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A CONFORMING FINITE ELEMENT METHOD WITH LAGRANGE MULTIPLIERS FOR THE BIHARMONIC PROBLEM (*)

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Abstract — *We consider a finite element method for solving the biharmonic problem $\Delta^2 u = f$ in Ω, $u = \partial u/\partial n = 0$ on $\partial\Omega$, $\Omega \subset R^2$, $\partial\Omega$ smooth We use the method of Lagrange multipliers to avoid the fulfillment of the Dirichlet boundary conditions in the subspaces Assuming the interior subspaces to be defined in terms of Argyris triangles, we show how the boundary subspaces in the Lagrange multiplier method can be defined so as to achieve a convergence rate of optimal order*

Resume — *On considere une methode d'elements finis pour resoudre le probleme biharmonique $\Delta^2 u = f$ dans Ω, $u = \partial u/\partial n = 0$ sur $\partial\Omega$, $\Omega \subset R^2$, $\partial\Omega$ regulier On utilise la methode des multiplicateurs de Lagrange pour eviter d'avoir a satisfaire les conditions aux limites de Dirichlet dans les sous-espaces Supposant les sous-espaces « a l'interieur » definis a l'aide de triangles d'Argyris, on montre comment definir les sous-espaces « a la frontiere » afin d'obtenir un ordre de convergence d'ordre optimal*

1. INTRODUCTION

Let Ω be a bounded, simply connected plane domain with a smooth boundary $\partial\Omega$ We consider a high-order displacement finite element method for the solution of the biharmonic problem

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

where f is some given function defined on Ω In our approximation method the fulfillment of the Dirichlet boundary conditions in the finite element subspaces is avoided by using Lagrange multipliers Thus, our approach is an analogue of the finite element method with Lagrange multipliers for solving the Dirichlet problem for a second-order elliptic equation, *see* [1, 6, 7, 8] Besides avoiding the boundary conditions we get here independent approximations for

$$\Delta u\big|_{\partial\Omega} \qquad \text{and} \qquad \frac{\partial}{\partial n}\Delta u\big|_{\partial\Omega},$$

which is sometimes of physical interest

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We consider in detail an approximation where the approximate solution of (1.1) is sought in a finite element space constructed by means of Argyris triangles [4]. We show how the boundary subspaces in the Lagrange multipliers method can be constructed so as to achieve a convergence rate of optimal order. Our method of proof is analogous to that used in [7]: we introduce a norm depending on the finite element partitioning and show that a quasioptimal errors estimate can be obtained in this norm.

2. THE APPROXIMATION METHOD

For $\Omega \subset R^2$, $\partial\Omega$ smooth, we use the symbol $H^m(\Omega)$, $m \geqq 0$, for a Sobolev space in its usual meaning. For non-integral s, $s \geqq 0$, one defineds $H^s(\Omega)$ by interpolation, and for $s<0$, $H^s(\Omega)$ in defined as the dual of $H^{-s}(\Omega)$ [5]. We also denote by $|D^k u|^2$ the sum of the squares of all the k-th order derivatives of u, u defined on Ω.

To define Sobolev spaces on the boundary, note that, since $\partial\Omega$ is a closed smooth curve, there exist the smooth periodic functions $J_1(t)$ and $J_2(t)$, $t \in R^1$, with period of length unity. such that $J(t)=(J_1(t), J_2(t))$ defines a 1-1 mapping of $(0, 1)$ onto $\partial\Omega$. Assuming J is such a mapping, we can define $H^s(\partial\Omega)$, $s \geqq 0$, as the closure of the set of all smooth functions on $\partial\Omega$ in the norm

$$\|\psi\|_{H^s(\partial\Omega)} = \|\varphi\|_{H^s(0,1)}, \qquad \varphi(t)=\psi(J(t)).$$

We consider the following weak formulation of problem (1.1): Find a triple $(u, \psi, \varphi) \in H^2(\Omega) \times L_2(\partial\Omega) \times L_2(\partial\Omega)$ such that

$$B(u, \psi, \varphi; v, \xi, \eta) = \int_\Omega f v\, dx$$

for all

$$(v, \xi, \eta) \in H^2(\Omega) \times L_2(\partial\Omega) \times L_2(\partial\Omega), \tag{2.1}$$

where

$$B(u, \psi, \varphi; v, \xi, \eta) = \int_\Omega \Delta u \Delta v\, dx + \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} \xi + u \eta + \frac{\partial v}{\partial n} \psi + v \varphi \right) ds. \tag{2.2}$$

If u is the solution of problem (1.1) for f sufficiently smooth, then the triple $(u, -\Delta u|_{\partial\Omega}, (\partial/\partial n) \Delta u|_{\partial\Omega})$ is the solution of (2.1). Noting that the weak solution of (1.1) in $H^2(\Omega)$ satisfies (*cf.* [5]):

$$\|u\|_{H^{s+4}(\Omega)} + \|\Delta u|_{\partial\Omega}\|_{H^{s+(3/2)}(\partial\Omega)} + \left\| \frac{\partial}{\partial n} \Delta u|_{\partial\Omega} \right\|_{H^{s+(1/2)}(\partial\Omega)} \leqq C \|f\|_{H^s(\Omega)}, \qquad s > -\frac{1}{2}, \tag{2.3}$$

we conclude that the assumption $f \in H^s(\Omega)$, $s > -1/2$, suffices for the solvability of (2.1).

If $M^h \subset H^2(\Omega)$, $U^h \subset L_2(\partial\Omega)$, $V^h \subset L_2(\partial\Omega)$ are finite-dimensional subspaces, one can define the approximate solution of (2.1) as the triple $(u_h, \psi_h, \varphi_h) \in M^h \times U^h \times V^h$ such that

$$B(u_h, \psi_h, \varphi_h; v, \xi, \eta) = \int_\Omega f v \, dx \tag{2.4}$$

for all

$$(v, \xi, \eta) \in M^h \times U^h \times V^h.$$

We define first the subspaces M^h. To this end, let $\{\Pi^h\}_{0<h<1}$ be a family of partitionings of Ω into disjoint open subsets T_i such that each $T_i \in \Pi^h$ is either a triangle, or a deformed triangle with one curved side on $\partial\Omega$. We assume that the partitionings are quasiuniform, i. e., the diameters of all the triangles in Π^h are proportional to h, and each $T \in \Pi^h$ contains a sphere of radius proportional to h (the minimal angle condition). Now let M^h be a finite-dimensional space of functions defined on Ω such that (i) for each $v \in M^h$ and $T \in \Pi^h$, $v_{|T}$ is a polynomial of degree $\leqq 5$, (ii) $M^h \subset H^2(\Omega)$, (iii) $D^2 v$ is continuous at the vertices of the triangulation Π^h.

