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L^∞-CONVERGENCE OF FINITE ELEMENT APPROXIMATION TO QUASILINEAR INITIAL BOUNDARY VALUE PROBLEMS (*)

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Communiqué par P -A RAVIART

Abstract — *Almost optimal L^∞-convergence of finite element approximation to nonlinear parabolic differential equations is proved by a weighted norm technique known from the elliptic case*

1. INTRODUCTION AND STATEMENT OF THE THEOREM

In this paper, we study the finite element approximation of nonlinear parabolic problems. Our aim is to obtain optimal uniform convergence for the discrete solution. As a model problem we consider the parabolic initial boundary value problem:

$$(\mathrm{P})\qquad \begin{cases} u_t-\sum_{i=1}^{n} \partial_i F_i(\nabla u)=f(x,t) & \text{in } \Omega\times[0,T], \\ u(x,0)=\psi(x) \quad \text{in } \Omega, & u(x,t)=0 \quad \text{on } \partial\Omega\times[0,T]. \end{cases}$$

In the above, Ω is a bounded domain in Euclidian $\mathbf{R}^n$ with sufficiently smooth boundary $\partial\Omega$.

Error estimates for quasilinear parabolic equations have been proved in various papers. J. Douglas Jr. and T. Dupont [6] have established optimal H^1-error estimates and V. Thomee and L. Wahlbin [15] have shown that the nonlinearity must not fulfil a global Lipschitz condition. Optimal L^2-error estimates, i. e. $\|e_h\|_{L^2}=O(h^m)$, have been obtained by Wheeler [16]. However the nonlinearity treated by Wheeler involves only u and not ∇u. Optimal pointwise estimates have been established by Bramble, Schatz, Thomée and Wahlbin [2] in the linear case ($u_t=A\,u$) and by Dobrowolski [5] in the quasilinear case treated by Wheeler for $n=2$, in the general linear case for $n\geqq 2$.

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$H^{j,p}(\Omega)$, $j \in \mathbf{N}$, $1 \leqq p < \infty$, denotes the completion of $C^\infty(\Omega)$ with respect to the norm:

$$\|w\|_{j,p} = \sum_{i=0}^{j} \left(\int_\Omega |\nabla^i w|^p \, dx \right)^{1/p}.$$

If this norm is taken in a domain $T \subset \mathbf{R}^n$ we shall write $\| \,.\, \|_{j,p,T}$. In the case $j=0$ and $p=2$ we will write $\| \,.\, \|_T$. Furthermore we use $H_0^1(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, and $H^{j,\infty}(\Omega)$, the space of functions with bounded generalized derivatives. Let X be a normed linear space consisting of a set of functions defined on Ω. If w is a function defined on $[0, T] \times \Omega$ we say $w \in L^p(X)$, $1 \leqq p < \infty$, if the norm:

$$\|w\|_{L^p(X)} = \left(\int_0^T |F(t)|^p \, dt \right)^{1/p},$$

is finite where:

$$F(t) = \|w(t)\|_X,$$

with the usual modification for $p = \infty$.

For abbreviation, we write $\| \,.\, \|_\infty$ instead of $\| \,.\, \|_{L^\infty(L^\infty)}$ and use a summation convention.

With the above notation we state our assumptions concerning problem (P):

(A1) $F_i \in C^3(\mathbf{R}^n)$, $i = 1, \ldots, n$, $f \in C^2(\Omega \times [0, T]$, $\psi \in H^{m,\infty}(\Omega)$.

(A2) Ellipticity (not necessarily uniform): for p, $\xi \in \mathbf{R}^n$, $\xi \neq 0$, and $|p| \leqq K$ there is a constant $C(K)$ with $F_{i,k}(p)\xi_i\xi_k \geqq C(K)|\xi|^2$ where (F_{ik}) denotes the matrix of the first derivatives of F_i.

(A3) $$\sum_{i=1}^{n} F_i(0) = 0.$$

(A4) There is a unique solution u of the problem (P) and $u(.,t)$, $u_t(.,t) \in H^{m,\infty}(\Omega)$, $t \in [0, T]$, $u_{tt} \in L^2(H^{m,\infty}(\Omega))$.

(A5) If the data of problem (P) are replaced by $rf(x, t)$ and $r\psi(x)$, $r \in [0, 1]$, the solution u^r sufficies assumption (A4) and is bounded in the norms of (A4) uniformly in r. In particular, we have $\|\nabla u^r\|_\infty$, $\|\nabla u_t^r\|_\infty \leqq \gamma$. Moreover u^r continuously depends on r.

REMARK: The m in the assumptions refers to the order of the spline space, which will be described below. (A3) is superfluous, but it allows us to give a more elegant presentation of the proof.

We shall assume that the spline space S_h, $0 < h \leqq h_0$, of order m satisfies the following conditions:

(S1) S_h is a finite dimensional subspace of $H_0^1(\Omega)$.

(S2) There is a partition τ_h of Ω into piecewise smooth subdomains such that the usual regularity condition is fulfilled: each $T \in \tau_h$ is contained in a ball of radius h and contains a ball of radius mh where the constant $m>0$ does not depend on h.

For all subdomains $T\in\tau_h$ and for all $z_h\in S_h$ we have $z_h\in H^m(T)$.

(S3) The following approximation and inverse properties hold:

(i) to each $z\in H^{m,p}\cap H_0^1$, there is a function $z_h\in S_h$ such that on each $T\in\tau_h$:

$$\|z-z_h\|_{j,p,T}\leqq ch^{m-j}\|z\|_{m,p,T'},\qquad 0\leqq j\leqq m,\quad 1\leqq p\leqq\infty,$$

where $T':=T\cup\{\text{neighbours of } T\}$;

(ii) for $z_h\in S_h$ and $T\in\tau_h$ we have:

$$\|z_h\|_{k,p,T}\leqq ch^{-k}\|z_h\|_{p,T},\qquad 0<k<m,\quad 1\leqq p\leqq\infty,$$

$$\|z_h\|_{j,\infty,T}\leqq ch^{-n/2}\|z_h\|_{j,2,T},\qquad 0\leqq j<m.$$

With these definitions the Galerkin approximation U of problem (P) is defined by:

$$(\mathrm{P}^h)\qquad (U_t, v_h)+\sum_{i=1}^{n}(F_i(\nabla U), v_{hi})=(f, v_h),\qquad v_h,\ U(.,t)\in S_h,\quad t\in[0,T],$$

$$(F_{ij}(\nabla\psi)(\psi-U(0))_j, v_{hi})=0,\qquad v_h\in S_h.$$

Theorem: *Let $m\geqq 3$. Under the conditions* (A1)-(A5) *and* (S1)-(S3) *there is a h_0 such that for all $0<h\leqq h_0$, the system* (P^h) *has a unique solution U in $\Omega\times[0,T]$. Furthermore, the following inequality holds:*

$$\|u-U\|_\infty\leqq ch^m|\ln h|^{(n/2)+2}.$$

Remark: Under similar assumptions the theorem is valid for the more general quasilinear initial boundary value problem:

$$u_t-\sum_{i=1}^{n}\partial_i F_i(x,t,u,\nabla u)+F(x,t,u,\nabla u)=f(x,t).$$

But all essential difficulties arising from the nonlinearity will be preserved in problem (P). In the case:

$$u_t-\sum_{i=1}^{n}\partial_i(F_i(x,t,u)\,u_i)+F(x,t,u)=f(x,t),$$

the method of our proof works for $m=2$ as well. For the general quasilinear equation, however, a much more complicated technique is used (*see* Frehse and Rannacher [8] for the elliptic case).

