|_{W_1^\infty} \|u\|_{r+2}\, \|v\|_1 h^{r+1}.$$

Since $\beta(W - U, v) = \beta(W - u, v)$ for $v \in \mathcal{M}^0_{k,r}$ and $W - U \in \mathcal{M}^0_{k,r}$, it follows that

$$(2.14)\qquad \| W - U\|_1 \leqslant c\, \|u\|_{r+2} h^{r+1}.$$

Similarly, if one defines $U^* \in \mathcal{M}_{k,r}$ by

$$\beta(u - U^*, V) = 0, \qquad V \in \mathcal{M}_{k,r},$$

we have

$$\|U^* - W\|_1 \leqslant C\,\|u\|_{r+2}h^{r+1}. \tag{2.15}$$

Obviously,

$$\|U^* - U\|_1 \leqslant C\,\|u\|_{r+2}h^{r+1}. \tag{2.16}$$

Now, it is clear that

$$\langle a\frac{\partial u}{\partial \nu}, v\rangle = \beta(U^*, v) - (f, v) \quad , \quad v \in \mathcal{M}_{k,r}.$$

Subtracting (1.4) from the above, we observe that

$$\langle \Gamma - a\frac{\partial u}{\partial \nu}, v\rangle = \beta(U - U^*, v) \quad , \quad v \in \mathcal{M}_{k,r}.$$

Thus,

$$\langle \Gamma - \hat{\Gamma}, v\rangle = \beta(U - U^*, v) \quad , \quad v \in \mathcal{M}_{k,r},$$

where $\hat{\Gamma} \in \partial\mathcal{M}_{k,r}$ and

$$\langle \hat{\Gamma}, v\rangle = \langle a\frac{\partial u}{\partial \nu}, v\rangle \quad , \quad v \in \mathcal{M}_{k,r}.$$

Let $Q \in \mathcal{M}_{k,r}$ be the extension of $\Gamma - \hat{\Gamma}$ given by Lemma 1.1. Then,

$$\begin{aligned}\|\Gamma - \hat{\Gamma}\|_{L^2(\partial R)} &\leqslant C\,\|U - U^*\|_1\,\|Q\|_1 \\ &\leqslant C\,\|u\|_{r+2}h^{r+1/2}.\end{aligned}$$

We have proved the following theorem.

Theorem 2.2. Let U and Γ be defined by (1.3) and (1.4). Then there exists a constant C such that

$$\left\|\Gamma - a\frac{\partial u}{\partial \nu}\right\|_{L^2(\partial R)} \leqslant C\,\|u\|_{r+2}h^{r+1/2}.$$

3. BOUNDARY FLUX ESTIMATES FOR PARABOLIC PROBLEMS

We first consider the case $a \equiv 1$ in (1.3). Let $W : [0, T] \to \mathcal{M}^0_{k,r}$ be given by $W(t) = P \otimes Pw(t)$. Then,

$$(\nabla(w - W), \nabla v) = \langle (I - P)\frac{\partial w}{\partial \nu}(., t), v\rangle - (\eta, v) \quad , \quad v \in \mathcal{M}_{k,r}, \tag{3.1}$$

where

$$\eta = (I \otimes (I - P))w_{xx} + ((I - P) \otimes I)w_{yy}.$$

It is easy to see that, for $v \in \mathcal{M}_{k,r}$,

$$\left(\rho \frac{\partial W}{\partial t}, v\right) + \beta(W, v) - (f, v)$$

$$(3.2) \qquad = \left\langle P \frac{\partial w}{\partial \nu}, v \right\rangle + \left(\eta + c(W - w) + \rho \frac{\partial(W - w)}{\partial t}, v\right).$$