The space M^h can be set up by means of Argyris triangles [4]; for h small enough, each $v \in M^h$ is defined uniquely by the values of $D^k v$, $k = 0, 1, 2$ at the vertices of the triangulation Π^h and by the values of $\partial v / \partial n$ at the midpoints of the sides of the triangles in Π^h.

To define the spaces U^h and V^h, let $\{x_1, \ldots, x_\nu\}$ be set of vertices of the triangulation Π^h on $\partial\Omega$ and let

$$t_i = J^{-1}(x_i), \qquad i = 1, \ldots, \nu,$$

$$I_i = (t_{i+1}, t_i), \qquad i = 1, \ldots, \nu - 1,$$

with J as above. We let N^h denote the third-degree Hermitean finite element space associated to the partitioning $\{I_i\}_1^{\nu-1}$ of [0, 1], i. e., N^h consists of continuously differentiable functions $\varphi(t)$ such that $\varphi_{|I_i}$ is a polynomial of degree $\leqq 3$ for all i. We further set $N_0^h = \{\varphi \in N^h, \varphi(0) = \varphi(1), \varphi'(0) = \varphi'(1)\}$ and define

$$U^h = V^h = \{\psi; \psi(J(t)) = \theta(t) \in N_0^h\}.$$

3. RATE OF CONVERGENCE

We start by introducing on $H^2(\Omega) \times L_2(\partial\Omega) \times L_2(\partial\Omega)$ the norm

$$\|(u, \psi, \varphi)\|_h^2 = \int_\Omega |\Delta u|^2 \, dx + h^{-1} \int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 ds + h^{-3} \int_{\partial\Omega} u^2 \, ds + h \int_{\partial\Omega} \psi^2 \, ds + h^3 \int_{\partial\Omega} \varphi^2 \, ds.$$

This is a norm, since the only harmonic function satisfying $u = 0$ on $\partial\Omega$ is zero. Our aim is to prove the following theorem.

THEOREM 1: *Let* $(u, \psi, \varphi) \in H^2(\Omega) \times L_2(\partial\Omega) \times L_2(\partial\Omega)$ *be the solution of problem* (1.2) *and let* M^h, U^h, V^h *be defined as above. Then if h is small enough, problem* (2.4) *has a unique solution* $(u_h, \psi_h, \varphi_h) \in M^h \times U^h \times V_h$ *and there exists a constant C independent of h such that*

$$\|(u-u_h, \psi-\psi_h, \varphi-\varphi_h)\|_h \leqq C \min_{(v,\xi,\eta)\in M^h \times U^h \times V^h} \|(u-v, \psi-\xi, \varphi-\eta)\|_h.$$

The proof is based on the following two results.

PROPOSITION 1: *Let* $v \in M^h$ *be such that*

$$\left.\begin{aligned} &\int_{\partial\Omega} \frac{\partial v}{\partial n} \xi \, ds = 0, \qquad \forall \xi \in U^h, \\ &\int_{\partial\Omega} v \eta \, ds = 0, \qquad \forall \eta \in V^h. \end{aligned}\right\} \tag{3.1}$$

Then if h is small enough, there is a constant C independent of h such that

$$h^{-1} \int_{\partial\Omega} \left| \frac{\partial v}{\partial n} \right|^2 ds + h^{-3} \int_{\partial\Omega} v^2 \, ds \leqq C \int_\Omega |\Delta v|^2 \, dx.$$

PROPOSITION 2: *For all* $(\xi, \eta) \in U^h \times V^h$, *h sufficiently small, there exists* $v \in M^h$ *such that*

$$\int_{\partial\Omega} \left(\frac{\partial v}{\partial n} \xi + v \eta \right) ds \geqq h \int_{\partial\Omega} \xi^2 \, ds + h^3 \int_{\partial\Omega} \eta^2 \, ds$$

and

$$\int_\Omega |\Delta v|^2 \, dx + h^{-1} \int_{\partial\Omega} \left| \frac{\partial v}{\partial n} \right|^2 ds + h^{-3} \int_{\partial\Omega} v^2 \, ds \leqq C \left\{ h \int_{\partial\Omega} \xi^2 \, ds + h^3 \int_{\partial\Omega} \eta^2 \, ds \right\},$$

where C is independent of h.

For a while, assume that the above propositions are true. Then we conclude, by comparing the propositions with the stability conditions of abstract Lagrange multiplier methods, as given in [3], that the bilinear form B of (2.2) satisfies

$$\inf_{(u,\psi,\varphi)\in M^h \times U^h \times V^h} \sup_{(v,\xi,\eta)\in M^h \times U^h \times V^h} \frac{B(u, \psi, \varphi; v, \xi, \eta)}{\|(u, \psi, \varphi)\|_h \|(v, \xi, \eta)\|_h} \geqq C > 0, \tag{3.2}$$

where C is independent of h. On the other hand, we note that B also satisfies

$$|B(u, \psi, \varphi; v, \xi, \eta)| \leqq \|(u, \psi, \varphi)\|_h \|(v, \xi, \eta)\|_h, \tag{3.3}$$

for all (u, ψ, φ),

$$(v, \xi, \eta) \in H^2(\Omega) \times L_2(\partial\Omega) \times L_2(\partial\Omega).$$

The assertion of theorem 1 now follows from (3.2) and (3.3) by classical reasoning (*see* [2], pp. 186-188). □