2. PROOF OF THE THEOREM

The proof of the theorem is obtained by the method of Dobrowolski [5] combined with a deformation technique first described by Frehse [7]. The method consists in the following.

For each $r \in [0, 1]$, consider the deformed problem:

$$(\mathrm{P}^r) \qquad \begin{cases} u_t^r - \sum_{i=1}^{n} \partial_i F_i(\nabla u^r) = rf & \text{in } \Omega \times [0, T], \\ u^r(x, 0) = r\psi \quad \text{in } \Omega, & u^r = 0 \quad \text{on } \partial\Omega \times [0, T] \end{cases}$$

and the corresponding Galerkin-approximation:

$$(U_t^r, v_h) + \sum_{i=1}^{n} (F_i(\nabla U^r), v_{hi}) = (rf, v_h), \qquad v_h, U^r(., t) \in S_h,$$

$$(F_{ij}(\nabla(r\psi))(r\psi - U(0))_j, v_{hi}) = 0, \qquad v_h \in S_h.$$

Because of the local Lipschitz continuity of $F(.)$, it is clear that there is a unique solution U^r in $[0, T_h]$. We intend to show that the solution exists in the whole interval $[0, T]$. In the case treated here this follows from the theory of ordinary differential equations anyhow, since the Dirichlet-form is positive definite. In the general case, i. e. $F_i = F_i(x, u, \nabla u)$, however this argument does not apply, but the existence of a global solution of the discrete problem follows as a by product of the deformation technique. Thus, illustrating how the proof proceeds in the general case, we have chosen this more complicated method to prove global solvability of the discrete problems. For each h, we define the set $\Xi_h \subset [0, 1]$ by:

$$\Xi_h = \{ r \in [0, 1] : (P_h^r) \text{ has a solution } U^r \text{ defined on } [0, T]$$
$$\text{and there holds } \|u^r - U^r\|_{\mathcal{T}} < 2 c_1 h^m |\ln h|^{(n/2)+2}$$
$$\text{and } \|\nabla U^r\|_{\mathcal{T}}, \|\nabla U_t^r\|_{\mathcal{T}} < 2\gamma \},$$

where c_1 is the constant appearing in lemma 1 and γ refers to (A5). Here and in the following every constant c or c_1 does not depend on h. We show that for $h \leqq h_0(c_1)$ the set Ξ_h is not empty, closed and open with respect to $[0, 1]$ and therefore must coincide with $[0, 1]$, which proves the theorem since $u = u^1$ and $U = U^1$.

(i) Ξ_h is not empty. For $r = 0$ we have $u^r = 0$ and $U^r = 0$ because of (A3);

(ii) Ξ_h is open in $[0, 1]$. Let $r \in \Xi_h$. From (A5) and the theory of ordinary differential equations using a truncation argument we obtain the strict inequalities:

$$\|u^{\rho}-U^{\rho}\|_{\infty}<2c_1 h^m|\ln h|^{(n/2)+2};$$
$$\|\nabla U^r\|_{\infty},\quad \|\nabla U_t^r\|_{\infty}<2\gamma$$

if ρ is in a neighbourhood of r;

(iii) Ξ_h is closed. Let $r(j)\in\Xi_h$ and $r(j)\to r_0$. Immediately we obtain:

$$\|u^{r_0}-U^{r_0}\|_{\infty}\leqq 2c_1 h^m|\ln h|^{(n/2)+2}$$

and

$$\|\nabla U^{r_0}\|_{\infty},\qquad \|\nabla U_t^{r_0}\|_{\infty}\leqq 2\gamma.$$

Now we have to prove the strict inequalities for r_0. This is done by the following:

LEMMA 1: *Suppose that* $\|\nabla U^r\|_{\infty}$, $\|\nabla U_t^r\|_{\infty}\leqq 2\gamma$. *Then, under the hypotheses of the theorem, there holds* ($e=u-U$):

$$h^4|\ln h|^{c(n)}\int_0^T|e_{tt}^r(x,t)|^2\,dt$$
$$+h^2|\ln h|^{c(n)}\int_0^T|\nabla e_t^r(x,t)|^2\,dt+\|e^r\|_{\infty}^2$$
$$\leqq c|\ln h|^{c(n)}\left\{h^{-5}\|e^r\|_{\infty}^4+\|\nabla e^r\|_{\infty}^2\int_0^T|\nabla e_t^r(x_1,t)|^2\,dt\right\}$$
$$+c_1 h^{2m}|\ln h|^{n+4}|,$$

where $x\in\Omega$ *is arbitrary and* $x_1=x_1(h,r)$ *is a fixed point in* Ω.

This lemma will be proved in the third section. From (S3) and

$$\|u^r-U^r\|_{\infty}\leqq 2c_1 h^m|\ln h|^{(n/2)+2}$$

we obtain for $u_h^r\in S_h$:

$$\begin{aligned}\|\nabla e^r\|_{\infty}&\leqq\|\nabla(u^r-u_h^r)\|_{\infty}+\|\nabla(U^r-u_h^r)\|_{\infty}\\&\leqq \mathrm{ch}^{m-1}+\mathrm{ch}^{-1}\|U^r-u_h^r\|_{\infty}\\&\leqq \mathrm{ch}^{m-1}+\mathrm{ch}^{-1}\|u^r-u_h^r\|_{\infty}+\mathrm{ch}^{-1}\|e^r\|_{\infty}\\&\leqq \mathrm{ch}^{m-1}+cc_1 h^{m-1}|\ln h|^{(n/2)+2}\end{aligned}$$

and therefore:

$$\|\nabla e^r\|_{\infty}^2\int_0^T|\nabla e_t^r(x_1,t)|^2\,dt\leqq cc_1^2 h^{2m-2}|\ln h|^{n+4}\int_0^T|\nabla e_t^r(x_1,t)|^2\,dt.$$