Subtracting (1.6) from (3.2), we obtain the equation

$$(3.3) \quad \left\langle \Gamma - P \frac{\partial u}{\partial \nu}, v \right\rangle = \left(\rho \frac{\partial \xi}{\partial t}, v\right) + \beta(\xi, v) - \left(\eta + c(W - w) + \rho \frac{\partial(W - w)}{\partial t}, v\right), \quad v \in \mathcal{M}_{r,k},$$

where $\xi = Z - W$. As in Lemma 1.1, let $Q \in \mathcal{M}_{k,r}$ be such that $Q = \Gamma - P \frac{\partial u}{\partial \nu}$ for $x \in \partial R$. Setting $V = Q$ in (3.3), we have

$$(3.4) \quad \left\| \Gamma - P \frac{\partial u}{\partial \nu} \right\| \leqslant C \left[\left\| \frac{\partial \xi}{\partial t} \right\| h^{1/2} + \|\xi\|_1 h^{-1/2} + h^{1/2} \left(\|\eta\| + \|W - w\| + \left\| \frac{\partial(W - w)}{\partial t} \right\| \right) \right].$$

Estimates on η and $W - w$ follow as in Section 2 with w replacing u. Note that

$$\left\| \frac{\partial(W - w)}{\partial t} \right\| \leqslant Ch^{r+1} \left\| \frac{\partial u}{\partial t} \right\|_{r+1}$$

We now obtain estimates of $\left\| \frac{\partial \xi}{\partial t} \right\|$ and $\|\xi\|_1$. For $V \in \mathcal{M}^0_{k\,r}$, we have

$$(3.5) \quad \left(\rho \frac{\partial \xi}{\partial t}, V\right) + \beta(\xi, V) = \left(\eta + c(W - w) + \rho \frac{\partial(W - w)}{\partial t}, V\right).$$

Differentiate (3.5) with respect to t :

$$(3.6) \quad \left(\rho \frac{\partial^2 \xi}{\partial t^2}, V\right) + \beta\left(\frac{\partial \xi}{\partial t}, V\right) = \left(\frac{\partial \eta}{\partial t} + c \frac{\partial(W - w)}{\partial t} + \rho \frac{\partial^2(W - w)}{\partial t^2}, V\right).$$

Setting $V = \dfrac{\partial \xi}{\partial t}$ in (3.6) we obtain the inequality

$$(3.7) \qquad \frac{1}{2}\frac{d}{dt}\left\|\rho^{1/2}\frac{\partial \xi}{\partial t}\right\|^2 + \left\|\frac{\partial \xi}{\partial t}\right\|_1^2 \leqslant C\left[\left\|\frac{\partial \eta}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial (W-w)}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial^2 (W-w)}{\partial t^2}\right\|_{-1}^2\right] \leqslant Ch^{2r+1}\left[\left\|\frac{\partial w}{\partial t}\right\|_{r+3/2}^2 + \left\|\frac{\partial^2 w}{\partial t^2}\right\|_{r-1/2}^2\right].$$

One can also show that

$$(3.8) \qquad \frac{1}{2}\frac{d}{dt}\left\|\rho^{1/2}\frac{\partial \xi}{\partial t}\right\|^2 \leqslant Ch^{2r+2}\left[\left\|\frac{\partial w}{\partial t}\right\|_{r+2}^2 + \left\|\frac{\partial^2 w}{\partial t^2}\right\|_{r}^2\right].$$

Integrate (3.7) with respect to t :

$$(3.9) \qquad \left\|\frac{\partial \xi}{\partial t}\right\|_{L_\infty(L^2)} \leqslant C\left\{\left\|\frac{\partial \xi}{\partial t}(0)\right\| + h^{r+1/2}\left(\left\|\frac{\partial w}{\partial t}\right\|_{L^2(H^{r+3/2})} + \left\|\frac{\partial^2 w}{\partial t^2}\right\|_{L^2(H^{r-1/2})}\right)\right\}$$
$$\leqslant C\left\{\left\|\frac{\partial w}{\partial t}(0)\right\|_{r+1/2} + \left\|\frac{\partial w}{\partial t}\right\|_{L^2(H^{r+3/2})} + \left\|\frac{\partial^2 w}{\partial t^2}\right\|_{L^2(H^{r-1/2})}\right\} h^{r+1/2}.$$