Proof of proposition 1: Let Γ_k be any connected subset of $\partial\Omega$ such that Γ_k is the union of k curved sides of triangles in Π^h, and let $S_k \subset \bar{\Omega}$ be the union of closed triangles $T \in \Pi^h$ that either have a side $\Gamma \subset \Gamma_k$ or have one vertex on Γ_k. We set

$$Q_k = \{ v_{|S_k};\ v \in M^h \}.$$

We further let A be a scaling mapping,

$$A(x) = h^{-1} x, \qquad x \in R^2,$$

and write

$$\hat{S}_k = A(S_k), \qquad \hat{\Gamma}_k = A(\Gamma_k),$$

$$\hat{Q}_k = \{ \hat{v};\ \hat{v}(h^{-1}x) = v(x) \in O_k,\ x \in S_k \}.$$

Let us first assume that Γ_k is a segment of a straight line and that the mapping J : $[0, 1] \to \partial\Omega$ introduced in section 2 is locally of the simple form

$$x \in \Gamma_k \Rightarrow x = J(t) = a + bt,$$

where $a,\ b \in R^2$ are some constant vectors. In this case the space

$$\hat{X}_k = \{ \varphi(x); \varphi(h^{-1}x) = \varphi_0(x) \in U^h|_{\Gamma_k} = V^h|_{\Gamma_k},\ x \in \Gamma_k \}$$

is simply the third-order Hermitean finite element space associated to the partitioning of Γ_k that is induced by the triangulation

$$\hat{\Pi}^h = \{ \hat{T};\ \hat{T} = A(T),\ T \in \Pi^n \}.$$

We let $\{\xi_i\}$ denote the set of ordinary local basis functions of $\hat{X}_k$ with $\|\xi_i\|_{L_\infty(\hat{\Gamma}_k)} = 1$, and let Λ_k be the index set such that if $i \in \Lambda_k$, then ξ_i and its tangential derivative on $\hat{\Gamma}_k$ vanish at the endpoints of $\hat{\Gamma}_k$. Obviously, if m_k is the number of vertices of $\hat{\Pi}^h$ in the interior of $\hat{\Gamma}_k$, then card $(\Lambda_k) = 2m_k$.

In the above notation, let us define on $\hat{Q}_k$ the seminorm $|.|_{\hat{Q}_k}$ as

$$|z|^2_{\hat{Q}_k} = \int_{\hat{S}_k} |\Delta z|^2\, dx + \sum_{i \in \Lambda_k} \left| \int_{\hat{\Gamma}_k} \xi_i \frac{\partial z}{\partial n} ds \right|^2 + \sum_{i \in \Lambda_k} \left| \int_{\hat{\Gamma}_k} \xi_i z\, ds \right|^2 .$$

Then we have:

LEMMA 1: *If k is large enough, $|.|_{\hat{Q}_k}$ is a norm on $\hat{Q}_k$.*

Proof: Let $z \in \hat{Q}_k$ be such that $|z|_{\hat{Q}_k} = 0$. We show first that z is a harmonic polynomial (of degree ≤ 5). To this end, let us number the triangles $\hat{T} \in \hat{\Pi}^h$, $\hat{T} \subset \hat{S}_k$, from 1 to l in such a way that $\hat{T}_i$ and $\hat{T}_{i+1}$ have a common side for

$i=1, \ldots, l-1$. This is possible by our definition of $\hat{S}_k$. Let p_i be a polynomial of degree $\leqq 5$ such that $z_{|\hat{T}_i}=p_i$. Then since $|z|_{\hat{Q}_k}=0$, p_i is a harmonic polynomial. Further, since z and $\partial z/\partial n$ are continuous, we conclude that $q_i=p_i-p_{i+1}$ is a harmonic polynomial satisfying $q_i=\partial q_i/\partial n=0$ on the common side of T_i and T_{i+1}. But then $q_i=0$. Hence, there is a harmonic polynomial p such that $z=p$ on $\hat{S}_k$.

We now have that z is a polynomial of degree $\leqq 5$ satisfying

$$\int_{\hat{\Gamma}_k} \xi_i z\, ds = \int_{\hat{\Gamma}_k} \xi_i \frac{\partial z}{\partial n} ds = 0, \qquad i \in \Lambda_k.$$

Since card (Λ_k) increases linearly with k, it is obvious that for k large enough we necessarily have $z=\partial z/\partial n=0$ on $\hat{\Gamma}_k$. But z was a harmonic polynomial, so $z=0$. $\square$

From lemma 1 we have in particular that

$$\|z\|^2_{H^2(\hat{S}_k)} \leqq C\left\{\int_{\hat{S}_k} |\Delta z|^2\, dx + \sum_{i\in\Lambda_k}\left|\int_{\hat{\Gamma}_k} \xi_i \frac{\partial z}{\partial n} ds\right|^2 + \sum_{i\in\Lambda_k}\left|\int_{\hat{\Gamma}_k} \xi_i z\, ds\right|^2\right\}, \qquad z \in \hat{Q}_k, \quad k \geqq k_0, \quad (3.4)$$

where C depends on $\hat{Q}_k$. Now it is easy to see, arguing by contradiction, that whenever the triangles composing $\hat{S}_k$ satisfy the minimal angle condition, (3.4) holds uniformly for all $\hat{S}_k$ constructed as above (with straight $\hat{\Gamma}_k$), with C depending only on the constant in the minimal angle condition and on k. (Note that, by the minimal angle condition, the number of triangles $\hat{T} \in \hat{\Pi}^h$ that touch $\hat{\Gamma}_k$ is at most a finite multiple of k.)

The next step of the proof is to verify that, for h small enough, (3.4) also holds when the actual curvature of $\hat{\Gamma}_k$ is taken into account. To this end, consider a given $\hat{\Gamma}_k$, $\hat{S}_k$ and choose an appropriate coordinate system $\{x_1, x_2\}$ to represent $\hat{\Gamma}_k$ as

$$\hat{\Gamma}_k = \{(x_1, x_2);\ x_2 = \theta(x_1),\ x_1 \in I = [0, d]\},$$

where $\theta(0)=\theta(d)=0$. Since $\partial\Omega$ is smooth, we may assume that if h is small enough, then θ also satisfies

$$|\theta'(x_1)| \leqq Ch, \qquad x_1 \in I, \qquad (3.5)$$

where C depends only on Ω for fixed k.