Applying this to the inequality in lemma 1 and choosing $h \leqq h_0(c_1)$ we obtain:

$$h^4 |\ln h|^{c(n)} \int_0^T |e_{tt}^r(x, t)|^2 \, dt + h^2 |\ln h|^{c(n)} \int_0^T |\nabla e_t^r(x, t)|^2 \, dt + \|e^r\|_\infty^2 \leqq \mathrm{c}h^{-5} |\ln h|^{c(n)} \|e^r\|_\infty^4 + c_1 h^{2m} |\ln h|^{n+4}. \tag{2.1}$$

In view of $\|e^r\|_\infty \leqq 2 c_1 h^m |\ln h|^{(n/2)+2}$ and $m \geqq 3$ this yields for $h \leqq h_0(c_1)$:

$$\|e^r\|_\infty < 2 c_1 h^m |\ln h|^{(n/2)+2}$$

and by the inverse relation:

$$\|\nabla U^r\|_\infty \leqq \|\nabla e^r\|_\infty + \|\nabla u^r\|_\infty < 2\gamma, \qquad h \leqq h_0(c_1).$$

It remains to prove $\|\nabla U_t^r\|_\infty < 2\gamma$. For this purpose let:

$$|\nabla e_t^r(x_0, t_0)| = \|\nabla e_t^r\|_\infty .$$

From the inequality:

$$|f(t)| \leqq \varepsilon \|f_t\|_{L^2([0, T])} + c\varepsilon^{-1} \|f\|_{L^2([0, T])},$$

$f \in H^1([0, T])$, which can be proved by integrating the relation $(d/dt) f^2 = 2 f f_t$, we obtain for any $\varepsilon > 0$:

$$\|\nabla e_t^r\|_\infty^2 \leqq \varepsilon \int_0^T |\nabla e_{tt}^r(x_0, t)|^2 \, dt + c\varepsilon^{-1} \int_0^T |\nabla e_t^r(x_0, t)|^2 \, dt. \tag{2.2}$$

Now (S3) yields:

$$\int_0^T |\nabla e_{tt}^r(x_0, t)|^2 \, dt \leqq \mathrm{c}h^{-2} \int_0^T |e_{tt}^r(x_2, t)|^2 \, dt + \mathrm{c}h^{2m-2},$$

where x_2 is a fixed point in Ω.

Choosing $\varepsilon = h^2$ in (2.2) we have:

$$\|\nabla e_t^r\|_\infty^2 \leqq \int_0^T |e_{tt}^r(x_2, t)|^2 \, dt + \mathrm{c}h^{-2} \int_0^T |\nabla e_t^r(x_0, t)|^2 \, dt + \mathrm{c}h^{2m-2}.$$

and with (2.1):

$$\|\nabla e_t^r\|_\infty^2 \leqq c_1 \mathrm{c}h^{2m-4} |\ln h|^{c(n)}.$$

Therefore we have for $h \leqq h_0(c_1)$:

$$\|\nabla U_t^r\|_\infty \leqq \|\nabla e_t^r\|_\infty + \|\nabla u_t^r\|_\infty < 2\gamma.$$

3. PROOF OF LEMMA 1

In this section we assume $\|\nabla U\|_{\infty}, \|\nabla U_t\|_{\infty} \leqq 2\gamma$ and suppress the parameter r. We begin by introducing some additional notation and technical tools. We use the weight functions:

$$\sigma(.)=(|\,.-x|^2+\rho^2)^{1/2}, \qquad \rho \geqq ch$$

and the corresponding weighted norms:

$$\|\,.\,\|_{(s)}=(\sum_{T\in\tau^h}\|\sigma^{s/2}\,.\,\|_T^2)^{1/2}, \qquad s\in\mathbf{R}.$$

Obviously:

$$\sigma^{-1}\leqq\rho^{-1}\leqq h^{-1}, \qquad |\nabla\sigma|\leqq 1, \quad \sigma\leqq c$$

and for $\rho\geqq ch$, c sufficiently large, it follows from (S1)-(S3) that:

$$\inf_{v_h\in S_h}\|\nabla^i(v-v_h)\|_{(s)}\leqq ch^{m-i}\sum_{j=1}^{m}\|\nabla^j v\|_{(s)}, \qquad 0\leqq i\leqq m-1, \tag{3.1}$$

$$\|\nabla^i v_h\|_{(s)}\leqq ch^{-i}\|v_h\|_{(s)}, \qquad 0\leqq i\leqq m-1 \tag{3.2}$$

(*see* Nitsche [13]).

We use the notation $\|\,.\,\|_{(s)\,i}$, $i\in\mathbf{N}$, to indicate that the norm is taken with a weight:

$$\sigma_i(.):=(|\,.-x|^2+\rho^2)^{1/2}+(|\,.-x_i|^2+\rho^2)^{1/2},$$

with:

$$\sigma_i^s(.):=(|\,.-x|^2+\rho^2)^{s/2}+(|\,.-x_i|^2+\rho^2)^{s/2}, \qquad s\in\mathbf{R}$$

where x_i is a fixed point in Ω.

For abbreviation, we define the functions:

$$a_{ij}^h(.,.)=\int_0^1 F_{ij}(\nabla U+(1-s)\nabla(u-U))\,ds, \qquad i,j=1,\ldots,n,$$

$$a_{ij}(.,.)=F_{ij}(\nabla u)\in C^1(\Omega\times[0,T]), \qquad i,j=1,\ldots,n$$

and corresponding bilinear forms:

$$a^h(v,w)=(a_{ij}^h v_j, w_i), \qquad a(v,w)=(a_{ij}v_j, w_i)$$

and the differential operator:

$$(Av,w)=a(v,w).$$

With these definitions we have:

$$(e_t, v_h)+a^h(e,v_h)=0, \qquad v_h\in S_h, \tag{3.3}$$

$$(e_{tt}, v_h)+a^h(e_t, v_h)+\int_\Omega a^h_{ijt}\, e_j v_{hi}\, dx=0, \qquad v_h\in S_h, \tag{3.4}$$

$$|a_{ij}-a^h_{ij}|\leqq c|\nabla e|, \tag{3.5}$$

$$|a_{ijt}-a^h_{ijt}|\leqq c\{|\nabla e|+|\nabla e_t|\}, \tag{3.6}$$

$$|a^h_{ijt}|,\ |a_{ijt}|\leqq c. \tag{3.7}$$

To fix the initial value $U(.,0)$, we have defined in the first section:

$$a(e, v_h)(0)=0, \qquad v_h\in S_l,$$

which implies that $U(x, 0)$ is an "elliptic" projection of $u(x, 0)$. From Nitsche [13] we obtain the asymptotic error estimate:

$$\|u(.,0)-U(.,0)\|_{L^\infty}\leqq \mathrm{ch}^m, \qquad m\geqq 3. \tag{3.8}$$

By the above choice of the initial value $U(.,0)$ we can show in the appendix, theorem A1:

$$\|u_t(.,0)-U_t(.,0)\|_{L^\infty}\leqq \mathrm{ch}^m, \qquad m\geqq 3. \tag{3.9}$$

Now lemma 1 will be proved by a series of further lemmas.