Similarly,

$$(3.10) \qquad \left\|\frac{\partial \xi}{\partial t}\right\|_{L^\infty(L^2)} \leqslant C\left\{\left\|\frac{\partial w}{\partial t}(0)\right\|_{r+1} + \left\|\frac{\partial w}{\partial t}\right\|_{L^2(H^{r+2})} + \left\|\frac{\partial^2 w}{\partial t^2}\right\|_{L^2(H^r)}\right\} h^{r+1}.$$

The use of $v = \dfrac{\partial \xi}{\partial t}$ in (3.3) and the inequality (2.8) lead to the estimates

$$(3.11) \qquad \|\xi\|_{L^\infty(H^1)} \leqslant C\left\{\|w_0\|_{r+\alpha+1} + \|w\|_{L^2(H^{r+\alpha+1})} + \left\|\frac{\partial w}{\partial t}\right\|_{L^2(H^{r+\alpha-1})}\right\} h^{r+\alpha}$$

for $\alpha = 3/2$ or 2. It then follows from (3.4), (3.9), and (3.11) with $\alpha = 3/2$

that

$$\left\| \Gamma - \frac{\partial u}{\partial \nu} \right\|_{L^\infty(L^2(\partial R))} \leqslant C \left\{ \|w_0\|_{r+5/2} + \left\| \frac{\partial w}{\partial t}(0) \right\|_{r+1/2} + \|w\|_{L^2(H^{r+5/2})} + \left\| \frac{\partial w}{\partial t} \right\|_{L^2(H^{r+3/2})} + \left\| \frac{\partial^2 w}{\partial t^2} \right\|_{L^2(H^{r-1/2})} \right\} h^{r+1} \tag{3.12}$$

and, with $\alpha = 2$ in (3.11) and (3.10) in place of (3.9),

$$\left\| \Gamma - \frac{\partial u}{\partial \nu} \right\|_{L^\infty(\partial R \times (0,T))} \leqslant C \left\{ \|w_0\|_{r+3} + \left\| \frac{\partial w}{\partial t}(0) \right\|_{r+1} + \|w\|_{L^2(H^{r+3})} + \left\| \frac{\partial w}{\partial t} \right\|_{L^2(H^{r+2})} + \left\| \frac{\partial^2 w}{\partial t^2} \right\|_{L^2(H^r)} + \|w\|_{L^\infty(W^\infty_{r+2})} \right\} h^{r+1}. \tag{3.13}$$

Theorem 3.1. Let w be the solution of

$$\rho \frac{\partial w}{\partial t} - \Delta w + cw = f, \quad x \in R, \quad t \in [0, T],$$

$$w = 0, \quad x \in \partial R, \quad t \in [0, T],$$

$$w(x, 0) = w_0(x), \quad x \in R.$$

If Z is the solution of (1.5) and $\Gamma(t)$ is defined by (1.6), then the error bounds (3.12) and (3.13) hold for the flux.

In the case $a \not\equiv 1$ we derive an $O(h^{r+1/2})$ estimate as in Theorem 2.2. Let $Z^*\colon [0, T] \to \mathcal{M}_{0,r}$ defined by

$$\left(\rho \left(\frac{\partial Z^*}{\partial t} - \frac{\partial w}{\partial t} \right), v \right) + \beta(Z^* - w, v) = 0, \quad t > 0, \quad v \in \mathcal{M}_{0,r}, \tag{3.14}$$

$$\beta(Z^* - w_0, v) = 0, \qquad v \in \mathcal{M}_{0,r}.$$

One can easily deduce from (3.14), (1.5), and (1.6) that

$$\left\langle \Gamma - a \frac{\partial u}{\partial \nu}, v \right\rangle = \beta(Z - Z^*, v) + \left(\rho \frac{\partial (Z - Z^*)}{\partial t}, v \right), \quad v \in \mathcal{M}_{0,r}. \tag{3.15}$$