We associate to each triangle $\hat{T} \in \hat{\Pi}^h$, $\hat{T} \subset \hat{S}_k$, another triangle $\hat{T}'$ as follows. Let $\hat{T}$ have the vertices x^k, $k=1, 2, 3$. Then $\hat{T}'$ is defined as a triangle with straight sides and with the vertices y^k such that if $x^k \notin \hat{\Gamma}_k$, then $y^k=x^k$ and if $x^k=(x_1^k,$

$\theta(x_1^k)) \in \hat{\Gamma}_k$, then $y^k = (x_1^k, 0)$. We denote the union of the closed triangles $\hat{T}'$ by $\hat{S}'_k$ and set $\hat{\Gamma}'_k = \{(x_1, x_2); x_1 \in I, x_2 = 0\}$. We further associate to $\hat{S}'_k$ and $\hat{\Gamma}'_k$ the spaces $\hat{Q}'_k$ and $\hat{X}'_k$ as above and let $\{\xi_i\}_{i \in \Lambda_k}$ denote the set of local basis functions for $\hat{X}'_k$ such that ξ_i and $d\xi_i/dx_1$ vanish at the end-points of $\hat{\Gamma}'_k$.

Noting that we have

$$\operatorname{dist}\{x, \partial\hat{T}\} \leqq Ch, \qquad x \in \partial\hat{T}', \tag{3.6}$$

where C is independent of the triangle $\hat{T}$, we conclude that the triangles $\hat{T}'$ satisfy the minimal angle condition if h is sufficiently small. Hence, we have from (3.4) that

$$\left.\begin{aligned} \|z\|^2_{H^2(\hat{S}'_k)} \leqq C\left\{\int_{\hat{S}_k} |\Delta z|^2\,dx + \sum_{i \in \Lambda_k}\left|\int_{\hat{\Gamma}'_k} \xi_i \frac{\partial z}{\partial n}\,ds\right|^2 + \sum_{i \in \Lambda_k}\left|\int_{\hat{\Gamma}'_k} \xi_i z\,ds\right|^2\right\}, \\ z \in \hat{Q}'_k, \quad k \geqq k_0. \end{aligned}\right\} \tag{3.7}$$

Now we need the following technical lemma.

LEMMA 2: *For any $z \in \hat{Q}_k$ and $\tilde{\varphi} = \tilde{\varphi}(x_1) \in \hat{X}'_k$ there exists $\tilde{z} \in \hat{Q}'_k$ and $\varphi \in \hat{X}_k$ such that*

$$\left|\, \|\Delta z\|^2_{L_2(\hat{S}_k)} - \|\Delta\tilde{z}\|^2_{L_2(\hat{S}'_k)} \right| \leqq Ch\|z\|^2_{H^2(\hat{S}_k)},$$
$$|\Phi(x_1, \theta(x_1)) - \tilde{\varphi}(x_1)| \leqq Ch\|\tilde{\Phi}\|_{L_\infty(I)}, \qquad x_1 \in I,$$

where C is independent of z, $\tilde{\Phi}$, x_1.

Proof: Let $z \in \hat{Q}_k$ be given, and let $\{a_i\}$ and $\{a'_i\}$ be the sets of the vertices of the triangulations of $\hat{S}_k$ and $\hat{S}'_k$, respectively, and let $\{b_i\}$ and $\{b'_i\}$ be the sets of the mid-points of the sides in the triangulations, with

$$|a_i - a'_i| \leqq Ch, \qquad |b_i - b'_i| \leqq Ch.$$

Define $\tilde{\mathbf{z}}$ so that

$$\frac{\partial^{l+m}\tilde{\mathbf{z}}}{\partial x_1^l\,\partial x_2^m}(a'_i) = \frac{\partial^{l+m} z}{\partial x_1^l\,\partial x_2^m}(a_1), \qquad l+m \leqq 2,$$

and

$$\frac{\partial\tilde{z}}{\partial n'}(b'_i) = \frac{\partial z}{\partial n}(b_i).$$

Then if p and $\tilde{\mathbf{p}}$ are polynomials such that $z_{|\hat{T}} = p$ and $\tilde{\mathbf{z}}_{|\hat{T}'} = \tilde{\mathbf{p}}$, it is easy to verify from (3.6) that

$$\|\Delta p\|^2_{L_2(\hat{T}\setminus\hat{T}')} + \|\Delta\tilde{\mathbf{p}}\|^2_{L_2(\hat{T}'\setminus\hat{T})} \leqq Ch\|p\|^2_{H^2(\hat{T})},$$
$$\|\Delta p - \Delta\tilde{\mathbf{p}}\|_{L_2(\hat{T}\cap\hat{T}')} \leqq Ch\|p\|_{H^2(\hat{T})}.$$

From this the first part of the assertion follows easily.

Next, let $\tilde{\Phi} \in \hat{X}'_k$ be given and define $\Phi \in \hat{X}_k$ so that if $(s, 0)$ is a vertex of the triangulation of $\hat{S}'_k$, then

$$\Phi(x_1, \theta(x_1)) = \tilde{\Phi}(x_1)$$

and

$$\frac{d}{dt}\Phi(x_1, \theta(x_1)) = \frac{d}{dx_1}\tilde{\Phi}(x_1) \qquad \text{at} \quad x_1 = s,$$

where d/dt denotes the tangential differentiation on $\hat{\Gamma}_k$. Recall from the definition of the subspace $U^h = V^h$ that if $\varphi \in \hat{X}_k$, then

$$\varphi(x_1, \theta(x_1)) = \varphi_0(x_1) = \eta(t(x_1)), \qquad x_1 \in I,$$

where η is a piecewise polynomial function, and the relation $x_1 = x_1(t)$ is of the form

$$x_1(t) = h^{-1} J_1(t), \qquad t \in I_0 = [t_1, t_2], \quad |t_1 - t_2| \leqq Ch,$$

where J_1 is a smooth mapping. Write J_1 locally as

$$J_1(t) = F(t) + \Delta(t),$$

where F is an affine mapping and Δ satisfies

$$\Delta(t_1) = \Delta(t_2) = 0, \qquad |\Delta(t)| \leqq Ch^2, \qquad t \in I_0.$$