LEMMA 3.1:

(i) $\int_0^T \|\nabla e\|^2_{(-n)}\,dt \leqq c\,\varepsilon^{-1}\int_0^T \|e\|^2_{(-n-2)}\,dt+\varepsilon h^2\int_0^T \|e_t\|^2_{(-n)}\,dt+\mathrm{ch}^{2m-2}|\ln h|$;

(ii)
$$\int_0^T \|\nabla e_t\|^2_{(-n)}\,dt$$
$$\leqq c\,\varepsilon^{-1}\int_0^T \|e_t\|^2_{(-n-2)}\,dt+\varepsilon h^2\int_0^T \|e_{tt}\|^2_{(-n)}\,dt$$
$$+c\int_0^T \|\nabla e\|^2_{(-n)}\,dt+c\int_0^T \|e\|^2_{(-n-2)}\,dt+\mathrm{ch}^{2m-2}|\ln h|.$$

Proof: (i) From (3.3) and the ellipticity of the form $a^h(.,.)$, it follows that:

$$\frac{1}{2}\frac{d}{dt}\|e\|^2_{(-n)}+c\|\nabla e\|^2_{(-n)}$$
$$\leqq (e_t, \sigma^{-n}e)+a^h(e, \sigma^{-n}e)+c\int_\Omega |\nabla e||e|\sigma^{-n-1}\,dx$$
$$=(e_t, \sigma^{-n}e-\varphi_h)+a^h(e, \sigma^{-n}e-\varphi_h)+c\int_\Omega |\nabla e||e|\sigma^{-n-1}\,dx$$
$$\leqq \|e_t\|_{(-n)}\|\sigma^{-n}e_t-\varphi_h\|_{(n)}+c\|\nabla e\|_{(-n)}\|\nabla(\sigma^{-n}e-\varphi_h)\|_{(n)}$$
$$+c\|\nabla e\|_{(-n)}\|e\|_{(-n-2)}. \tag{3.10}$$

Now let φ_h be the approximation of $\sigma^{-n} e$ in the sense of (S3).

Then we have:

$$\| \sigma^{-n} e - \varphi_h \|^2_{(n)} \leq ch^{2m} \sum_{j=0}^{m} \| \nabla^j (\sigma^{-n} e) \|^2_{(n)} .$$

From (3.1) and (3.2) we obtain for the term of order m:

$$\| \nabla^m (\sigma^{-n} e) \|^2_{(n)} \leq c \sum_{j=0}^{m} \| \nabla^j e \, \sigma^{-n-m+j} \|^2_{(n)}$$
$$\leq c \sum_{j=0}^{m-1} h^{-2j} \| e \, \sigma^{-n-m+j} \|^2_{(n)} + c \, |\ln h| .$$

Estimating $\| \nabla (\sigma^{-n} e - \varphi_h) \|_{(n)}$ analogously, we obtain:

$$h^{-1} \| \sigma^{-n} e - \varphi_h \|_{(n)} + \| \nabla (\sigma^{-n} e - \varphi_h) \|_{(n)}$$
$$\leq c \| e \|_{(-n-2)} + ch^{m-1} |\ln h|^{1/2} \tag{3.11}$$

By replacing (3.11) in (3.10), integrating from 0 to T, with accounting for $\| e \|_{(-n)}(0) \leq ch^m |\ln h|^{1/2}$, we obtain (i).

(ii) With the aid of (3.4) we get:

$$\frac{1}{2} \frac{d}{dt} \| e_t \|^2_{(-n)} + c \| \nabla e_t \|^2_{(-n)}$$
$$\leq (e_{tt}, \sigma^{-n} e_t) + a^h (e_t, \sigma^{-n} e_t) + c \int_{\Omega} |e_t| |\nabla e_t| \sigma^{-n-1} \, dx$$
$$= (e_{tt}, \sigma^{-n} e_t - \varphi_h) + a^h (e_t, \sigma^{-n} e_t - \varphi_h) + \int_{\Omega} a^h_{ijt} (\sigma^{-n} e_t - \varphi_h)_j \, e_i \, dx$$
$$- \int_{\Omega} a^h_{ijt} (\sigma^{-n} e_t)_j \, e_i \, dx + c \int_{\Omega} |\nabla e_t| \, . \, |e_t| \sigma^{-n-1} \, dx.$$

We omit the rest of the proof, because from now on we can estimate similarly to (i).

LEMMA 3.2:

(i) $$\int_0^T \| e_t \|^2_{(-n)} \, dt \leq ch^{-2} \int_0^T \| \nabla e \|^2_{(-n)} \, dt + ch^{2m-4} |\ln h| ;$$

(ii) $$\int_0^T \| e_{tt} \|^2_{(-n)} \, dt \leq ch^{-2} \int_0^T \{ \| \nabla e_t \|^2_{(-n)} + \| \nabla e \|^2_{(-n)} \} \, dt + ch^{2m-4} |\ln h| .$$

Proof: (i) From (3.3) we obtain:

$$\|e_t\|^2_{(-n)} = (e_t, \sigma^{-n} e_t) = (e_t, \sigma^{-n} e_t - \varphi_h) + a^h(e, \sigma^{-n} e_t - \varphi_h) - a^h(e, \sigma^{-n} e_t)$$
$$\leqq \|e_t\|_{(-n)} \|\sigma^{-n} e_t - \varphi_h\|_{(n)} + c\|\nabla e\|_{(-n)} \|\nabla(\sigma^{-n} e_t - \varphi_h)\|_{(n)} + \|\nabla e\|_{(-n)} \|\nabla e_t\|_{(-n)}.$$

By a technique used in the proof of lemma 3.1 (i) we can show:

$$h^{-1}\|\sigma^{-n} e_t - \varphi_h\|_{(n)} + \|\nabla(\sigma^{-n} e_t - \varphi_h)\|_{(n)} \leqq c\|e_t\|_{(-n-2)} + ch^{m-1}|\ln h|^{1/2}.$$

Then we have with the inequality:

$$ab \leqq \frac{1}{2}\varepsilon a^2 + \frac{1}{2}\varepsilon^{-1} b^2 \qquad \text{and} \qquad \rho \geqq ch,\ c \text{ large:}$$

$$\|e_t\|^2_{(-n)} \leqq ch^{-2}\|\nabla e\|^2_{(-n)} + \varepsilon h^2 \|\nabla e_t\|^2_{(-n)}.$$

By the technical tool:

$$\|\nabla v\|_{(-n)} \leqq ch^{-1}\|v\|_{(-n)} + ch^{m-1}|\ln h|^{1/2}, \qquad v \in H^m(T), \quad T \in \tau^h, \tag{3.12}$$

which we have already used in the first section and by integrating from 0 to T, we have shown the lemma.