Let $\hat{\Gamma}$ denote the L^2 projection of $a \frac{\partial u}{\partial \nu}$ into $\partial \mathcal{M}_{0,r}$ and let $Q \in \mathcal{M}_{0,r}$ be the extension of $\Gamma - \hat{\Gamma}$ provided by Lemma 1.1. Then,

$$(3.16)\quad \|\Gamma - \hat{\Gamma}\|_{L^\infty(L^2(\partial R))} \leqslant C\left[\|Z - Z^*\|_1 h^{-1/2} + h^{1/2}\left\|\frac{\partial(Z - Z^*)}{\partial t}\right\|\right]$$
$$\leqslant C\left[(\|Z - W\|_1 + \|W - Z^*\|_1)h^{-1/2} + \left(\left\|\frac{\partial(Z - W)}{\partial t}\right\| + \left\|\frac{\partial(W - Z^*)}{\partial t}\right\| h^{1/2}\right)\right].$$

where $W(\cdot, t) = P \otimes Pw(\cdot, t)$.

We observe that for $v \in \mathcal{M}_{0,r}$

$$(3.17)\quad \left(\rho\frac{\partial\theta}{\partial t}, v\right) + \beta(\theta, v) = \left(\rho\frac{\partial}{\partial t}(w - W), v\right) + \beta(w - W, v),$$

where $\theta = Z^* - W$. One can easily verify that

$$(3.18)\quad \left\|\frac{\partial\theta}{\partial t}\right\|_{L^\infty(L^2)} \leqslant Ch^{r+1/2}\left[\left\|\frac{\partial^2 w}{\partial t^2}\right\|_{L^2(H^{r-1/2})} + \left\|\frac{\partial w}{\partial t}\right\|_{L^2(H^{r+1/2})} + \left\|\frac{\partial w}{\partial t}(0)\right\|_{r+1/2}\right]$$

and

$$(3.19)\quad \|\theta\|_{L^\infty(H^1)} \leqslant Ch^{r+1}\left[\left\|\frac{\partial w}{\partial t}\right\|_{L^2(H^r)} + \|w\|_{L^2(H^{r+2})} + \|w_0\|_{r+2}\right].$$

Similar estimates hold for $Z - W$. Thus, from (3.16)-(3.19),

$$\|\Gamma - \hat{\Gamma}\|_{L^\infty(L^2(\partial R))} \leqslant Ch^{r+1/2}\left[\|w\|_{L^2(H^{r+2})} + \left\|\frac{\partial w}{\partial t}\right\|_{L^2(H^{r+1/2})} + \left\|\frac{\partial^2 w}{\partial t^2}\right\|_{L^2(H^{r-1/2})} + \|w_0\|_{r+2} + \left\|\frac{\partial w}{\partial t}(0)\right\|_{r+1/2}\right].$$

Theorem 3.2. Let Z and Γ be defined by (1.5) and (1.6). Then

$$\left\|\Gamma - a\frac{\partial u}{\partial \nu}\right\|_{L^\infty(L^2(\partial R))} \leqslant Ch^{r+1/2}\left[\|w\|_{L^\infty(H^{r+2})} + \left\|\frac{\partial w}{\partial t}\right\|_{L^2(H^{r+1/2})} + \left\|\frac{\partial^2 w}{\partial t^2}\right\|_{L^2(H^{r-1/2})} + \|w_0\|_{r+2} + \left\|\frac{\partial w}{\partial t}(0)\right\|_{r+1/2}\right]$$

The discretization in time of (1.5) can be carried out in many ways. A very detailed treatment of a collocation-in-time method has been given in [2] for the single space variable analogue of (1.5). It is clear that the same approach is applicable in the present case, and analogous results can be obtained. This will be left to the reader.

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