Taking the inverse we then have

$$t = F^{-1}(hx_1) + \Delta_1(x_1), \qquad x_1 \in I,$$

with $\Delta_1(0) = \Delta_1(d) = 0$, $|\Delta_1(x_1)| \leqq Ch^2$, $x_1 \in I$. Thus, we may write

$$\begin{aligned} \varphi_0(x_1) &= \eta(F^{-1}(hx_1) + \Delta_1(x_1)) \\ &= \eta(F^{-1}(hx_1)) + \Delta_2(x_1) = \eta_0(x_1) + \Delta_2(x_1), \qquad x_1 \in I, \end{aligned}$$

where $\eta_0 \in \hat{X}'_k$ and Δ_2 satisfies

$$|\Delta_2(x_1)| \leqq Ch \|\eta\|_{L_\infty(I_0)} = Ch \|\varphi_0\|_{L_\infty(I)}.$$

Setting $\varphi = \Phi$ and using (3.5) we now easily find that

$$\Phi(x_1, \theta(x_1)) = \Phi_0(x_1) = \eta(x_1) + \Delta(x_1), \qquad x_1 \in I,$$

where

$$|\Delta(x_1)| \leqq Ch \|\Phi_0\|_{L_\infty(I)} \leqq C_1 h \|\tilde{\Phi}\|_{L_\infty(I)},$$

and $\eta \in \hat{X}'_k$ is such that if $(s, 0)$ is a vertex of the triangulation of $\hat{S}'_k$, then

$$|\eta(s) - \tilde{\Phi}(s)| \leqq Ch \|\tilde{\varphi}\|_{L_\infty(I)}$$

and

$$\left| \frac{d}{dx_1}[\eta(x_1) - \tilde{\Phi}(x_1)]_{|x_1 = s} \right| \leqq Ch \|\tilde{\Phi}\|_{L_\infty(I)}.$$

The second part of the assertion then follows. □

Now let $z \in \hat{Q}_k$ be given, let $\tilde{z}$ be as in lemma 2, and let $\xi_i \in \hat{X}_k$ be local basis functions such that

$$|\tilde{\xi}_i(x_1) - \xi_i(x_1, \theta(x_1))| \leqq Ch \| \tilde{\xi}_i \|_{L_\infty(I)} = Ch.$$

Then lemma 2 and (3.7) imply:

$$\begin{aligned}
\| z \|^2_{H^2(\hat{S}_k)} &\leqq C \| \tilde{z} \|^2_{H^2(\hat{S}_k)} \\
&\leqq C_1 \left\{ \int_{\hat{S}_k} |\Delta \tilde{z}|^2 \, dx + \sum_{i \in \Lambda_k} \left| \int_{\hat{\Gamma}_k} \tilde{\xi}_i \frac{\partial \tilde{z}}{\partial x_2} dx_1 \right|^2 + \sum_{i \in \Lambda_k} \left| \int_{\hat{\Gamma}'_k} \tilde{\xi}_i \tilde{z} \, dx_1 \right|^2 \right\} \\
&\leqq C_1 \left\{ \int_{\hat{S}_k} |\Delta z|^2 \, dx + \sum_{i \in \Lambda_k} \left| \int_{\hat{\Gamma}_k} \xi_i \frac{\partial z}{\partial n} ds \right|^2 + \sum_{i \in \Lambda_k} \left| \int_{\hat{\Gamma}_k} \xi_i z \, ds \right|^2 \right\} \\
&\qquad + C_2 h \left\{ \| z \|^2_{H^2(\hat{S}_k)} + \int_{\hat{\Gamma}_k} \left| \frac{\partial z}{\partial n} \right|^2 ds + \int_{\hat{\Gamma}_k} z^2 \, ds \right\}.
\end{aligned}$$

On the other hand, within the assumptions made on $\hat{S}_k$ we certainly have:

$$\int_{\hat{\Gamma}_k} \left| \frac{\partial z}{\partial n} \right|^2 ds + \int_{\hat{\Gamma}_k} z^2 \, ds \leqq C \| z \|^2_{H^2(\hat{S}_k)}, \qquad z \in \hat{Q}_k. \tag{3.8}$$

Thus, we conclude that (3.4) holds also in the case of a curved $\hat{\Gamma}_k$ if h is small enough.

As a consequence of (3.4) and (3.8) we have in particular that

$$\left.\begin{aligned}
&\int_{\hat{\Gamma}_k} \left| \frac{\partial z}{\partial n} \right|^2 ds + \int_{\hat{\Gamma}_k} z^2 \, ds \\
&\qquad \leqq C \left\{ \int_{\hat{S}_k} |\Delta z|^2 \, dx + \sum_{i \in \Lambda_k} \left| \int_{\hat{\Gamma}_k} \xi_i \frac{\partial z}{\partial n} ds \right|^2 + \sum_{i \in \Lambda_k} \left| \int_{\hat{\Gamma}_k} \xi_i z \, ds \right|^2 \right\}, \\
&\qquad\qquad z \in \hat{Q}_k, \quad k \geqq k_0.
\end{aligned}\right\} \tag{3.9}$$

Using this inequality it is now easy to complete the proof: Take $v \in M^h$ to be such that (3.1) is satisfied, and choose a collection $\{ S_k^{(j)}, \Gamma_k^{(j)} \}_{j=1}^{\nu}$, $k \geqq k_0$, such that $\bigcup_{j=1}^{\nu} \Gamma_k^{(j)} = \partial\Omega$ and for all j, $S_k^{(j)} \cap S_k^{(l)} = \emptyset$ for all except three values of l. Then if

$$v_j(x) = v(hx), \qquad x \in \hat{S}_k^{(j)} = A(S_k^{(j)}),$$

we have, setting $z = v_j$ *in* (3.9), *that*

$$\int_{\hat{\Gamma}_k^{(j)}} \xi_i \frac{\partial v_j}{\partial n} ds = \int_{\hat{\Gamma}_k^{(j)}} \xi_i v_j \, ds = 0,$$

and hence

$$\int_{\hat{\Gamma}_k^{(j)}} \left| \frac{\partial v_j}{\partial n} \right|^2 ds + \int_{\hat{\Gamma}_k^{(j)}} v_j^2 \, ds \leqq C \int_{\hat{S}_k^{(j)}} |\Delta v_j|^2 \, dx, \qquad j = 1, \ldots, \nu.$$