(ii) Here we use (3.4):

$$\|e_{tt}\|^2_{(-n)} = (e_{tt}, \sigma^{-n} e_{tt} - \varphi_h) + a^h(e_t, \sigma^{-n} e_{tt} - \varphi_h) - a^h(e_t, \sigma^{-n} e_{tt}) + \int_\Omega a^h_{ijt} e_j (\sigma^{-n} e_{tt} - \varphi_h)_i \, dx - \int_\Omega a^h_{ijt} e_j (\sigma^{-n} e_{tt})_i \, dx.$$

Again we can omit the easy estimates.

LEMMA 3.3:

$$\int_0^T \|e\|^2_{(-n-2)} dt \leqq ch^{-4}\|e\|^3_\infty + ch^{2m-4}\|e\|_\infty + ch^{2m-2}|\ln h|.$$

Proof: We define the problem:

$$\left.\begin{aligned} -v_t - Av = \sigma^{-n-2} e \quad \text{in } \Omega \times [0, T], \\ v(x, t) = 0 \quad \text{on } \partial\Omega \times [0, T], \qquad v(x, T) = 0 \quad \text{in } \Omega, \end{aligned}\right\} \tag{3.13}$$

which arises from the usual parabolic equation in $\tau\in[0, T]$ by the transformation $t(\tau)=T-\tau$ so that the *a priori* estimates stated in the appendix can be used. With the aid of (3.13), we have:

$$\begin{aligned}\int_0^T \|e\|_{(-n-2)}^2\,dt &= -\int_0^T (e, v_t)\,dt+\int_0^T a(e, v)\,dt\\ &= \int_0^T (e_t, v-v_h)\,dt+\int_0^T a^h(e, v-v_h)\,dt\\ &\quad+\int_0^T (a-a^h)(e, v)\,dt+(e, v)(0)\\ &\leqq \int_0^T \|e_t\|_{(-n)}\|v-v_h\|_{(n)}\,dt+c\int_0^T \|\nabla e\|_{(-n)}\|\nabla(v-v_h)\|_{(n)}\\ &\quad+\int_0^T\int_\Omega |\nabla e|^2|\nabla v|\,dt+\|e\|_{L^\infty}(0)\|v\|_{L^1(0)}\\ &=A+B+C+D.\end{aligned}$$

From the approximation property in weighted norms (3.1) and from theorem A2 in the appendix, we obtain:

$$\begin{aligned}A&\leqq ch^2\int_0^T\sum_{i=0}^2 \|e_t\|_{(-n)}\|\nabla^i v\|_{(n)}\,dt\\ &\leqq \varepsilon h^2\int_0^T \|e_t\|_{(-n)}^2\,dt+c\,\varepsilon^{-1}h^2\rho^{-2}|\ln h|\int_0^T \|e\|_{(-n-2)}^2\,dt\end{aligned}$$

and similarly

$$\begin{aligned}B&\leqq ch\int_0^T\sum_{i=0}^2 \|\nabla e\|_{(-n)}\|\nabla^i v\|_{(n)}\,dt\\ &\leqq \varepsilon\int_0^T \|\nabla e\|_{(-n)}^2\,dt+c\,\varepsilon^{-1}h^2\rho^{-2}|\ln h|\int_0^T \|e\|_{(-n-2)}^2\,dt.\end{aligned}$$

Theorems A2 (i) and (3.12) yield:

$$\begin{aligned}C&\leqq \|\nabla e\|_\infty^2\int_0^T \|\nabla v\|_{L^1}\,dt\leqq \|\nabla e\|_\infty^2 . ch^{-2}\|e\|_\infty\\ &\leqq ch^{-4}\|e\|_\infty^3+ch^{2m-4}\|e\|_\infty\end{aligned}$$

and theorems A2 and (3.8) yield:

$$D\leqq ch^m\int_0^T\int_\Omega |e\,\sigma^{-n-2}|\,dx\,dt\leqq c\,\varepsilon^{-1}h^{2m-2}+\varepsilon\int_0^T \|e\|_{(-n-2)}^2\,dt.$$

Choosing $\rho = ch|\ln h|^{1/2}$, c sufficiently large, and combining the above estimates with lemmas (3.1) (i) and (3.2) (i) we obtain lemma 3.3.

LEMMA 3.4:

$$\int_0^T \|e_t\|_{(-n-2)}^2 dt \leq ch^{-7}\|e\|_\infty^4 + ch^{-1}|\ln h|^{n+5}\|e\|_\infty^2 + ch^{-2}\|\nabla e\|_\infty^2 \int_0^T \|\nabla e_t\|_{(-n)\,0}^2 dt + ch^{2m-2}|\ln h|^{n+5}.$$

Proof: Again we use the problem:

$$-v_t - Av = \sigma^{-n-2} e_t \quad \text{in } \Omega\times[0, T],$$
$$v(x, t) = 0 \quad \text{on } \partial\Omega\times[0, T], \qquad v(x, T) = 0 \quad \text{in } \Omega,$$

as a device to obtain:

$$\begin{aligned}\int_0^T \|e_t\|_{(-n-2)}^2 dt &= -\int_0^T (e_t, v_t)\,dt + \int_0^T a(e_t, v)\,dt \\ &= \int_0^T (e_{tt}, v - v_h)\,dt + \int_0^T a^h(e_t, v - v_h) \\ &\quad + \int_0^T\int_\Omega (a_{ijt}^h - a_{ijt})\, e_j (v - v_h)_i \,dx\,dt \\ &\quad + \int_0^T\int_\Omega a_{ijt}\, e_j (v - v_h)_i \,dx\,dt + \int_0^T\int_\Omega (a_{ijt}^h - a_{ijt})\, e_j v_i \,dx\,dt \\ &\quad - \int_0^T\int_\Omega a_{ijt}\, e_j v_i \,dx\,dt + \int_0^T (a - a^h)(v, e_t)\,dt + (v, e_t)(0) \\ &= A + B + C + D + E + F + G + H.\end{aligned}$$

We only estimate C, E, F and G.