Upon scaling scaling back to the original size we get

$$h^{-1}\int_{\Gamma_k^{(j)}}\left|\frac{\partial v}{\partial n}\right|^2 ds+h^{-3}\int_{\Gamma_k^{(j)}} v^2\,ds \leqq C\int_{S_k^{(j)}}|\Delta v|^2\,dx, \qquad j=1,\ \dots,\ \nu.$$

Summing over j, we now obtain the desired inequality, and the proof of proposition 1 is complete. □

Proof of proposition 2: Let $(\xi,\eta)\in U^h\times V^h$ be given, let $\{x^1,\ \dots,\ x^\nu\}$ be the set of vertices of the triangulation Π^h on $\partial\Omega$, and let $\{y^1,\ \dots,\ y^\nu\}$ be the set of midpoints of the sides on $\partial\Omega$ of the triangles in Π^h. We consider functions $u,\ v\in M^h$, which satisfy

$$\left.\begin{aligned} &u_n(x^i)=h\xi(x^i), \qquad u_n(y^i)=h\xi(y^i), \qquad u_{nt}(x^i)=h\xi_t(x^i),\\ &v(x^i)=h^3\eta(x^i), \qquad v_t(x^i)=h^3\eta_t(x^i),\\ &i=1,\ \dots,\ \nu. \end{aligned}\right\} \tag{3.10}$$

Here u_n and u_t are respectively the normal and the tangential derivative of u on $\partial\Omega$.

Among the functions $u,\ v\in M^h$ that satisfy (3.10), let u_0 and v_0 be those obtained by setting all the remaining degress of freedom (in the Argyris triangles) equal to zero. We prove first some estimates for u_0, v_0 and $w_0=u_0+v_0$.

LEMMA 3: *If h is small enough, then*

$$\int_\Omega|\Delta w_0|^2\,dx\leqq C\left\{h^{-1}\int_{\partial\Omega}\left|\frac{\partial w_0}{\partial n}\right|^2 ds+h^{-3}\int_{\partial\Omega} w_0^2\,ds\right\},$$

$$\left\|\frac{\partial u_0}{\partial n}\right\|_{L_2(\partial\Omega)}+h^{-2}\|u_0\|_{L_2(\partial\Omega)}\leqq Ch\|\xi\|_{L_2(\partial\Omega)},$$

$$\left\|\frac{\partial v_0}{\partial n}\right\|_{L_2(\partial\Omega)}+\|v_0\|_{L_2(\partial\Omega)}\leqq Ch^3\|\eta\|_{L_2(\partial\Omega)}.$$

Proof: Let $T\in\Pi^h$ be such that T has a curved side Γ on $\partial\Omega$, let $\hat T=A(T)$, $\hat\Gamma=A(\Gamma)$, where $A(x)=h^{-1}x$, $x\in R^2$, and let $\hat v(x)=v(hx)$ for v defined on T or Γ. We choose a coordinate system $\{x_1,\ x_2\}$ so as to represent $\hat\Gamma$ as

$$\left.\begin{aligned} &\hat\Gamma=\{(x_1,\ x_2);\ x_2=\theta(x_1),\ x_1\in I=[0,\ d]\},\\ &\theta(0)=\theta(d)=0, \qquad |\theta'(x_1)|\leqq Ch, \qquad x_1\in I. \end{aligned}\right\} \tag{3.11}$$

One can verify from (3.11) and from the minimal angle condition that if p is any polynomial of degree $\leqq 5$, then

$$\left|\frac{\partial^{k+l}p}{\partial x_1^k\,\partial x_2^l}(x_1,0)-\frac{\partial^{k+l}p}{\partial t^k\,\partial n^l}(x_1,\ \theta(x_1))\right| \leqq Ch\|p\|_{H^2(\hat T)}, \qquad x_1\in I, \quad k,\ l\geqq 0. \tag{3.12}$$

Further, since $p(x_1, 0)$ and $\partial p/\partial x_2(x_1, 0)$ are polynomials in x_1 of degree 5 and 4, respectively, we have, for some positive constants C_1 and C_2,

$$C_1 \int_0^d |p(x_1, 0)|^2 dx_1 \geqq \sum_{k=0}^{2} \left\{ \left| \frac{\partial^k p}{\partial x_1^k}(0,0) \right|^2 + \left| \frac{\partial^k p}{\partial x_1^k}(d, 0) \right|^2 \right\} \geqq C_2 \int_0^d |p(x_1, 0)|^2 dx_1, \quad (3.13)$$

and

$$C_1 \int_0^d \left| \frac{\partial p}{\partial x_2}(x_1, 0) \right|^2 dx_1 \geqq \left| \frac{\partial p}{\partial x_2}(0, 0) \right|^2 + \left| \frac{\partial p}{\partial x_2}(d, 0) \right|^2 + \left| \frac{\partial p}{\partial x_2}(y_1, 0) \right|^2 + \left| \frac{\partial^2 p}{\partial x_1 \partial x_2}(0, 0) \right|^2 + \left| \frac{\partial^2 p}{\partial x_1 \partial x_2}(d, 0) \right|^2 \geqq C_2 \int_0^d \left| \frac{\partial p}{\partial x_2}(x_1, 0) \right|^2 dx_1, \quad (3.14)$$

where $(y_1, \theta(y_1))$ is the midpoint of Γ.

We now apply the above inequalities in the particular case where $p = \hat{w}_0$. First, note that $\hat{w}_0$ is defined uniquely by the values of

$$\frac{\partial \hat{w}_0}{\partial x_2}(y_1, 0), \qquad \frac{\partial^{k+l} \hat{w}_0}{\partial x_1^k \partial x_2^l}(0, 0),$$

and

$$\frac{\partial^{k+l} \hat{w}_0}{\partial x_1^k \partial x_2^l}(d, 0), \qquad k+l \leqq 2,$$

and that

$$\frac{\partial^2 \hat{w}_0}{\partial n^2}(0, 0) = \frac{\partial^2 \hat{w}_0}{\partial t^2}(0, 0) = \frac{\partial^2 \hat{w}_0}{\partial n^2}(d, 0) = \frac{\partial^2 \hat{w}_0}{\partial t^2}(d, 0) = 0.$$

Using (3.12) through (3.14) we then have

$$\begin{aligned} \| \hat{w}_0 \|^2_{H^2(\hat{T})} &\leqq C \left\{ \sum_{k=0}^{2} [| D^k \hat{w}_0(0, 0) |^2 + | D^k \hat{w}_0(d, 0) |^2] + \left| \frac{\partial \hat{w}_0}{\partial x_2}(y_1, 0) \right|^2 \right\} \\ &\leqq C_1 \left\{ \int_0^d \left| \frac{\partial \hat{w}_0}{\partial x_2}(x_1, 0) \right|^2 dx_1 + \int_0^d \left| \hat{w}_0(x_1, 0) \right|^2 dx_1 \right\} + C_2 h^2 \| \hat{w}_0 \|^2_{H^2(\hat{T})} \\ &\leqq C_1 \left\{ \int_{\hat{\Gamma}} \left| \frac{\partial \hat{w}_0}{\partial n} \right|^2 ds + \int_{\hat{\Gamma}} \hat{w}_0^2 \, ds \right\} + C_3 h^2 \| \hat{w}_0 \|^2_{H^2(\hat{T})}, \end{aligned}$$

and so, for h small enough,

$$\|\hat{w}_0\|^2_{H^2(\hat{T})} \leqq C\left\{\int_{\hat{\Gamma}}\left|\frac{\partial \hat{w}_0}{\partial n}\right|^2 ds + \int_{\hat{\Gamma}} \hat{w}_0^2\, ds\right\}.$$

Next, let $p=\hat{u}_0$. Then (3.10) and (3.12) through (3.14) imply

$$\|\hat{u}_0\|_{L_2(\hat{\Gamma})} \leqq Ch\|\hat{u}_0\|_{H^2(\hat{T})} \leqq C_1 h\left\|\frac{\partial \hat{u}_0}{\partial n}\right\|_{L^2(\hat{\Gamma})}.$$

Further, using (3.10) and repeating some of the arguments used in the proof of lemma 2, we have

$$\left\|\frac{\partial \hat{u}_0}{\partial n}\right\|_{L_2(\hat{\Gamma})} \leqq Ch^2\|\hat{\xi}\|_{L_2(\hat{\Gamma})}.$$

By a similar logic, one can verify that

$$\left\|\frac{\partial \hat{v}_0}{\partial n}\right\|_{L_2(\hat{\Gamma})} \leqq Ch\|\hat{v}_0\|_{L_2(\hat{\Gamma})} \leqq C_1 h^4\|\hat{\eta}\|_{L_2(\hat{\Gamma})}.$$

Consider finally a triangle $T\in\Pi^h$ which has only a vertex on $\partial\Omega$. Let this vertex be shared by the triangles $T_1, T_2\in\Pi^h$, both of which have a side on $\partial\Omega$. Then if $\hat{T}=A(T)$, $\hat{T}_i=A(T_i)$, it is easy to verify from the definition of w_0 that

$$\|\hat{w}_0\|^2_{H^2(\hat{T})} \leqq C\{\|\hat{w}_0\|^2_{H^2(\hat{T}_1)} + \|\hat{w}_0\|^2_{H^2(\hat{T}_2)}\}.$$

Upon scaling in the last five inequalities obtained above, summing over T and Γ, and noting that w_0 vanishes on any triangle $T\in\Pi^h$ that does not touch $\partial\Omega$, the asserted inequalities follow. □

In view of lemma 3, if we set $v=w_0$, the second inequality of proposition 2 is proved. To prove the first inequality, note first that we have

$$\left\|\frac{\partial u_0}{\partial n} - h\xi\right\|_{L_2(\partial\Omega)} \leqq Ch\|h\xi\|_{L_2(\partial\Omega)}.$$

This follows again from local arguments similar to those used in the proof of lemma 2. Using this we have that, for h small enough,

$$\int_{\partial\Omega}\frac{\partial u_0}{\partial n}\xi\, ds \geqq Ch\int_{\partial\Omega}\xi^2\, ds, \qquad C>0. \tag{3.15}$$

To continue, we need the following lemma. The proof is given in the Appendix.

LEMMA 4: *Let $p(t)$ be any polynomial of degree $\leqq 3$, and let $q(t)$ be a polynomial of degree $\leqq 5$ such that*

$$q(0)=p(0), \qquad q'(0)=p'(0), \qquad q(1)=p(1),$$
$$q'(1)=p'(1), \qquad q''(0)=q''(1)=0.$$

Then

$$\int_0^1 pq\,dt \geqq C \int_0^1 p^2\,dt, \qquad C>0.$$

Using lemma 4 and once again repeating arguments from the proof of lemma 2, we get that for h small enough,

$$\int_{\partial\Omega} v_0 \eta\,ds \geqq Ch^3 \int_{\partial\Omega} \eta^2\,ds, \qquad C \geqq 0. \tag{3.16}$$

Combining (3.15) and (3.16) with the inequalities of lemma 3 we now have

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial w_0}{\partial n}\xi\,ds + \int_{\partial\Omega} w_0 \eta\,ds &\geqq Ch \int_{\partial\Omega} \xi^2\,ds + Ch^3 \int_{\partial\Omega} \eta^2\,ds \\ &\qquad + \int_{\partial\Omega} \frac{\partial v_0}{\partial n}\xi\,ds + \int_{\partial\Omega} u_0 \eta\,ds \\ &\geqq Ch \int_{\partial\Omega} \xi^2\,ds + Ch^3 \int_{\partial\Omega} \eta^2\,ds - C_1 h^3 \|\xi\|_{L_2(\partial\Omega)} \|\eta\|_{L_2(\partial\Omega)} \\ &\geqq \left(C - \frac{1}{2} C_1 h\right) \left\{ h \int_{\partial\Omega} \xi^2\,ds + h^3 \int_{\partial\Omega} \eta^2\,ds \right\}, \qquad C>0.\end{aligned}$$

This proves the first inequality in proposition 2, with $v = w_0$, h sufficiently small. The proof is then complete. □

Using theorem 1, we can now evaluate the rate of convergence of the Lagrange multiplier method (2.4).