From (3.6) we conclude:

$$\begin{aligned}|C| &\leq \int_0^T\int_\Omega (|\nabla e| + |\nabla e_t|)\,|\nabla e|\,|\nabla(v - v_h)|\,dx\,dt \\ &\leq ch\|\nabla e\|_\infty \int_0^T (\|\nabla e\|_{(-n)} + \|\nabla e_t\|_{(-n)}) \sum_{k=0}^{2} \|\nabla^k v\|_{(n)}\end{aligned}$$

and from theorem A2 and the inequality:

$$ab \leq \frac{1}{2}\varepsilon a^2 + \frac{1}{2}\varepsilon^{-1} b^2,$$

$$C \leqq c|\ln h| \|\nabla e\|_\infty^4 + c\|\nabla e\|_\infty^2 \int_0^T \|\nabla e_t\|_{(-n)}^2 dt + \mathrm{ch}^2 \rho^{-2} |\ln h| \int_0^T \|e\|_{(-n-2)}^2 dt.$$

Choosing $\rho = \mathrm{ch}|\ln h|^{1/2}$, c sufficiently large, and applying (3.12) we see that C can be estimated appropriately.

For the next term we obtain similarly:

$$|E| \leqq \int_0^T \int_\Omega (|\nabla e| + |\nabla e_t|) |\nabla e| |\nabla v| \, dx \, dt$$

$$\leqq \mathrm{ch}^{-2} \|\nabla e\|_\infty^2 \int_0^T \{\|\nabla e\|_{(-n)}^2 + \|\nabla e_t\|_{(-n)}^2\} dt$$

$$+ \mathrm{ch}^2 \rho^{-2} |\ln h| \int_0^T \|e\|_{(-n-2)}^2 dt.$$

We now estimate F. Partial integration leads us to:

$$|F| \leqq c \int_0^T \int_\Omega |e| |\nabla^2 v| \, dx \, dt \leqq c |\ln h|^{1/2} \|e\|_\infty \int_0^T \|\nabla^2 v\|_{(n)} dt$$

and by theorem A1:

$$|F| \leqq \mathrm{ch}^{-2} |\ln h| \|e\|_\infty^2 + \mathrm{ch}^2 \rho^{-2} |\ln h| \int_0^T \|e\|_{(-n-2)}^2 dt.$$

The second term can be cancelled by altering the constant c in $\rho = \mathrm{ch}|\ln h|^{1/2}$. Let $|e(x_2, t_2)| = \|e\|_\infty$. Then we get by the inequality:

$$|f(t)| \leqq c \varepsilon^{-1} \|f\|_{L^2([0, T])} + \varepsilon \|f_t\|_{L^2([0, T])},$$

$f \in H^1([0, T])$, and by the definition of σ:

$$\|e\|_\infty^2 \leqq \varepsilon \int_0^T e_t(x_2, t)^2 dt + c \varepsilon^{-1} \int_0^T e(x_2, t)^2 dt$$

$$\leqq c \varepsilon h^{-n} \rho^{n+2} \int_0^T \|e_t\|_{(-n-2)\,1}^2 dt$$

$$+ c \varepsilon^{-1} h^{-n} \rho^{n+2} \int_0^T \|e\|_{(-n-2)\,0}^2 dt + \mathrm{ch}^{2m}.$$

Choosing $\varepsilon = \varepsilon_0 h^2 |\ln h|^{(n/2)+2}$, we arrive at:

$$\|e\|_\infty^2 \leqq \varepsilon_0 h^2 |\ln h|^{-1} \int_0^T \|e_t\|_{(-n-2)\,1}^2 dt$$

$$+ c \varepsilon_0^{-1} h^2 |\ln h|^{n+3} \int_0^T \|e\|_{(-n-2)\,0}^2 dt + \mathrm{ch}^{2m}$$

and therefore:

$$|F| \leqq c|\ln h|^{n+4} \int_0^T \|e\|^2_{(-n-2)\,0}\, dt + \mathrm{c}h^{2m-2}|\ln h|$$

and by lemma 3.3:

$$|F| \leqq \mathrm{c}h^{-4}|\ln h|^{c(n)}\|e\|^3_\infty + \mathrm{c}h^{2m-4}|\ln h|^{c(n)}\|e\|_\infty + \mathrm{c}h^{2m-2}|\ln h|^{n+5}.$$

Finally we have:

$$|G| \leqq c\int_0^T\int_\Omega |\nabla e||\nabla e_t||\nabla v|\, dx\, dt.$$

Note that $|G|$ is similar to $|E|$.

Utilizing the above estimates and lemmas 3.1-3.3 we have:

$$\begin{aligned}\int_0^T \|e_t\|^2_{(-n-2)}\, dt &\leqq \mathrm{c}h^{-4}|\ln h|^{c(n)}\|e\|^3_\infty + \mathrm{c}h^{2m-4}|\ln h|^{c(n)}\|e\|_\infty \\ &+ \mathrm{c}h^{-2}|\ln h|^{c(n)}\|\nabla e\|^4_\infty + \mathrm{c}h^{-2}\|\nabla e\|^2_\infty \int_0^T \|\nabla e_t\|^2_{(-n)\,0}\, dt \\ &+ \mathrm{c}h^{2m-2}|\ln h|^{n+5}.\end{aligned}$$

Now the technique from (3.12) and the inequality:

$$ab \leqq \frac{1}{2}\varepsilon a^2 + \frac{1}{2}\varepsilon^{-1} b^2,$$

will complete the proof.

LEMMA 1:

$$h^4|\ln h|^{c(n)}\int_0^T |e_{tt}(x,t)|^2\, dt + \mathrm{c}h^2|\ln h|^{c(n)}\int_0^T |\nabla e_t(x,t)|^2\, dt + \|e\|^2_\infty$$

$$\leqq c|\ln h|^{c(n)}\left\{h^{-5}\|e\|^4_\infty + \|\nabla e\|^2_\infty \int_0^T |\nabla e_t(x_1,t)|^2\, dt\right\} + c_1 h^{2m}|\ln h|^{n+4},$$

where $x_1 = x_1(h)$ is a fixed point in Ω.

Proof: From the definition of weighted norms we conclude:

$$\int_0^T |e_{tt}(x,t)|^2\, dt + \int_0^T |\nabla e_t(x,t)|^2\, dt$$

$$\leqq c|\ln h|^{c(n)}\int_0^T \{\|e_{tt}\|^2_{(-n)} + \|\nabla e_t\|^2_{(-n)}\}\, dt + \mathrm{c}h^{2m-2}|\ln h|^{c(n)}.$$

Now lemmas 3.1 (ii), 3.2 (ii) and 3.4 yield the statement for the first two terms. From the inequality:

$$|f(t)| \leqq \varepsilon \| f_t \|_{L^2([0,T])} + c\,\varepsilon^{-1} \| f \|_{L^2([0,T])}, \qquad f \in H^1([0, T]),$$

we conclude for $|e(x_0, t_0)| = \| e \|_\infty$:

$$\| e \|_\infty^2 \leqq \varepsilon \int_0^T e_t(x_0, t)^2\, dt + c\,\varepsilon^{-1} \int_0^T e(x_0, t)^2\, dt + \mathrm{ch}^{2m}$$

and from $\varepsilon = |\ln h|^{(-n/2)-2}$ and the definition of σ, it follows that

$$\| e \|_\infty^2 \leqq \mathrm{ch}^2 |\ln h|^{-1} \int_0^T \| e_t \|_{(-n-2)\,0}^2\, dt + \mathrm{ch}^2 |\ln h|^{n+3} \int_0^T \| e \|_{(-n-2)\,0}^2\, dt + \mathrm{ch}^{2m}.$$