THEOREM 2: *Let (u, ψ, φ) be the solution of (2.1) for $f \in H^s(\Omega)$, $s > -1/2$, and let (u_h, ψ_h, φ_h) be the solution of (2.4) with the subspaces M^h, U^h, V^h defined as above. Then we have the error bound*

$$\begin{aligned}\sum_{k=0}^{2} h^{2k-4} \int_\Omega |D^k(u-u_h)|^2\,dx + h^{-1} \int_{\partial\Omega} \left| \frac{\partial}{\partial n}(u-u_h) \right|^2 ds \\ + h^{-3} \int_{\partial\Omega} |u-u_h|^2\,ds + h \int_{\partial\Omega} |\psi - \psi_h|^2\,ds + h^3 \int_{\partial\Omega} |\varphi - \varphi_h|^2\,ds \\ \leqq Ch^{2\mu} \|f\|^2_{H^s(\Omega)},\end{aligned}$$

$$\mu = \min\{4, s+2\}.$$

Proof: For u defined on Ω and sufficiently smooth, let $\tilde{u}$ be the interpolant of u on M^h. Then we have, by classical results of approximation theory (*cf.* [4]), the estimates

$$\sum_{k=0}^{2} h^{2k-4} \int_\Omega |D^k(u-\tilde{u})|^2\,dx \leqq Ch^{2s-4} \|u\|^2_{H^s(\Omega)},$$

$$u \in H^s(\Omega), \qquad 3 < s \leqq 6.$$

Reasoning by a local scaling argument analogous to that used in [7] one can also verify that

$$h^{-1}\int_{\partial\Omega}\left|\frac{\partial}{\partial n}(u-\tilde{u})\right|^2 ds+h^{-3}\int_{\partial\Omega}|u-\tilde{u}|^2 ds$$
$$\leqq C\sum_{k=0}^{2} h^{2k-4}\int_{\Delta}|D^k(u-\tilde{u})|^2 dx,$$

where Δ is the union of the triangles in Π^h that have a side on $\partial\Omega$.

On the other hand, by the definition of the space $U^h=V^h$ and again by classical results of approximation theory, we have

$$\min_{\xi\in U^h}\|\psi-\xi\|_{L_2(\partial\Omega)}\leqq Ch^s\|\psi\|_{H^s(\partial\Omega)},$$
$$\psi\in H^s(\partial\Omega),\qquad 0\leqq s\leqq 4.$$

Upon combining the above estimates with theorem 1 and with the *a priori* estimate (2.3) we have proved:

$$\|(u-u_h,\ \psi-\psi_h,\ \varphi-\varphi_h)\|_h\leqq Ch^{\mu}\|f\|_{H^s(\Omega)},$$
$$s>-\frac{1}{2},\qquad \mu=\min\{4,\ s+2\}.$$

To complete the proof, we use the Aubin-Nitsche duality argument together with (2.3), (3.3), and the above approximation results to conclude that

$$\|u-u_h\|_{L_2(\Omega)}\leqq Ch^2\|(u-u_h,\ \psi-\psi_h,\ \varphi-\varphi_h)\|_h.$$

Finally, since partitioning Π^h is quasiuniform, we have the inverse estimates

$$\int_{\Omega}|D^k(u-u_h)|^2 dx\leqq C\Big\{h^{-2k}\|u-u_h\|^2_{L_2(\Omega)}$$
$$+\min_{v\in M^h}\Big\{h^{-2k}\|u-v\|^2_{L_2(\Omega)}+\int_{\Omega}|D^k(u-v)|^2 dx\Big\}\Big\},$$
$$k=1,2.$$

Upon combining the last three estimates, the assertion of the theorem follows. □

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APPENDIX

PROOF OF LEMMA 4

Let

$$p(t)=\sum_{i=0}^{3} \alpha_i t^i, \qquad t\in[0, 1].$$

Then the polynomial $q(t)$ of degree $\leqq 5$ which satisfies

$$q(t_0)=p(t_0), \qquad q'(t_0)=p'(t_0), \qquad q''(t_0)=0, \qquad t_0=0, 1,$$

is given by

$$q(t)=\alpha_0+\alpha_1 t+\alpha_2(2t^3-t^4)+\alpha_3(-2t^3+6t^4-3t^5).$$

We then have

$$\int_0^1 pq\,dt=[\alpha]^T [A] [\alpha],$$

where $[\alpha]^T=[\alpha_0, \ldots, \alpha_3]$ and the 4×4 matrix $[A]$ is given by

$$[A]=\begin{bmatrix} 1 & \frac{1}{2} & \frac{19}{60} & \frac{9}{40} \\ & \frac{1}{3} & \frac{29}{120} & \frac{13}{70} \\ & sym. & \frac{4}{21} & \frac{13}{84} \\ & & & \frac{11}{84} \end{bmatrix}$$

By a direct computation, the characteristic equation of $[A]$ can be written into the form

$$\sum_{i=0}^{4} (-1)^i c_i \lambda^i = 0, \qquad c_i > 0$$

Hence, all the eigenvalues of $[A]$ are positive If in particular $\lambda_0 > 0$ is the smallest eigenvalue, we have

$$\int_0^1 pq\,dt = [\alpha]^T [A] [\alpha] \geqq \lambda_0 [\alpha]^T [\alpha] \geqq \lambda_0 C \int_0^1 p^2\,dt, \qquad C > 0,$$

which proves the assertion $\square$