Now by the lemmas 3.1-3.4:

$$\begin{aligned} \| e \|_\infty^2 \leqq {} & \mathrm{ch}^{-5} |\ln h|^{c(n)} \| e \|_\infty^4 + \mathrm{ch}\, |\ln h|^{c(n)} \| e \|_\infty^2 \\ & + c\, |\ln h|^{c(n)} \| e \|_\infty \int_0^T \| \nabla e_t \|_{(-n)\,0}\, dt \\ & + \mathrm{ch}^{-2} |\ln h|^{n+3} \| e \|_\infty^3 \\ & + \mathrm{ch}^{2m-2} |\ln h|^{c(n)} \| e \|_\infty + \mathrm{ch}^{2m} |\ln h|^{n+4}. \end{aligned}$$

Finally, we obtain from the theorem of Fubini:

$$\int_0^T \| \nabla e_t \|_{(-n)\,0}^2\, dt \leqq c\, |\ln h| \int_0^T |\nabla e_t(x_1, t)|\, dt.$$

APPENDIX

THEOREM A1: *For the solution u of problem* (P) *and the corresponding Galerkin-approximation U of* (P_h), *we have the asymptotic error estimate:*

$$\| u_t(.\,, 0) - U_t(.\,, 0) \|_{L^\infty} \leqq \mathrm{ch}^m, \qquad m \geqq 3.$$

Proof: In view of:

$$a(u - U, v_h)(0) = 0, \qquad v_h \in S_h,$$

we have with $e = u - U$ for $t = 0$:

$$(e_t, v_h) + (a - a^h)(e, v_h) = 0, \qquad v_h \in S_h.$$

Using the weight function σ and $q=n+\delta$, $\delta\in(0, 1)$, we obtain:

$$(e_t, \sigma^{-q} e_t)=(e_t, \sigma^{-q} e_t-\varphi_h)+(a-a^h)(e, \sigma^{-q} e-\varphi_h)$$
$$-(a-a_h)(e, \sigma^{-q} e), \qquad t=0.$$

Choosing φ_h = approximation of $\sigma^{-q} e$ we can estimate:

$$\| e_t\|^2_{(-q)}\leqq \mathrm{ch}\| e_t\|_{(-q)} \sum_{i=0}^{1} \| \nabla^i (\sigma^{-q} e_t)\|_{(q)}$$
$$+c\| \nabla e\|_{\infty}\| \nabla e\|_{(-q)}\| \nabla (\sigma^{-q} e)\|_{(q)}, \qquad t=0.$$

Now we choose $\rho=\mathrm{ch}$ with c large and get:

$$\| e_t\|^2_{(-q)}\leqq \mathrm{ch}^{2m-\delta}. \tag{A1}$$

If

$$| e_t(x_0, 0)|=\| e_t(.\,, 0)\|_{\infty}$$

and

$$\sigma(.)=(|\,.-x_0|^2+\rho^2)^{1/2},$$

we obtain from (A1):

$$\| e_t(.\,, 0)\|_{L^\infty}\leqq \mathrm{ch}^m.$$

THEOREM A2: *Let* $-A v:=-(a_{ij}(x, t) v_j)_i$ *be a sufficiently regular elliptic differential operator and let* $v(x, t)$ *be the solution of the problem:*

$$\begin{cases} v_t-A v=f(x, t) & \text{in } \Omega\times(0, T], \\ v(x, t)=0 \quad \text{on } \partial\Omega\times[0, T], & v(x, 0)=0 \quad \text{in } \Omega, \end{cases} \tag{A2}$$

where $f\in L^2(\Omega\times[0, T])$. *Then:*

$$\int_0^T \sum_{i=0}^{2} \| \nabla^i v\|^2_{(n)}\, dt\leqq c\,\rho^{-2}|\ln h| \int_0^T \| f\|^2_{(n+2)}\, dt.$$

Proof: Let us start with the case $n\geqq 3$. We have:

$$(\sigma^{n/2} v)_t-A(\sigma^{n/2} v) =\sigma^{n/2} v_t-\sigma^{n/2} A v-v A \sigma^{n/2}-2 a_{ij}(\sigma^{n/2})_j v_i.$$

Because $(\sigma^{n/2} v)(x, 0)=0$ and the boundary condition is fulfilled the standard *a priori* estimate yields:

$$\int_0^T \| \nabla^2 (\sigma^{n/2} v)\|^2\, dt \leqq c \int_0^T \{ \| v_t-A v\|^2_{(n)}+\| \nabla v\|^2_{(n-2)}+\| v\|^2_{(n-4)} \}\, dt$$

and therefore:

$$\text{(A3)} \qquad \int_0^T \|\nabla^2 v\|_{(n)}^2 \, dt \leqq c \int_0^T \{\|f\|_{(n)}^2 + \|\nabla v\|_{(n-2)}^2 + \|v\|_{(n-4)}^2\} \, dt.$$

Further we conclude by partial integration:

$$\|\nabla v\|_{(n-2)}^2 \leqq \int_\Omega \{|\nabla^2 v| |v| \sigma^{n-2} + |\nabla v| |v| |\nabla(\sigma^{n-2})|\} \, dx$$

$$\leqq c \|v\|_{(n-4)} \{\|\nabla^2 v\|_{(n)} + \|\nabla v\|_{(n-2)}\}$$

and by the inequality:

$$ab \leqq \frac{1}{2} \varepsilon a^2 + \frac{1}{2} \varepsilon^{-1} b^2,$$

$$\text{(A4)} \qquad \|\nabla v\|_{(n-2)} \leqq \varepsilon \|\nabla^2 v\|_{(n)} + c\,\varepsilon^{-1} \|v\|_{(n-4)}.$$

For abbreviation, let $\Omega_T := \Omega \times [0, T]$. Denoting by Γ the Green's function of $(d/dt) - A$ over Ω_T we obtain:

$$\int_0^T \|v\|_{(n-4)}^2 = \int_{\Omega_T} \sigma^{n-4} \left| \int_{\Omega_t} \left(\frac{d}{dt} - A\right) v \,\Gamma \, dy \, d\tau \right|^2 dx \, dt$$

and by Hölder's inequality and an interchange of the order of integration:

$$\int_0^T \|v\|_{(n-4)}^2 \leqq \int_{\Omega_T} \sigma^{n+2} f^2 \left\{ \int_{\Omega_T} \sigma^{n-4} |\Gamma| \left(\int_{\Omega_T} \sigma^{-n-2} |\Gamma| \, dx' \, dt' \right) dy \, d\tau \right\} dx \, dt.$$

It is well known that the Green's function Γ can be estimated by the Green's function of $d/dt - c\,\Delta$, $c > 0$, over $\mathbf{R}^n \times [0, T]$ (*see* [9]). Then the solution w of the initial value problem:

$$w_t - c\,\Delta w = \sigma^{-\mu}(x),$$

$$w(x, 0) = 0,$$

yields an estimate of $\int_{\Omega_T} \sigma^{-\mu}(x) |\Gamma| \, dx \, dt$. Denoting by w_1 the solution of the corresponding elliptic problem:

$$-c\,\Delta w_1 = \sigma^{-\mu}(x),$$

it is clear that:

$$w(x, t) \leqq w_1(x).$$

From:

$$-\Delta \sigma^{2-n} = n(n-2)\,\rho^2\,\sigma^{-n-2}$$

and

$$-\Delta(\ln\sigma^{-2})=(2(n-2)|y-x_0|^2+2n\rho^2)\sigma^{-4},$$

we easily obtain:

$$\int_{\Omega_T}\sigma^{-n-2}|\Gamma|\,dx'\,dt'\leqq c\,\rho^{-2}\,\sigma^{2-n}$$

and

$$\int_{\Omega_T}\sigma^{-2}|\Gamma|\,dy\,d\tau\leqq c\,|\ln\rho|$$

and therefore:

$$\int_0^T\|v\|_{(n-4)}^2\,dt\leqq c\,\rho^{-2}\,|\ln h|\int_0^T\|f\|_{(n+2)}^2\,dt.$$

Together with (A3), (A4) this completes the proof for $n\geqq 3$. Denoting by $y^k=x^k-x_0^k$, $j=1, 2$, the components of the vector $x-x_0$, $x\in\Omega$, we get for $n=2$:

$$\|\nabla^2 v\|_{(2)}^2=\sum_{j=1}^{2}\|y^k\nabla^2 v\|^2+\rho^2\|\nabla^2 v\|^2$$

and by the standard *a priori* estimate, we have:

$$\int_0^T\rho^2\|\nabla^2 v\|^2\,dt\leqq c\int_0^T\rho^2\|f\|^2\,dt\leqq c\,\rho^{-2}\int_0^T\|f\|_{(4)}^2\,dt.$$

Now we observe that $y^k v$, $k=1, 2$, is the solution of the problem:

$$y^k v_t-A(y^k v)=y^k v_t-y^k Av-v\,A\,y^k-2\,a_{ij}\,y_j^k\,v_i,\qquad k=1, 2$$

and again from the standard *a priori* estimate we conclude:

$$\int_0^T\|\nabla^2(y^k v)\|^2\,dt\leqq\int_0^T\|y^k f\|^2\,dt+c\int_0^T\|v\|_{1,2}^2\,dt,\qquad k=1, 2.$$

Hence:

$$\int_0^T\|y^k\nabla^2 v\|^2\,dt\leqq c\int_0^T\{\|\nabla^2(y^k v)\|^2+\|\nabla v\|^2\}\,dt$$

$$\leqq c\int_0^T\{\|f\|_{(2)}^2+\|v\|_{1,2}^2\}\,dt,\qquad k=1, 2$$

and thus by Poincaré's inequality and $\sigma^{-1}\leqq\rho^{-1}$:

$$\text{(A5)}\qquad\int_0^T\|\nabla^2 v\|_{(2)}^2\,dt\leqq c\int_0^T\{\|f\|_{(2)}^2+\|\nabla v\|^2\}\,dt.$$

By partial integration, it follows that:

$$\int_0^T \|\nabla v\|^2 dt \leqq c\int_0^T a(v, v)\,dt \leqq -c\int_0^T\int_\Omega (v_t - A v)v\,dx\,dt$$
$$\leqq c\rho^{-2}|\ln \rho| \int_0^T \|f\|_{(4)}^2 dt + c\rho^2|\ln\rho|^{-1}\int_0^T \|v\|_{(-4)}^2 dt.$$

Denoting by Γ the Green's function of $(d/dt)-A$ over $\Omega\times[0, T]$ we obtain:

$$\int_0^T \|v\|_{(-4)}^2 dt = \int_{\Omega_T} \sigma^{-4}\left|\int_{\Omega_T}\left(\frac{d}{dt}-A\right)v\,\Gamma\,dy\,d\tau\right|^2 dx\,dt$$
$$\leqq \int_{\Omega_T} \sigma^4 f\left\{\int_{\Omega_T}\sigma^{-4}|\Gamma|\int_{\Omega_T}\sigma^{-4}|\Gamma|dx'\,dt')\,dy\,d\tau\right\}dx\,dt.$$

Analogously to the case $n\geqq 3$, it follows that:

$$\int_0^T \|v\|_{(-4)}^2 dt \leqq c\rho^{-4}(1+|\ln\rho|)^2\int_0^T \|f\|_{(4)}^2 dt$$

and thus:

$$\int_0^T \|\nabla v\|^2 dt \leqq c\rho^{-2}\,(1+|\ln\rho|)\int_0^T \|f\|_{(4)}^2\,dt.$$

Now the theorem is proved by (A5) and $\rho(h)\geqq ch$.

THEOREM A3: *If v is the solution of problem* (A1), *then:*

(i) $$\int_0^T \|\nabla v\|_{L^1} dt \leqq c\int_0^T \|f\|_{L^1} dt;$$

(ii) $$\|v\|_{L^1}(t) \leqq c\int_0^T \|f\|_{L^1} dt.$$

Proof: (i) Denoting by Γ the Green's function of problem (A1) we have:

$$\int_0^T\int_\Omega |\nabla v|\,dx\,dt \leqq \int_0^T\int_\Omega\int_0^T\int_\Omega |\nabla_x\Gamma(x-y, t-\tau)f(y,\tau)|\,dy\,d\tau\,dx\,dt$$

and by the theorem of Fubini:

$$\leqq \int_0^T\int_\Omega |f(y,\tau)|\left\{\int_0^T\int_\Omega |\nabla_x\Gamma(x-y, t-\tau)|\,dx\,dt\right\}dy\,d\tau.$$

It is well known that:

$$|\nabla_x\Gamma(x, t)| \leqq ct^{-(n+1)/2}\exp\left(-c\frac{|x|^2}{t}\right)$$

and the right hand side is integrable over $\mathbf{R}^n\times[0, T]$ (*see* [9]).

(ii) The proof is similar to (i).